A class of transversal polymatroids with Gorenstein base ring

by

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Abstract

In this paper, the principal tool to describe transversal polymatroids with Gorenstein base ring is polyhedral geometry, especially the Danilov-Stanley theorem for the characterization of canonical module. Also, we compute the $a$-invariant and the Hilbert series of base ring associated to this class of transversal polymatroids.

Key Words: Base ring, transversal polymatroid, equations of a cone, $a$-invariant, canonical module, Hilbert series.

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1 Introduction

In this paper we determine the facets of the polyhedral cone generated by the exponent set of the monomials defining the base ring associated to a transversal polymatroid. The importance of knowing those facets comes from the fact that the canonical module of the base ring can be expressed in terms of the relative interior of the cone. This would allow one to compute the $a$-invariant of those base rings. The results presented were discovered by extensive computer algebra experiments performed with Normaliz [4].

2 Preliminaries

Let $n \in \mathbb{N}$, $n \geq 3$, $\sigma \in S_n$, $\sigma = (1, 2, \ldots, n)$ the cycle of length $n$, $[n] := \{1, 2, \ldots, n\}$ and $\{e_i\}_{1 \leq i \leq n}$ be the canonical base of $\mathbb{R}^n$. For a vector $x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, we will denote $\|x\| := x_1 + \ldots + x_n$. If $x^a$ is a monomial in $K[x_1, \ldots, x_n]$ we set $\log(x^a) = a$. Given a set $A$ of monomials, the log set of $A$, denoted $\log(A)$, consists of all $\log(x^a)$ with $x^a \in A$. 
We consider the following set of integer vectors of $\mathbb{N}^n$:

$$
\nu_{\sigma^0[i]} := \{-n-i-1, -(n-i-1), \ldots, -(n-i-1), (i+1), (i+1)\},
$$

$$
\nu_{\sigma^1[i]} := \{(i+1), -(n-i-1), \ldots, -(n-i-1), (i+1), (i+1)\},
$$

$$
\nu_{\sigma^2[i]} := \{(i+1), (i+1), -(n-i-1), \ldots, -(n-i-1), (i+1), (i+1)\},
$$

where $\sigma^k[i] := \{\sigma^k(1), \ldots, \sigma^k(i)\}$ for all $1 \leq i \leq n-1$ and $0 \leq k \leq n-1$.

Remark: $\nu_{\sigma^k[n-1]} = n \in [n] \setminus \sigma^k[n-1]$ for all $0 \leq k \leq n$.

For example, if $n = 4$, $\sigma = (1, 2, 3, 4) \in S_4$ then we have the following set of integer vectors:

$$
\nu_{\sigma^0[1]} = \nu_{\{1\}} = (-2, 2, 2, 2), \quad \nu_{\sigma^0[2]} = \nu_{\{1, 2\}} = (-1, -1, 3, 3),
$$

$$
\nu_{\sigma^1[1]} = \nu_{\{2\}} = (2, -2, 2, 2), \quad \nu_{\sigma^1[2]} = \nu_{\{2, 3\}} = (3, -1, -1, 3),
$$

$$
\nu_{\sigma^2[1]} = \nu_{\{3\}} = (2, 2, -2, 2), \quad \nu_{\sigma^2[2]} = \nu_{\{3, 4\}} = (3, 3, -1, -1),
$$

$$
\nu_{\sigma^3[1]} = \nu_{\{4\}} = (2, 2, 2, -2), \quad \nu_{\sigma^3[2]} = \nu_{\{1, 4\}} = (-1, 3, 3, -1).
$$

$$
\nu_{\sigma^0[3]} = \nu_{\{1, 2, 3\}} = (0, 0, 0, 4), \quad \nu_{\sigma^1[3]} = \nu_{\{2, 3, 4\}} = (4, 0, 0, 0),
$$

$$
\nu_{\sigma^2[3]} = \nu_{\{1, 3, 4\}} = (0, 4, 0, 0), \quad \nu_{\sigma^3[3]} = \nu_{\{1, 2, 4\}} = (0, 0, 4, 0).
$$
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If $0 \neq a \in \mathbb{R}^n$, then $H_a$ will denote the hyperplane of $\mathbb{R}^n$ through the origin with normal vector $a$, that is,

$$H_a = \{ x \in \mathbb{R}^n \mid \langle x, a \rangle = 0 \},$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^n$. The two closed halfspaces bounded by $H_a$ are:

$$H^+_a = \{ x \in \mathbb{R}^n \mid \langle x, a \rangle \geq 0 \} \quad \text{and} \quad H^-_a = \{ x \in \mathbb{R}^n \mid \langle x, a \rangle \leq 0 \}.$$

We will denote by $H_{\sigma^k[i]}$ the hyperplane of $\mathbb{R}^n$ through the origin with normal vector $\nu_{\sigma^k[i]}$, that is,

$$H_{\nu_{\sigma^k[i]}} = \{ x \in \mathbb{R}^n \mid \langle x, \nu_{\sigma^k[i]} \rangle = 0 \},$$

for all $1 \leq i \leq n - 1$ and $0 \leq k \leq n - 1$.

An affine space in $\mathbb{R}^n$ is the translate of a linear subspace of $\mathbb{R}^n$. Let $A \subset \mathbb{R}^n$, we denote by $\text{aff}(A)$ the affine space generated by $A$. There is a unique linear subspace $V$ of $\mathbb{R}^n$ such that $\text{aff}(A) = x_0 + V$, for some $x_0 \in \mathbb{R}^n$. The dimension of $\text{aff}(A)$ is $\dim(\text{aff}(A)) = \dim_{\mathbb{R}}(V)$.

Recall that a polyhedral cone $Q \subset \mathbb{R}^n$ is the intersection of a finite number of closed subspaces of the form $H^+_a$. If $Q = H^+_{a_1} \cap \ldots \cap H^+_{a_m}$ is a polyhedral cone, then $\text{aff}(Q)$ is the intersection of those hyperplanes $H_{a_i}$, $i = 1, \ldots, m$, that contain $Q$ (see [3, Proposition 1.2.]) The dimension of $Q$ is the dimension of $\text{aff}(Q)$, $\dim(Q) = \dim(\text{aff}(Q))$.

If $A = \{ \gamma_1, \ldots, \gamma_r \}$ is a finite set of points in $\mathbb{R}^n$ the cone generated by $A$, denoted by $\mathbb{R}_+A$, is defined as

$$\mathbb{R}_+A = \{ \sum_{i=1}^r a_i \gamma_i \mid a_i \in \mathbb{R}_+ \text{ for all } 1 \leq i \leq n \}.$$

An important fact is that $Q$ is a polyhedral cone in $\mathbb{R}^n$ if and only if there exists a finite set $A \subset \mathbb{R}^n$ such that $Q = \mathbb{R}_+A$ (see [3] or [10, Theorem 4.1.1.]).

Next we give some important definitions and results (see [1], [2], [3], [8], [9]).

**Definition 2.1.** A proper face of a polyhedral cone is a subset $F \subset Q$ such that there is a supporting hyperplane $H_a$ satisfying:

1. $F = Q \cap H_a \neq 0$,
2. $Q \not\subseteq H_a$ and $Q \subset H^+_a$.

The dimension of a proper face $F$ of a polyhedral cone $Q$ is: $\dim(F) = \dim(\text{aff}(F))$.

**Definition 2.2.** A cone $C$ is pointed if 0 is a face of $C$. Equivalently we can require that $x \in C$ and $-x \in C \Rightarrow x = 0$.

**Definition 2.3.** The 1-dimensional faces of a pointed cone are called extremal rays.
Definition 2.4. A proper face $F$ of a polyhedral cone $Q \subset \mathbb{R}^n$ is called a facet of $Q$ if $\dim(F) = \dim(Q) - 1$.

Definition 2.5. If a polyhedral cone $Q$ is written as

$$Q = H^+_{a_1} \cap \ldots \cap H^+_{a_r}$$

such that no $H^+_{a_i}$ can be omitted, then we say that this is an irreducible representation of $Q$.

Theorem 2.6. Let $Q \subset \mathbb{R}^n$, $Q \neq \mathbb{R}^n$, be a polyhedral cone with $\dim(Q) = n$.

Then the halfspaces $H^+_{a_1}, \ldots, H^+_{a_m}$ in an irreducible representation $Q = H^+_{a_1} \cap \ldots \cap H^+_{a_m}$ are uniquely determined. In fact, the sets $F_i = Q \cap H_{a_i}$, $i = 1, \ldots, n$, are the facets of $Q$.

Proof: See [3, Theorem 1.6.]

Definition 2.7. Let $Q$ be a polyhedral cone in $\mathbb{R}^n$ with $\dim Q = n$ and such that $Q \neq \mathbb{R}^n$. Let

$$Q = H^+_{a_1} \cap \ldots \cap H^+_{a_r}$$

be the irreducible representation of $Q$. If $a_i = (a_{i1}, \ldots, a_{in})$, then we call

$$H_{a_i}(x) := a_{i1}x_1 + \ldots + a_{in}x_n = 0, \quad i \in [r],$$

the equations of the cone $Q$.

Definition 2.8. The relative interior $ri(Q)$ of a polyhedral cone is the interior of $Q$ with respect to the embedding of $Q$ into its affine space aff($Q$), in which $Q$ is full-dimensional.

The following result gives us the description of the relative interior of a polyhedral cone when we know its irreducible representation.

Theorem 2.9. Let $Q \subset \mathbb{R}^n$, $Q \neq \mathbb{R}^n$, be a polyhedral cone with $\dim(Q) = n$ and let

$$Q = H^+_{a_1} \cap \ldots \cap H^+_{a_m}$$

be an irreducible representation of $Q$ with $H^+_{a_1}, \ldots, H^+_{a_m}$ pairwise distinct, where $a_i \in \mathbb{R}^n \setminus \{0\}$ for all $i$. Set $F_i = Q \cap H_{a_i}$ for $i \in [r]$. Then:

a) $ri(Q) = \{x \in \mathbb{R}^n \mid \langle x, a_i \rangle > 0, \ldots, \langle x, a_r \rangle > 0\}$, where $ri(Q)$ is the relative interior of $Q$, which in this case is just the interior.

b) Each facet $F$ of $Q$ is of the form $F = F_i$ for some $i$.

c) Each $F_i$ is a facet of $Q$.

Proof: See [1, Theorems 8.2.15 and 3.2.1].
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Theorem 2.10. (Danilov, Stanley) Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field $K$ and $F$ a finite set of monomials in $R$. If $K[F]$ is normal, then the canonical module $\omega_{K[F]}$ of $K[F]$, with respect to standard grading, can be expressed as an ideal of $K[F]$ generated by monomials

$$\omega_{K[F]} = (\{x^a | a \in \mathbb{N}A \cap \text{ri}(R_+ A)\}),$$

where $A = \log(F)$ and $\text{ri}(R_+ A)$ denotes the relative interior of $R_+ A$.

The formula above represents the canonical module of $K[F]$ as an ideal of $K[F]$ generated by monomials. For a comprehensive treatment of the Danilov-Stanley formula see [2], [8] or [9].

3 Polymatroids

Let $K$ be an infinite field, $n$ and $m$ be positive integers, $[n] = \{1, 2, \ldots, n\}$. A nonempty finite set $B$ of $\mathbb{N}^n$ is the base set of a discrete polymatroid $\mathcal{P}$ if for every $u = (u_1, u_2, \ldots, u_n), v = (v_1, v_2, \ldots, v_n) \in B$ one has $u_1 + u_2 + \ldots + u_n = v_1 + v_2 + \ldots + v_n$ and for all $i$ such that $u_i < v_i$ there exists $j$ such that $u_j < v_j$ and $u + e_j - e_i \in B$, where $e_k$ denotes the $k^{\text{th}}$ vector of the standard basis of $\mathbb{N}^n$. The notion of discrete polymatroid is a generalization of the classical notion of matroid, see [5], [6], [7], [11]. Associated with the base $B$ of a discrete polymatroid $\mathcal{P}$ one has a $K-$algebra $K[B]$, called the base ring of $\mathcal{P}$, defined to be the $K-$subalgebra of the polynomial ring in $n$ indeterminates $K[x_1, x_2, \ldots, x_n]$ generated by the monomials $x^u$ with $u \in B$. From [7] the algebra $K[B]$ is known to be normal and hence Cohen-Macaulay.

If $A_i$ are some nonempty subsets of $[n]$ for $1 \leq i \leq m$, $\mathcal{A} = \{A_1, \ldots, A_m\}$, then the set of the vectors $\sum_{k=1}^{m} e_{i_k}$ with $i_k \in A_k$ is the base of a polymatroid, called the transversal polymatroid presented by $\mathcal{A}$. The base ring of a transversal polymatroid presented by $\mathcal{A}$ is the ring

$$K[\mathcal{A}] := K[x_{i_1} \cdots x_{i_m} | i_j \in A_j, 1 \leq j \leq m].$$

4 Cones of dimension $n$ with $n + 1$ facets

Lemma 4.1. Let $1 \leq i \leq n - 2$, $A := \{\log(x_{j_1} \cdots x_{j_k}) | j_k \in A_k, \text{ for all } 1 \leq k \leq n\} \subset \mathbb{N}^n$ the exponent set of generators of the $K-$algebra $K[\mathcal{A}]$, where $\mathcal{A} = \{A_1 = [n], \ldots, A_i = [n], A_{i+1} = [n] \setminus [i], \ldots, A_{n-1} = [n] \setminus [i], A_n = [n]\}$. Then the cone generated by $A$ has the irreducible representation

$$\mathbb{R}_+ A = \bigcap_{a \in N} H_a^+,$$

where $N = \{\nu_{a[i]}, \nu_{a[k]} | 0 \leq k \leq n - 1\}$. **
Proof: We denote $J_k = \{(i+1) e_k + (n-i-1) e_{i+1}, \text{ if } 1 \leq k \leq i \}
\text{ and } J = n e_n$. Since $A_t = [n]$ for any $t \in \{1, \ldots, i\} \cup \{n\}$ and $A_r = [n] \setminus \{i\}$ for any $r \in \{i+1, \ldots, n-1\}$, it is easy to see that for any $k \in \{1, \ldots, i\}$ and $r \in \{i+2, \ldots, n\}$ the set of monomials $x_k^{i+1} x_{i+1}^{n-i-1}$, $x_k^{i+1} x_r^{n-i-1}$, $x_n^i$ is a subset of the generators of the $K$-algebra $K[A]$. Thus one has

$\left\{J_1, J_2, J_{i+2}, \ldots, J_n, J\right\} \subset A$.

If we denote by $C$ the $n \times n$-matrix whose rows are the entries of the vectors $J_1, J_2, J_{i+2}, \ldots, J_n$, then by a simple computation we get $|\det(C)| = (i+1)^i (n-i-1)^{n-i-1}$. Therefore the set

$\left\{J_1, J_2, J_{i+2}, \ldots, J_n, J\right\}$

is linearly independent and it follows that $\dim \mathbb{R}_+ A = n$. Since $\{J_1, J_2, J_{i+2}, \ldots, J_n\}$ is linearly independent and lies on the hyperplane $H_\sigma[i]$, we have that $\dim (H_\sigma[i] \cap \mathbb{R}_+ A) = n - 1$.

Now we will prove that $\mathbb{R}_+ A \subset H_a^+$ for all $a \in N$. It is enough to show that for all vectors $P \in A$, $\langle P, a \rangle \geq 0$ for all $a \in N$. Since $\nu_{\sigma[n-1]} = n \in [n] \setminus \sigma[k[n-1]]$, where $\{e_i\}_{1 \leq i \leq n}$ is the canonical base of $\mathbb{R}^n$, we get that $\langle P, \nu_{\sigma[k[n-1]]} \rangle \geq 0$. Let $P \in A$, $P = \log(x_{j_1} \cdots x_{j_1} x_{j_{n-1}} \cdots x_{j_{n-1}} x_{j_n})$ and let $t$ be the number of $j_k$, such that $1 \leq k \leq i$ and $j_k \in [i]$. Thus $1 \leq t \leq i$. Now we have only two cases to consider:

1) If $j_n \in [i]$, then

$\langle P, \nu_{\sigma[i]} \rangle = -t(n-i-1)+(i-t)(i+1)+(n-i-1)(i+1) - (n-i-1) = n(i-t) \geq 0$.

2) If $j_n \in [n] \setminus [i]$, then

$\langle P, \nu_{\sigma[i]} \rangle = -t(n-i-1)+(i-t)(i+1)+(n-i-1)(i+1) + (i+1) = n(i-t+1) > 0$.

Thus

$\mathbb{R}_+ A \subseteq \bigcap_{a \in N} H_a^+$.

To prove the converse inclusion, it is clearly enough to prove that all the extremal rays of the cone $\bigcap_{a \in N} H_a^+$ are contained in $\mathbb{R}_+ A$. Any extremal ray of the cone $\bigcap_{a \in N} H_a^+$ can be written as the intersection of $n - 1$ hyperplanes $H_a$, with $a \in N$. There are two possibilities to obtain extremal rays by intersection of $n - 1$ hyperplanes.

First case.

Let $1 \leq i_1 < \ldots < i_{n-1} \leq n$ be a sequence and $\{t\} = [n] \setminus \{i_1, \ldots, i_{n-1}\}$. The system of equations (*) admits the solution $x \in \mathbb{Z}_+^n$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$,
with $|x| = n$ and $x_k = n \cdot \delta_{kt}$ for all $1 \leq k \leq n$, where $\delta_{kt}$ is the Kronecker symbol.

There are two possibilities:
1) If $1 \leq t \leq i$, then $H_{\sigma^{|t|}}(x) < 0$ and therefore $x \notin \bigcap_{a \in N} H_a^+$.
2) If $i + 1 \leq t \leq n$, then $H_{\sigma^{|t|}}(x) > 0$ and thus $x \in \bigcap_{a \in N} H_a^+$ and is an extremal ray.

Hence, there exist $n-i$ sequences $1 \leq i_1 < \ldots < i_{n-1} \leq n$ such that the system of equations $(*)$ has a solution $x \in \mathbb{Z}_+^n$ with $|x| = n$ and $H_{\sigma^{|i|}}(x) > 0$. The extremal rays are: \{\langle n e_k \mid i + 1 \leq k \leq n \rangle\}.

**Second case.**

Let $1 \leq i_1 < \ldots < i_{n-2} \leq n$ be a sequence and $\{j, k\} = [n] \setminus \{i_1, \ldots, i_{n-2}\}$, with $j < k$, and

\[
\begin{cases}
  z_{i_1} = 0, \\
  \vdots \\
  z_{i_{n-2}} = 0, \\
  -(n-i-1)z_1 - \ldots - (n-i-1)z_i + (i+1)z_{i+1} + \ldots + (i+1)z_n = 0
\end{cases}
\]

be the system of linear equations associated to this sequence.

There are two possibilities:
1) If $1 \leq j \leq i$ and $i + 1 \leq k \leq n$, then the system of equations $(**)$ admits the solution $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{Z}_+^n$, with $|x| = n$ and $x_t = (i+1)\delta_{jt} + (n-i-1)\delta_{kt}$ for all $1 \leq t \leq n$.
2) If $1 \leq j, k \leq i$ or $i + 1 \leq j, k \leq n$, then there exist no solutions $x \in \mathbb{Z}_+^n$ with $|x| = n$ for the system of equations $(**)$. Because otherwise $H_{\sigma^{|j|}}(x) > 0$ or $H_{\sigma^{|k|}}(x) < 0$.

Thus, there exist $i(n-i)$ sequences $1 \leq i_1 < \ldots < i_{n-2} \leq n$ such that the system of equations $(**)$ has a solution $x \in \mathbb{Z}_+^n$ with $|x| = n$ and the extremal rays are: \{(i+1)e_j + (n-i-1)e_k \mid 1 \leq j \leq i \text{ and } i + 1 \leq k \leq n\}.

In conclusion, there exist $(i+1)(n-i)$ extremal rays of the cone $\bigcap_{a \in N} H_a^+$:

\[R := \{ne_k \mid i+1 \leq k \leq n\} \cup \{(i+1)e_j + (n-i-1)e_k \mid 1 \leq j \leq i \text{ and } i+1 \leq k \leq n\}.
\]

Since $R \subset A$ we have $\mathbb{R}_+A = \bigcap_{a \in N} H_a^+$.

It is easy to see that the representation is irreducible because if we delete, for some $k$, the hyperplane with the normal $\nu_{\sigma^{|i|}}$, then a coordinate of a log($x_{j_1} \cdots x_{j_k}$) would be negative, which is impossible; and if we delete the hyperplane with the normal $\nu_{\sigma^{|j|}}$, then the cone $\mathbb{R}_+A$ would be generated by $A = \{\log(x_{j_1} \cdots x_{j_n}) \mid j_k \in [n], \text{ for all } 1 \leq k \leq n\}$, which again is impossible. Thus the representation $\mathbb{R}_+A = \bigcap_{a \in N} H_a^+$ is irreducible.

**Lemma 4.2.** Let $1 \leq i \leq n-2$, $1 \leq t \leq n-1$, $A := \{\log(x_{j_1} \cdots x_{j_n}) \mid j_{\sigma^{|t|}}(k) \in A_{\sigma^{|t|}}(k), 1 \leq k \leq n\} \subset \mathbb{N}^n$ the exponent set of generators of $K$-algebra $K[A]$, where
\( A = \{ A_{\sigma^t(k)} \mid A_{\sigma^t(k)} = [n], \text{ for } 1 \leq k \leq i \text{ and } A_{\sigma^t(k)} = [n] \setminus \sigma^t[i], \text{ for } i + 1 \leq k \leq n - 1, \text{ and } A_{\sigma^t(n)} = [n] \}. \) Then the cone generated by \( A \) has the irreducible representation

\[ \mathbb{R}^+_A = \bigcap H_a^+, \]

where \( N = \{ \nu_{\sigma^t[i]}, \nu_{\sigma^t[n-1]} \mid 0 \leq k \leq n - 1 \} \).

**Proof:** The proof goes as that for Lemma 4.1. \( \square \)

## 5 The \( \sigma \)-invariant and the canonical module

**Lemma 5.1.** The \( K \)-algebra \( K[A] \), where \( A = \{ A_{\sigma^t(k)} \mid A_{\sigma^t(k)} = [n], \text{ for } 1 \leq k \leq i \text{ and } A_{\sigma^t(k)} = [n] \setminus \sigma^t[i], \text{ for } i + 1 \leq k \leq n - 1, \text{ and } A_{\sigma^t(n)} = [n] \}, \) is a Gorenstein ring for all \( 0 \leq t \leq n - 1 \) and \( 1 \leq i \leq n - 2 \).

**Proof:** Since the algebras from Lemmas 4.1 and 4.2 are isomorphic, it is enough to prove the case \( t = 0 \).

We will show that the canonical module \( \omega_{K[A]} \) is generated by \( (x_1 \cdots x_n)K[A] \).

Since \( K \)-algebra \( K[A] \) is normal, using the Danilov-Stanley theorem we get that the canonical module \( \omega_{K[A]} \) is

\[ \omega_{K[A]} = \{ x^n \mid \alpha \in NA \cap ri(\mathbb{R}^+_A) \}. \]

Let \( d = \gcd(n, i + 1) \) be the greatest common divisor of \( n \) and \( i + 1 \), then the equation of the facet \( H_{\nu_{\sigma^t[i]}} \) is

\[ H_{\nu_{\sigma^t[i]}} : \quad -\frac{(n - i - 1)}{d} \sum_{k=1}^{i} x_k + \frac{(i + 1)}{d} \sum_{k=i+1}^{n} x_k = 0. \]

The relative interior of the cone \( \mathbb{R}^+_A \) is

\( ri(\mathbb{R}^+_A) = \{ x \in \mathbb{R}^n \mid x_k > 0, \text{ } \forall k \in [n], \quad -\frac{(n - i - 1)}{d} \sum_{k=1}^{i} x_k + \frac{(i + 1)}{d} \sum_{k=i+1}^{n} x_k > 0 \} \).

We will show that \( NA \cap ri(\mathbb{R}^+_A) = (1, \ldots, 1) + (NA \cap \mathbb{R}^+_A) \).

It is clear that \( ri(\mathbb{R}^+_A) \supset (1, \ldots, 1) + \mathbb{R}^+_A \). If \( (\alpha_1, \alpha_2, \ldots, \alpha_n) \in NA \cap ri(\mathbb{R}^+_A), \) then \( \alpha_k \geq 1 \) for all \( 1 \leq k \leq n \) and

\[ -\frac{(n - i - 1)}{d} \sum_{k=1}^{i} \alpha_k + \frac{(i + 1)}{d} \sum_{k=i+1}^{n} \alpha_k \geq 1 \text{ and } \sum_{k=1}^{n} \alpha_k = tn \text{ for some } t \geq 1. \]
We claim that there exists \((\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}A \cap \mathbb{R}_+ A\) such that \((\alpha_1, \alpha_2, \ldots, \alpha_n) = (\beta_1 + 1, \beta_2 + 1, \ldots, \beta_n + 1)\). Let \(\beta_k = \alpha_k - 1\) for all \(1 \leq k \leq n\). It is clear that \(\beta_k \geq 0\) and
\[
- \frac{(n-i-1)}{d} \sum_{k=1}^{i} \beta_k + \frac{(i+1)}{d} \sum_{k=i+1}^{n} \beta_k = - \frac{(n-i-1)}{d} \sum_{k=1}^{i} \alpha_k + \frac{(i+1)}{d} \sum_{k=i+1}^{n} \alpha_k - \frac{n}{d}.
\]
If
\[
- \frac{(n-i-1)}{d} \sum_{k=1}^{i} \alpha_k + \frac{(i+1)}{d} \sum_{k=i+1}^{n} \alpha_k = j \text{ with } 1 \leq j \leq \frac{n}{d} - 1,
\]
then we will get a contradiction. Indeed, since \(n\) divides \(\sum_{k=1}^{n} \alpha_k\), it follows that \(\frac{n}{d}\) divides \(j\), which is false. So we have
\[
- \frac{(n-i-1)}{d} \sum_{k=1}^{i} \beta_k + \frac{(i+1)}{d} \sum_{k=i+1}^{n} \beta_k =
\]
\[
- \frac{(n-i-1)}{d} \sum_{k=1}^{i} \alpha_k + \frac{(i+1)}{d} \sum_{k=i+1}^{n} \alpha_k - \frac{n}{d} \geq 0.
\]

Thus \((\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}A \cap \mathbb{R}_+ A\) and \((\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}A \cap ri(\mathbb{R}_+ A)\).

Since \(\mathbb{N}A \cap ri(\mathbb{R}_+ A) = (1, \ldots, 1) + (\mathbb{N}A \cap \mathbb{R}_+ A)\), we get that \(\omega_{K[A]} = (x_1 \cdots x_n) K[A]\).

Let \(S\) be a standard graded \(K\)-algebra over a field \(K\). Recall that the \(a\)-\textit{invariant} of \(S\), denoted \(a(S)\), is the degree as a rational function of the Hilbert series of \(S\), see for instance ([9, p. 99]). If \(S\) is Cohen-Macaulay and \(\omega_S\) is the canonical module of \(S\), then
\[
a(S) = - \min\{i \mid (\omega_S)_i \neq 0\},
\]
see [2, p. 141] and [9, Proposition 4.2.3]. In our situation \(S = K[A]\) is normal \([7]\) and consequently Cohen-Macaulay, thus this formula applies. We have the following consequence of Lemma 5.1.

**Corollary 5.2.** The \(a\)-\textit{invariant} of \(K[A]\) is \(a(K[A]) = -1\).

**Proof:** Let \(\{x_1, \ldots, x_n\}\) be generators of the \(K\)-algebra \(K[A]\). Then \(K[A]\) is a standard graded algebra with the grading
\[
K[A]_i = \sum_{|c| = i} K(x^c_1)^{c_1} \cdots (x^c_n)^{c_n}, \text{ where } |c| = c_1 + \ldots + c_q.
\]

Since \(\omega_{K[A]} = (x_1 \cdots x_n) K[A]\), it follows that \(\min\{i \mid (\omega_{K[A]})_i \neq 0\} = 1\), thus \(a(K[A]) = -1\).
6 Ehrhart function

We consider a fixed set of distinct monomials $F = \{x^{α_1}, \ldots, x^{α_r}\}$ in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$. Let

$$P = \text{conv}(\log(F))$$

be the convex hull of the set $\log(F) = \{α_1, \ldots, α_r\}$. The normalized Ehrhart ring of $P$ is the graded algebra

$$A_P = \bigoplus_{j=0}^{∞} (A_P)_j \subset R[T],$$

where the $j^{th}$ component is given by

$$(A_P)_j = \sum_{α \in ℤ \log(F) \cap jP} K x^α T^j.$$

The normalized Ehrhart function of $P$ is defined as

$$E_P(j) = \dim_K (A_P)_j = | ℤ \log(F) \cap jP |.$$

From [9, Proposition 7.2.39 and Corollary 7.2.45] we have the following important result.

**Theorem 6.1.** If $K[F]$ is a standard graded subalgebra of $R$ and $h$ is the Hilbert function of $K[F]$, then:

a) $h(j) \leq E_P(j)$ for all $j \geq 0$, and

b) $h(j) = E_P(j)$ for all $j \geq 0$ if and only if $K[F]$ is normal.

In this section we will compute the Hilbert function and the Hilbert series for the $K$–algebra $K[A]$, where $A$ satisfies the hypothesis of Lemma 4.1.

**Proposition 6.2.** In the hypothesis of Lemma 4.1, the Hilbert function of the $K$–algebra $K[A]$ is

$$h(t) = \sum_{k=0}^{(i+1)t} \binom{k + i - 1}{k} \binom{nt - k + n - i - 1}{nt - k}.$$ 

**Proof:** From [7] we know that the $K$–algebra $K[A]$ is normal. Therefore, to compute the Hilbert function of $K[A]$ is equivalent to compute the Ehrhart function of $P$, where $P = \text{conv}(A)$. It is clearly enough to show that $P$ is the intersection of the cone $ℝ_+ A$ with the hyperplane $x_1 + \ldots + x_n = n$, that is,

$$P = \{α \in ℜ^n | α_k \geq 0 \text{ for any } k \in [n], 0 \leq α_1+\ldots+α_i \leq i+1 \text{ and } α_1+\ldots+α_n = n\},$$
whence it follows that

\[ tP = \{ \alpha \in \mathbb{R}^n \mid \alpha_k \geq 0, \forall k \in [n], 0 \leq \alpha_1 + \ldots + \alpha_i \leq (i+1) t \text{ and } \alpha_1 + \ldots + \alpha_n = n t \}. \]

Since for any \( 0 \leq k \leq (i+1) t \) the equation \( \alpha_1 + \ldots + \alpha_i = k \) has \( \binom{k+i-1}{i} \) nonnegative integer solutions and the equation \( \alpha_{i+1} + \ldots + \alpha_n = n t - k \) has \( \binom{nt-k+n-i-1}{nt-k} \) nonnegative integer solutions, we get that

\[ E_P(t) = | \mathbb{Z} A \cap tP | = \sum_{k=0}^{(i+1)t} \binom{k+i-1}{k} \binom{nt-k+n-i-1}{nt-k}. \]

Corollary 6.3. The Hilbert series of the \( K \)-algebra \( K[A] \), where \( A \) satisfies the hypothesis of Lemma 4.1, is

\[ H_{K[A]}(t) = \frac{1 + h_1 t + \ldots + h_{n-1} t^{n-1}}{(1-t)^n}, \]

where

\[ h_j = \sum_{s=0}^{j} (-1)^s h(j-s) \binom{n}{s} \]

and \( h(s) \) is the Hilbert function of \( K[A] \).

Proof: Since the \( a \)-invariant of \( K[A] \) is \( a(K[A]) = -1 \), it follows that to compute the Hilbert series of \( K[A] \) it is necessary to know the first \( n \) values of the Hilbert function of \( K[A] \), \( h(i) \) for \( 0 \leq i \leq n - 1 \). Since \( \dim(K[A]) = n \), applying \( n \) times the difference operator \( \Delta \) (see [2]) on the Hilbert function of \( K[A] \) we get the conclusion.

Let \( \Delta^0(h)_j := h(j) \) for any \( 0 \leq j \leq n - 1 \). For \( k \geq 1 \) let \( \Delta^k(h)_0 := 1 \) and \( \Delta^k(h)_j := \Delta^{k-1}(h)_j - \Delta^{k-1}(h)_{j-1} \) for any \( 1 \leq j \leq n - 1 \). We claim that

\[ \Delta^k(h)_j = \sum_{s=0}^{k} (-1)^s h(j-s) \binom{k}{s} \]

for any \( k \geq 1 \) and \( 0 \leq j \leq n - 1 \). We proceed by induction on \( k \).

If \( k = 1 \), then

\[ \Delta^1(h)_j = \Delta^0(h)_j - \Delta^0(h)_{j-1} = h(j) - h(j-1) = \sum_{s=0}^{1} (-1)^s h(j-s) \binom{1}{s} \]

for any \( 1 \leq j \leq n - 1 \).

If \( k > 1 \), then

\[ \Delta^k(h)_j = \Delta^{k-1}(h)_j - \Delta^{k-1}(h)_{j-1} = \]
\[
\sum_{s=0}^{k-1} (-1)^s h(j-s) \binom{k-1}{s} - \sum_{s=0}^{k-1} (-1)^s h(j-1-s) \binom{k-1}{s} = h(j) \binom{k-1}{0} + \\
\sum_{s=1}^{k-1} (-1)^s h(j-s) \binom{k-1}{s} - \sum_{s=0}^{k-2} (-1)^s h(j-1-s) \binom{k-1}{s} + (-1)^k h(j-k) \binom{k-1}{k-1} \\
= h(j) + \sum_{s=1}^{k-1} (-1)^s h(j-s) \left[ \binom{k-1}{s} + \binom{k-1}{s-1} \right] + (-1)^k h(j-k) \binom{k-1}{k-1} \\
= h(j) + \sum_{s=1}^{k-1} (-1)^s h(j-s) \binom{k}{s} + (-1)^k h(j-k) \binom{k-1}{k-1} = \sum_{s=0}^{k} (-1)^s h(j-s) \binom{k}{s}.
\]

Thus, if \( k = n \) it follows that

\[
h_j = \Delta^n(h)_j = \sum_{s=0}^{n} (-1)^s h(j-s) \binom{n}{s} = \sum_{s=0}^{j} (-1)^s h(j-s) \binom{n}{s}
\]

for any \( 1 \leq j \leq n-1 \).

\[\square\]

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**References**


A class of transversal polymatroids


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