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Taylor-Couette flow of an Oldroyd-B fluid in a circular cylinder subject to a time-dependent rotation

by
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Abstract

The velocity and the shear stress, corresponding to the unsteady flow of an Oldroyd-B fluid in an infinite circular cylinder subject to a time-dependent couple, are established by means of the Hankel transform. The similar solutions for Maxwell, Second grade and Newtonian fluids can be obtained as limiting cases of general solutions. Finally, the influence of the material parameters on the velocity profile and the shear stress is spotlighted by means of graphical illustrations.

Key Words: Oldroyd-B fluid; Taylor-Couette flow; Velocity field; Shear stress.
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1 Introduction

The study of the motion of a fluid in the neighbourhood of a rotating or sliding body is of great interest for industry. The flow between rotating cylinders or through a rotating cylinder has applications in the food industry, it being one of the most important and interesting problems of motion near rotating bodies. As early as 1886, Stokes [1] established an exact solution for the rotational oscillations of an infinite rod immersed in a linearly viscous fluid. However, such motions have been intensively studied since G.I. Taylor (1923) reported the results of his famous investigations [2]. For Newtonian fluids, the velocity distribution for a fluid contained in a circular cylinder can be found in [3]. The first exact solutions corresponding to different motions of non-Newtonian fluids, in cylindrical domains, seem to be those of Ting [4], Srivastava [5] and Waters and King [6]. In the meantime a lot of papers regarding such motions have been published. The interested readers can see for instance the papers [7-14] and their related references.

However, it is worthy pointing out that all above mentioned works dealt with problems in which the velocity is given on the boundary. To the best of our knowledge, the first exact solutions for flows into cylindrical domains when the shear stress is given on the boundary are those obtained by Bandelli and Rajagopal [15] for Second grade fluids. Similar solutions, corresponding to a time-dependent shear stres on the boundary, have been recently obtained in [16-19].

The aim of this paper is to present exact solutions corresponding to the flow of an Oldroyd-B fluid due to an infinite circular cylinder subject to a time-dependent shear stress. Such solutions, in addition to serving as approximations to some specific initial value problems also serve to a very important purpose, namely, they can be used as tests to verify numerical schemes that are developed to study complex unsteady flow problems. Just as in the case of Newtonian fluids, it is necessary to develop a large class of exact and approximate solutions for fluids such as the Oldroyd-B fluid as it has been found to approximate the response of many dilute polymeric liquids. The solutions that have been here obtained tend to the similar solutions for Maxwell, Second grade and Newtonian fluids by taking appropriate limits. Furthermore, our limiting solutions (25) and (26) are different of those obtained in [19, Eqs. (47) and (49)] which are wrong. Finally, the influence of physical parameters on the velocity profile is shown by graphical illustrations.

2 Governing equations

The incompressible Oldroyd-B fluids are characterized by the constitutive equations [6,7,9,10,14]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S},\ \mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T) = \mu[\mathbf{A} + \lambda_r(\dot{\mathbf{A}} - \mathbf{L}\mathbf{A} - \mathbf{A}\mathbf{L}^T)], \tag{1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $-p\mathbf{I}$ denotes the indeterminate spherical stress, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{V}$ is the velocity, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin-Ericksen tensor, μ is the dynamic viscosity, λ and λ_r are relaxation and retardation times and the superposed dot indicates the material time derivative. This model includes as special cases the Maxwell model and the linearly viscous fluid model. In some special flows, like those to be considered here, the governing equations for Oldroyd-B fluids resemble those for second grade fluids.

For the problem under consideration we assume a velocity field and an extra-stress tensor of the form

$$\mathbf{V} = \mathbf{V}(r,t) = \omega(r,t)\mathbf{e}_\theta,\ \mathbf{S} = \mathbf{S}(r,t), \tag{2}$$

where $\mathbf{e}_\theta$ is the unit vector in the θ-direction of a cylindrical coordinate system r, θ, z. For these flows the constraint of incompressibility is automatically satisfied. Substituting (2) into $(1)_2$ and having in mind the initial condition

$$\mathbf{S}(y,0) = \mathbf{0}\ \text{(the fluid being at rest up to the moment } t=0\text{)}, \tag{3}$$

we find that $S_{rr} = S_{zz} = S_{rz} = S_{\theta z} = 0$. In the absence of a pressure gradient in the axial direction and neglecting body forces, the linear momentum and the constitutive equation $(1)_2$ lead to the relevant partial differential equations[14]

$$(1+\lambda\partial_t)\tau(r,t) = \mu(1+\lambda_r\partial_t)\left(\partial_r - \frac{1}{r}\right)\omega(r,t)\,,\ \rho\partial_t\omega(r,t) = \left(\partial_r + \frac{2}{r}\right)\tau(r,t), \quad (4)$$

where $\tau = S_{rz}$ is the shear stress that is different of zero and ρ is the constant density of the fluid.

Eliminating $\tau(r,t)$ between Eqs. (4) we obtain the governing equation

$$\lambda\partial_t^2\omega(r,t) + \partial_t\omega(r,t) = (\nu + \alpha\partial_t)\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\omega(r,t), \quad (5)$$

where $\alpha = \nu\lambda_r$ and $\nu = \mu/\rho$ is the kinematic viscosity of the fluid. The partial differential equation (5) with adequate initial and boundary conditions, can be solved in principle by several methods, the integral transforms technique representing a systematic, efficient and powerful tool. The Laplace transform can be used to eliminate the time variable, while the finite Hankel transform can be employed to eliminate the spatial variable. Bandelli and Rajagopal [15] as well as Bandelli et al [20] showed that the Laplace transform does not work for some problems regarding second grade fluids. More exactly, the obtained solutions by means of the Laplace transform do not satisfy the initial conditions due to the incompatibility between the prescribed data. Furthermore, the inversion procedure for obtaining the solution is not always a trivial matter. Consequently, we shall use the Hankel transform here.

3 Taylor-Couette flow within a circular cylinder subject to a time-dependent couple

Consider an Oldroyd-B fluid at rest in an infinitely long circular cylinder of radius R. At time $t = 0^+$ the cylinder is set in rotation about its axis by a time-dependent torque

$$\tau(R,t) = f[t - \lambda(1 - e^{-t/\lambda})];\ f = \text{constant}, \quad (6)$$

per unit length. Due to the shear, the fluid is gradually moved its velocity being of the form (2). The governing equation is given by Eq. (5), while the appropriate initial and boundary conditions are

$$\omega(r,0) = \partial_t\omega(r,0) = 0\,;\quad r \in [0,R), \quad (7)$$

respectively,

$$(1+\lambda\partial_t)\tau(R,t) = \mu(1+\lambda_r\partial_t)\left(\partial_r - \frac{1}{r}\right)\omega(r,t)\mid_{r=R} = ft\,;\ t \geq 0. \quad (8)$$

In order to solve the linear partial differential equation (5), with the initial and boundary conditions (7) and (8), we denote by

$$\omega_{nH}(t) = \int_0^R r\omega(r,t)J_1(rr_n)dr, \tag{9}$$

the finite Hankel transform of $\omega(r,t)$, where $r_n\,(n = 1, 2, 3, ...)$ are the positive roots of the transcendental equation $J_2(Rr) = 0$, and use the known relation [21, Eq. (13.4.31)]

$$\int_0^R r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\omega(r,t)J_1(rr_n)dr =$$

$$RJ_1(Rr_n)\left(\partial_r - \frac{1}{r}\right)\omega(r,t)\mid_{r=R} - r_n^2\omega_{nH}(t). \tag{10}$$

Multiplying now Eq. (5) by $rJ_1(rr_n)$, integrating the result with respect to r from 0 to R and using the identity (10), it results that

$$\lambda\ddot{\omega}_{nH}(t) + (1 + \alpha r_n^2)\dot{\omega}_{nH}(t) + \nu r_n^2\omega_{nH}(t) = \frac{f}{\rho}tRJ_1(Rr_n);\;\; t > 0. \tag{11}$$

Equalities (7) also imply

$$\omega_{nH}(0) = \dot{\omega}_{nH}(0) = 0. \tag{12}$$

The solution of the ordinary differential equation (11) with the initial conditions (12) is

$$\omega_{nH}(t) =$$

$$\frac{f}{\mu}\frac{RJ_1(Rr_n)}{r_n^2}\left[t - \frac{e^{q_{2n}t} - e^{q_{1n}t}}{q_{2n} - q_{1n}} - \frac{1 + \alpha r_n^2}{\nu r_n^2}\left(1 - \frac{q_{2n}e^{q_{1n}t} - q_{1n}e^{q_{2n}t}}{q_{2n} - q_{1n}}\right)\right], \tag{13}$$

$$\text{where } q_{1n}, q_{2n} = \frac{-(1 + \alpha r_n^2) \pm \sqrt{(1 + \alpha r_n^2)^2 - 4\nu\lambda r_n^2}}{2\lambda}.$$

Finally, applying the inverse Hankel transform formula [21, Eq. (13.4.30)] and using the identity

$$r^3 = 4R\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^2 J_1(Rr_n)},$$

we find the velocity field $\omega(r,t)$ under the form

$$\begin{aligned}\omega(r,t) &= \frac{fr^3}{2\mu R^2}(t-\lambda_r) - \frac{2f}{\mu\nu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^4 J_1(Rr_n)} + \\ &+\frac{2f}{\mu R}\sum_{n=1}^{\infty}\left[\frac{1+\alpha r_n^2}{\nu r_n^2}\frac{q_{2n}e^{q_{1n}t}-q_{1n}e^{q_{2n}t}}{q_{2n}-q_{1n}} - \right.\\ &\left. -\frac{e^{q_{2n}t}-e^{q_{1n}t}}{q_{2n}-q_{1n}}\right]\frac{J_1(rr_n)}{r_n^2 J_1(Rr_n)},\end{aligned} \tag{14}$$

or equivalently

$$\omega(r,t) = \frac{fr^3}{2\mu R^2}(t-\lambda_r) - \frac{2f}{\mu\nu R}\sum_{n=1}^{\infty}\left(1-\lambda\frac{q_{1n}^2 e^{q_{2n}t}-q_{2n}^2 e^{q_{1n}t}}{q_{2n}-q_{1n}}\right)\frac{J_1(rr_n)}{r_n^4 J_1(Rr_n)}. \tag{15}$$

In order to determine the shear stress, we use Eqs. (3) and (4), and get

$$\tau(r,t) = \frac{\mu}{\lambda}e^{-t/\lambda}\int_0^t e^{\tau/\lambda}\left(1+\lambda_r\frac{\partial}{\partial\tau}\right)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\omega(r,\tau)d\tau. \tag{16}$$

Substituting (14) into (16) and using the identities

$$q_{1n}q_{2n} = \frac{\nu r_n^2}{\lambda},\ q_{3n}q_{4n} = \frac{\nu(\lambda-\lambda_r)r_n^2}{\lambda^2},\ q_{1n}q_{3n} = -\nu r_n^2\frac{1+\lambda_r q_{1n}}{\lambda},$$

$$q_{2n}q_{4n} = -\nu r_n^2\frac{1+\lambda_r q_{2n}}{\lambda},\ q_{1n}q_{4n} = \frac{\nu r_n^2+q_{1n}}{\lambda},\ q_{2n}q_{3n} = \frac{\nu r_n^2+q_{2n}}{\lambda},$$

where $\lambda q_{3n} = 1+\lambda q_{1n}$ and $\lambda q_{4n} = 1+\lambda q_{2n}$, we obtain, after lengthy but straightforward computations, the next suitable form for the shear stress

$$\begin{aligned}\tau(r,t) &= \frac{fr^2}{R^2}\left[t-\lambda(1-e^{-t/\lambda})\right] + \frac{2f}{\nu R}(1-e^{-t/\lambda})\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{r_n^3 J_1(Rr_n)} - \\ &-\frac{2f}{\nu R}\frac{1}{\lambda-\lambda_r}\sum_{n=1}^{\infty}\left[\frac{e^{q_{2n}t}-e^{q_{1n}t}}{q_{2n}-q_{1n}} - \lambda_r\frac{q_{2n}e^{q_{1n}t}-q_{1n}e^{q_{2n}t}}{q_{2n}-q_{1n}} + \right.\\ &\left.+\lambda_r e^{-t/\lambda}\right]\frac{1+\alpha r_n^2}{r_n^3}\frac{J_2(rr_n)}{J_1(Rr_n)} - \\ &-\frac{2f}{\nu R}\frac{\lambda}{\lambda-\lambda_r}\sum_{n=1}^{\infty}\left[\frac{q_{4n}e^{q_{1n}t}-q_{3n}e^{q_{2n}t}}{q_{2n}-q_{1n}} - e^{-t/\lambda}\right]\frac{J_2(rr_n)}{r_n^3 J_1(Rr_n)} - \\ &-\frac{2f}{\nu R}\frac{\lambda_r}{\lambda-\lambda_r}\sum_{n=1}^{\infty}\left[\frac{q_{4n}e^{q_{1n}t}-q_{3n}e^{q_{2n}t}}{q_{2n}-q_{1n}} - e^{-t/\lambda}\right]\frac{1+\alpha r_n^2}{r_n^3}\frac{J_2(rr_n)}{J_1(Rr_n)} + \\ &+\frac{2f}{\nu R}\frac{\lambda_r}{\lambda-\lambda_r}\sum_{n=1}^{\infty}\left[\frac{q_{2n}e^{q_{2n}t}-q_{1n}e^{q_{1n}t}}{q_{2n}-q_{1n}} + \nu r_n^2\frac{e^{q_{2n}t}-e^{q_{1n}t}}{q_{2n}-q_{1n}} - \right.\\ &\left.-e^{-t/\lambda}\right]\frac{J_2(rr_n)}{r_n^3 J_1(Rr_n)}.\end{aligned} \tag{17}$$

Of course, Eq. (17) can be processed to give the simple form

$$\tau(r,t) = \frac{fr^2}{R^2}\left[t - \lambda(1 - e^{-t/\lambda})\right] +$$

$$+\frac{2f}{\nu R}\sum_{n=1}^{\infty}\left[1 - \frac{q_{2n}e^{q_{1n}t} - q_{1n}e^{q_{2n}t}}{q_{2n} - q_{1n}}\right]\frac{J_2(rr_n)}{r_n^3 J_1(Rr_n)}. \tag{18}$$

4 Limiting cases

1.Making the limit of Eqs. (15) and (18) as $\lambda_r \to 0$, we obtain the solutions

$$\omega(r,t) = \frac{fr^3}{2\mu R^2} - \frac{2f}{\mu\nu R}\sum_{n=1}^{\infty}\left[1 - \lambda\frac{q_{5n}^2 e^{q_{6n}t} - q_{6n}^2 e^{q_{5n}t}}{q_{6n} - q_{5n}}\right]\frac{J_1(rr_n)}{r_n^4 J_1(Rr_n)}, \tag{19}$$

$$\tau(r,t) = \frac{fr^2}{R^2}\left[t - \lambda(1 - e^{-t/\lambda})\right] +$$

$$\frac{2f}{\nu R}\sum_{n=1}^{\infty}\left[1 - \frac{q_{6n}e^{q_{5n}t} - q_{5n}e^{q_{6n}t}}{q_{6n} - q_{5n}}\right]\frac{J_2(rr_n)}{r_n^3 J_1(Rr_n)}, \tag{20}$$

corresponding to a Maxwell fluid performing the same motion. Into above relations

$$q_{5n}, q_{6n} = \frac{-1 \pm \sqrt{1 - 4\nu\lambda r_n^2}}{2\lambda}.$$

2. By now letting $\lambda \to 0$ into Eqs. (15) and (18), the similar solutions

$$\omega(r,t) = \frac{fr^3}{2\mu R^2}(t - \lambda_r) - \frac{2f}{\mu\nu R}\sum_{n=1}^{\infty}\Bigg[1 - (1 + \alpha r_n^2)exp\left(-\frac{\nu r_n^2 t}{1 + \alpha r_n^2}\right)\Bigg]\frac{J_1(rr_n)}{r_n^4 J_1(Rr_n)}, \tag{21}$$

$$\tau(r,t) = \frac{fr^2}{R^2}t + \frac{2f}{\nu R}\sum_{n=1}^{\infty}\left[1 - exp\left(-\frac{\nu r_n^2 t}{1 + \alpha r_n^2}\right)\right]\frac{J_2(rr_n)}{r_n^4 J_1(Rr_n)}, \tag{22}$$

corresponding to a second grade fluid are obtained.
3. Finally, taking the limit of Eqs. (19), (20) or (21), (22) when $\lambda \to 0$, respectively, $\lambda_r \to 0$ the simple solutions [18, Eqs. (29) and (37)]

$$\omega(r,t) = \frac{fr^3}{2\mu R^2}t - \frac{2f}{\mu\nu R}\sum_{n=1}^{\infty}\left(1 - e^{-\nu r_n^2 t}\right)\frac{J_1(rr_n)}{r_n^4 J_1(Rr_n)}, \tag{23}$$

$$\tau(r,t) = \frac{fr^2}{R^2}t + \frac{2f}{\nu R}\sum_{n=1}^{\infty}\left(1 - e^{-\nu r_n^2 t}\right)\frac{J_2(rr_n)}{r_n^3 J_1(Rr_n)}, \tag{24}$$

for a Newtonian fluid are obtained. Of course, the boundary condition corresponding to the last two cases, as it results from Eq. (6) for $\lambda \to 0$ is

$$\tau(R,t) = ft. \tag{25}$$

5 Numerical Results and Conclusions

The main purpose of this paper is to provide exact solutions for the velocity field $\omega(r,t)$ and the shear stress $\tau(r,t)$ corresponding to the non-steady rotational flow of an Oldroyd-B fluid induced by an infinite circular cylinder subject to a time-dependent shear stress. These solutions, obtained by means of the Hankel transforms and presented under series form in terms of Bessel functions $J_0(.)$, $J_1(.)$ and $J_2(.)$, satisfy all imposed initial and boundary conditions. To the best of our knowledge, the solutions that have been here obtained seem to be the first exact solutions for motions of Oldroyd-B fluids corresponding to a problem with time-dependent shear stress on the boundary.

For $\lambda_r \to 0$, the general solutions (15) and (18) reduce to the solutions (19) and (20) for a Maxwell fluid performing the same motion. It is remarkable the fact that the similar solutions (21) and (22) for second grade fluids, can be also obtained as limiting cases (for $\lambda \to 0$) from general solutions, although the constitutive equations for Oldroyd-B fluids do not contain as a special case the constitutive equations of second grade fluids. Furthermore, making $\lambda \to 0$ into (19) and (20) or $\lambda_r \to 0$ into (21) and (22), the solutions (23) and (24) for Newtonian fluids are recovered. Of course, these solutions are different of those established in [17], which are completely wrong.

Finally, in order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity field $\omega(r,t)$ and the shear stress $\tau(r,t)$, given by Eqs. (15), (23) and (17), (24), have been drawn against r for different values of t and of the material parameters. Figs. 1 and 4 show the influence of time on the velocity and the shear stress, respectively. From these figures it is clearly seen that the flow velocity as well as the shear stress increases with time. The influence of the retardation time λ_r on the velocity is shown in Fig. 2. From diagrams it results that, in accordance with our expectations, the velocity of the fluid decreases for increasing λ_r. The corresponding results for the shear stress are given in Fig. 5. They are in agreement with those coming from Fig. 2 for the velocity field. Last figures 3 and 6, show that for λ and $\lambda_r \to 0$, the velocity as well as the shear stress corresponding to the non-Newtonian fluids, as it was to be expected, are going to those belonging to a Newtonian fluid.

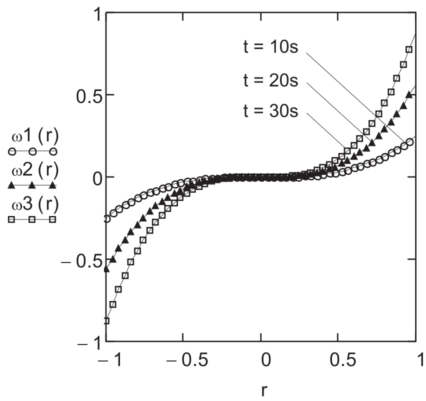

Fig. 1. Profiles of the velocity field $\omega(r, t)$ given by Eq. (16) - curves $\omega 1(r)$, $\omega 2(r)$, $\omega 3(r)$, for $\nu = 0.0357541$, $\mu = 32$, $R = 1$, $f = 2$, $\lambda = 10$, $\lambda_r = 2$ and different values of t.

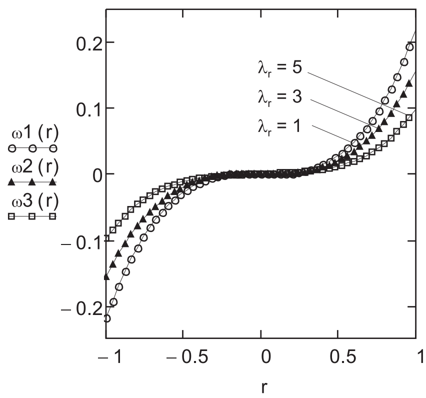

Fig. 2. Profiles of the velocity field $\omega(r, t)$ given by Eq. (16) - curves $\omega 1(r)$, $\omega 2(r)$, $\omega 3(r)$, for $\nu = 0.0357541$, $\mu = 32$, $R = 1$, $f = 2$, $\lambda = 10$, $t = 8s$ and different values of λ_r.

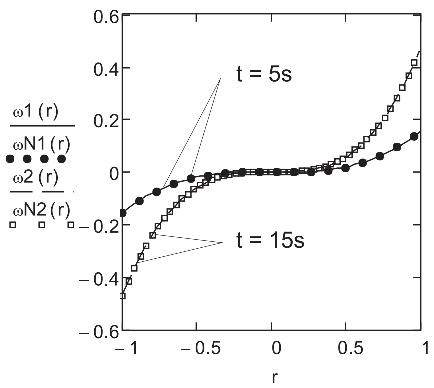

Fig. 3. Profiles of the velocity field ω(r, t) given by Eq. (16) - curves ω1(r), ω2(r) and Eq. (23) - curves ωN1(r), ωN2(r), for ν = 0.0357541, μ = 32, R = 1, f = 2, λ = 0.01, λ_r = 0.01 and different values of t.

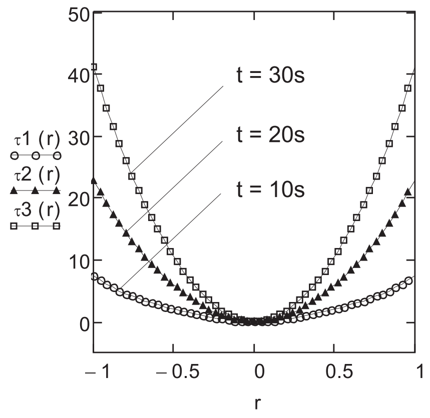

Fig. 4. Profiles of the shear stress τ(r, t) given by Eq. (18) - curves τ1(r), τ2(r), τ2(r), for ν = 0.0357541, μ = 32, R = 1, f = 2, λ = 10, λ_r = 2 and different values of t.

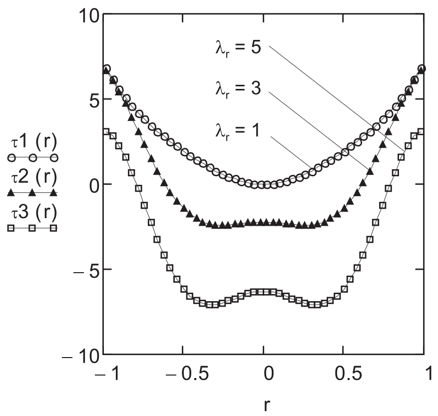

Fig. 5. Profiles of the shear stress $\tau(r, t)$ given by Eq. (18) - curves $\tau 1(r)$, $\tau 2(r)$, $\tau 2(r)$, for $\nu = 0.0357541$, $\mu = 32$, $R = 1$, $f = 2$, $\lambda = 10$, $t = 8s$ and different values of λ_r.

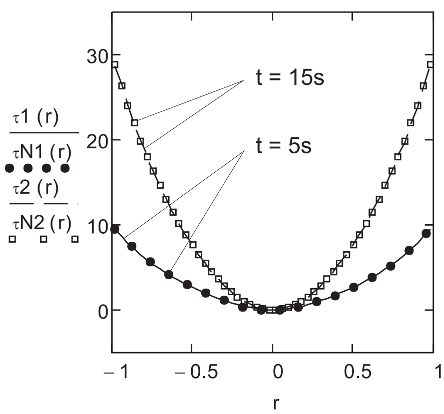

Fig. 6. Variation of the shear stress $\tau(y, t)$ given by Eq. (18) - curves $\tau 1(r)$, $\tau 2(r)$ and Eq. (24) - curves $\tau N1(r)$, $\tau N2(r)$, for $\nu = 0.0357541$, $\mu = 32$, $R = 1$, $f = 2$, $\lambda = 0.02$, $\lambda_r = 0.01$ and different values of t.

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