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Hamiltonicity in directed Toeplitz graphs $T_n\langle 1, 3, 5; t\rangle$

by

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Dedicated to our beloved PhD supervisor, Dr. Tudor Zamfirescu, on his eightieth birthday

Abstract

A directed Toeplitz graph $T_n\langle s_1, \ldots, s_k; t_1, \ldots, t_l\rangle$ with vertices $1, 2, \ldots, n$, where the edge $(i,\, j)$ occurs if and only if $j - i = s_p$ or $i - j = t_q$ for some $1 \le p \le k$ and $1 \le q \le l$, is a digraph whose adjacency matrix is a Toeplitz matrix (a square matrix that has constant values along all diagonals parallel to the main diagonal). In this paper, we study hamiltonicity in directed Toeplitz graphs $T_n\langle 1, 3, 5; t\rangle$.

Key Words: Adjacency matrix, Toeplitz graph, Hamiltonian graph, length of an edge.
2020 Mathematics Subject Classification: Primary 05C20; Secondary 05C45.

1 Introduction

A directed Toeplitz graph $T_n\langle s_1, \ldots, s_k; t_1, \ldots, t_l\rangle$ with vertices $1, 2, \ldots, n$, where the edge $(i,\, j)$ occurs if and only if $j - i = s_p$ or $i - j = t_q$ for some $1 \le p \le k$ and $1 \le q \le l$, is a digraph, of order $n > max\{s_k, t_l\}$, whose adjacency matrix is a Toeplitz matrix. We use [22] for terminology and notations not defined here, and consider finite directed graphs without multiple edges and loops. Since all graphs will be directed, we shall omit mentioning it.

Properties of Toeplitz graphs, such as colourability, planarity, bipartiteness, connectivity, cycle discrepancy, edge irregularity strength, decomposition, labeling, and metric dimension have been studied in [1]-[6], [8]-[12], [14]-[15], and [26]. Hamiltonian properties of Toeplitz graphs were first investigated by R. van Dal et al. in [7] and then studied in [13, 25, 27], while the hamiltonicity in directed Toeplitz graphs was first studied by S. Malik and T. Zamfirescu in [24], by S. Malik in [16], by S. Malik and A.M. Qureshi in [23], and then by S. Malik in [17]-[22].

In [17] and [22], the hamiltonicity of the Toeplitz graphs $T_n\langle 1, 3; 1, t\rangle$ was investigated. In [20] and [21], the hamiltonicity of the Toeplitz graphs $T_n\langle 1, 3, 4; t\rangle$ was investigated. In this paper we still keep $s_1 = 1$ and $s_2 = 3$ but then we consider $s_3 = 5$, that is, we investigate the hamiltonicity in Toeplitz graphs $T_n\langle 1, 3, 5; t\rangle$.

For a vertex a of $T_n\langle 1, 3, 5; t\rangle$, We define paths $A_{a\to a-10}$, $B_{a\to a+10}$, $C_{a\to a-4}$ and $D_{a\to a+4}$ in $T_n\langle 1, 3, 5; t\rangle$ as $A_{a\to a-10} = (a, a-3, a-6, a-1, a-4, a-7, a-10)$, $B_{a\to a+10} = (a, a+3, a+6, a+9, a+10)$, $C_{a\to a-4} = (a, a-5, a-4)$ and $D_{a\to a+4} = (a, a+3, a+4)$, respectively, see Figure 1.

Remark 1: If the Toeplitz graph $T_n\langle 1, 3, 5; t\rangle$ has a hamiltonian cycle containing the path $(n-2,\, n-1,\, n)$, then $T_{n+(t-1)}\langle 1, 3, 5; t\rangle$ enjoys the same property. Because such a hamiltonian cycle in $T_n\langle 1, 3, 5; t\rangle$ can be transformed into a hamiltonian cycle in $T_{n+(t-1)}\langle 1, 3, 5; t\rangle$,

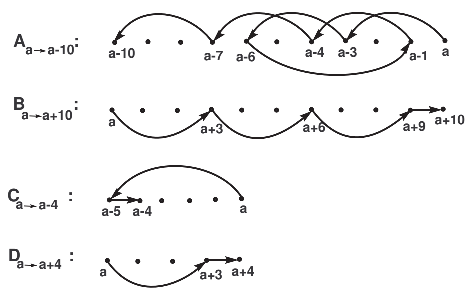

Figure 1: Paths $A_{a\to a-10}$, $B_{a\to a+10}$, $C_{a\to a-4}$ and $D_{a\to a+4}$ in $T_n\langle 1,3,5;t\rangle$

by replacing the edge $(n-2,\, n-1)$ with the path $(n-2, n+1, n+2, \ldots, n+(t-1), n-1)$, which preserves the same property. For example, see Figure 2, where a hamiltonian cycle in $T_{10}\langle 1,3,5;5\rangle$ is transformed into a hamiltonian cycle in $T_{14}\langle 1,3,5;5\rangle$ by replacing the edge $(8,9)$ with the path $(8,11,12,13,14,9)$, which preserves the same property so $T_{14}\langle 1,3,5;5\rangle$ can be transformed into a hamiltonian cycle in $T_{18}\langle 1,3,5;5\rangle$, and so on.

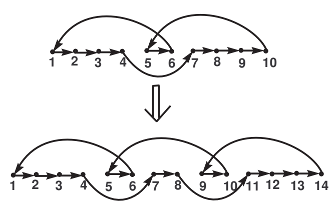

Figure 2: Hamiltonian cycles in $T_{10}\langle 1,3,5;5\rangle$ and $T_{14}\langle 1,3,5;5\rangle$

2 Toeplitz graphs $T_n\langle 1,3,5;t\rangle$ for odd t

For $t=1$, clearly $T_n\langle 1,3,5;1\rangle$ is hamiltonian if and only if $n=6$, because the decreasing edges (the edges of the type (a,b) where $a>b$) are of length one only and this is only possible when $n=6$ and it is easily seen that $T_6\langle 1,3,5;1\rangle$ has a unique hamiltonian cycle $(1,6,5,4,3,2,1)$.

Theorem 2.1. *$T_n\langle 1,3,5;3\rangle$ is hamiltonian if and only if n is even.*

Proof. For $n=6,8,10,12$, hamiltonian cycles in $T_n\langle 1,3,5;3\rangle$ are $(1,2,5,6,3,4,1)$, $(1,2,7,8,5,6,3,4,1)$, $(1,2,5,10,7,8,9,6,3,4,1)$ and $(1,2,7,12,9,10,11,8,5,6,3,4,1)$, in order, see Figure 3. Now for even $n\geq 14$. If $n\cong 0\, mod\, 10$, then a hamiltonian cycle in $T_n\langle 1,3,5;t\rangle$ is $(1,2,3)\cup B_{3\to 13}\cup B_{13\to 23}\cup\cdots\cup B_{n-27\to n-17}\cup(n-17, n-14, n-11, n-8, n-5, n, n-3, n-2, n-1, n-4, n-7, n-6, n-9)\cup A_{n-9\to n-19}\cup A_{n-19\to n-29}\cup\cdots\cup A_{11\to 1}$, see Figure 4. If $n\cong 2\, mod\, 10$, then a hamiltonian cycle in $T_n\langle 1,3,5;t\rangle$ is $(1,2,3)\cup B_{3\to 13}\cup$

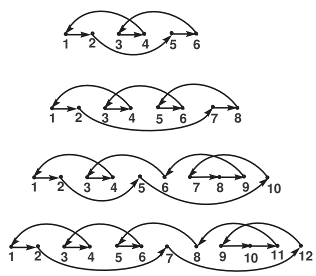

Figure 3: Hamiltonian cycles in $T_6\langle 1,3,5;3\rangle$, $T_8\langle 1,3,5;3\rangle$, $T_{10}\langle 1,3,5;3\rangle$, and $T_{12}\langle 1,3,5;3\rangle$

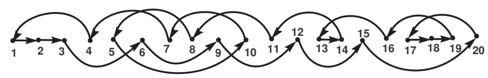

Figure 4: A hamiltonian cycle in $T_{20}\langle 1,3,5;3\rangle$

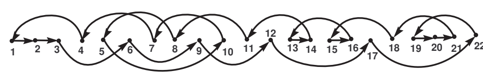

Figure 5: A hamiltonian cycle in $T_{22}\langle 1,3,5;3\rangle$

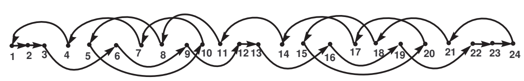

Figure 6: A hamiltonian cycle in $T_{14}\langle 1,3,5;3\rangle$

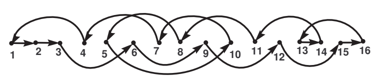

Figure 7: A hamiltonian cycle in $T_{16}\langle 1,3,5;3\rangle$

$B_{13\to 23}\cup\cdots\cup B_{n-29\to n-19}\cup(n-19,n-16,n-13,n-10,n-5,n,n-3,n-2,n-1,n-4,n-7,n-6,n-9,n-8,n-11)\cup A_{n-11\to n-21}\cup A_{n-21\to n-31}\cup\cdots\cup A_{11\to 1}$, see Figure 5. If $n\cong 4\,mod\,10$, then a hamiltonian cycle in $T_n\langle 1,3,5;t\rangle$ is $(1,2,3)\cup B_{3\to 13}\cup B_{13\to 23}\cup\cdots\cup B_{n-11\to n-1}\cup(n-1,n,n-3)\cup A_{n-3\to n-13}\cup A_{n-13\to n-23}\cup\cdots\cup A_{11\to 1}$, see Figure 6. If $n\cong 6\,mod\,10$, then a hamiltonian cycle in $T_n\langle 1,3,5;t\rangle$ is $(1,2,3)\cup B_{3\to 13}\cup B_{13\to 23}\cup\cdots\cup B_{n-23\to n-13}\cup(n-13,n-10)\cup B_{n-10\to n}\cup(n,n-3,n-2,n-5)\cup A_{n-5\to n-15}\cup A_{n-15\to n-25}\cup\cdots\cup A_{11\to 1}$, see Figure 7. If $n\cong 8\,mod\,10$, then a hamiltonian cycle in $T_n\langle 1,3,5;t\rangle$ is $(1,2,3)\cup B_{3\to 13}\cup B_{13\to 23}\cup\cdots\cup B_{n-25\to n-15}\cup(n-15,n-12,n-9,n-6,n-1,n,n-3,n-2,n-5,n-4,n-7)\cup A_{n-7\to n-17}\cup A_{n-17\to n-27}\cup\cdots\cup A_{11\to 1}$, see Figure 8. Thus $T_n\langle 1,3,5;3\rangle$ is hamiltonian for all even n.

Conversely, $T_n\langle 1,3,5;3\rangle$ is bipartite and, being hamiltonian, n must be even. □

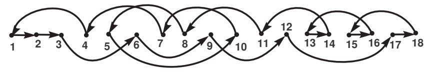

Figure 8: A hamiltonian cycle in $T_{18}\langle 1,3,5;3\rangle$

Theorem 2.2. *$T_n\langle 1,3,5;5\rangle$ is hamiltonian if and only if n is even.*

Proof. For $n=6$ and $n=8$, unique hamiltonian cycles in $T_n\langle 1,3,5;5\rangle$ are $(1,2,3,4,5,6,1)$ and $(1,2,7,8,3,4,5,6,1)$, respectively.

Now for even $n \geq 10$. If $n \cong 0\, mod\, 4$, then a hamiltonian cycle in $T_n\langle 1,3,5;5\rangle$ is $(1,2,3,4)\cup D_{4\to 8}\cup D_{8\to 12}\cup\cdots\cup D_{n-12\to n-8}\cup(n-8,n-7,n-4,n-3,n,n-5,n-2,n-1,n-6)\cup C_{n-6\to n-10}\cup C_{n-10\to n-14}\cup\cdots\cup C_{10\to 6}\cup(6,1)$, see Figure 9. If $n\cong 2\, mod\, 4$,

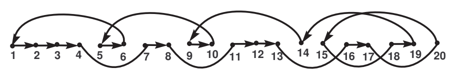

Figure 9: A hamiltonian cycle in $T_{20}\langle 1,3,5;5\rangle$

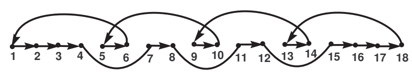

Figure 10: A hamiltonian cycle in $T_{18}\langle 1,3,5;5\rangle$

then a hamiltonian cycle in $T_n\langle 1,3,5;5\rangle$ is $(1,2,3,4)\cup D_{4\to 8}\cup D_{8\to 12}\cup\cdots\cup D_{n-6\to n-2}\cup(n-2,n-1,n)\cup C_{n\to n-4}\cup C_{n-4\to n-8}\cup C_{10\to 6}\cup(6,1)$, see Figure 10. Thus $T_n\langle 1,3,5;5\rangle$ is hamiltonian for all even n.

Conversely, $T_n\langle 1,3,5;5\rangle$ is bipartite and, being hamiltonian, n must be even. □

Theorem 2.3. *For odd $t\geq 7$, $T_n\langle 1,3,5;t\rangle$ is hamiltonian if and only if n is even.*

Proof. We consider $t\geq 7$ is odd and n is even.

Case 1. $n\cong 0\, mod\,(t-1)$.

The smallest n, different from $t-1$, is $2t-2$. A hamiltonian cycle in $T_{n=2t-2}\langle 1,3,5;t\rangle$ is $(1,2,3,\ldots,t-3,t+2,t+3,\ldots,n-2,n-1,n,n-t=t-2,t-1,t,t+1,1)$, which contains the path $(n-2,n-1,n)$, see Figure 11. By Remark 1, this hamiltonian cycle in $T_{2t-2}\langle 1,3,5;t\rangle$ can be extended to a hamiltonian cycle in $T_{3t-3=2t-2+(t-1)}\langle 1,3,5;t\rangle$, which preserves the same property. Suppose $T_n\langle 1,3,5;t\rangle$, with $n=(2t-2)+r(t-1)$, has a hamiltonian cycle

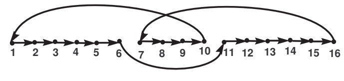

Figure 11: A hamiltonian cycle in $T_{16}\langle 1,3,5;9\rangle$

containing the path $(n-2, n-1, n)$, for some non-negative integer r, then $T_{n+t-1}\langle 1,3,5;t\rangle$ enjoys the same property. Thus $T_n\langle 1,3,5;t\rangle$ is hamiltonian for all even $n \cong 0\, mod\,(t-1)$.

Case 2. $n \cong 2\, mod\,(t-1)$.

The smallest n is $t+1$. A hamiltonian cycle in $T_{n=t+1}\langle 1,3,5;t\rangle$ is $(1,2,3,\ldots,n-2,n-1,n,1)$ which contains the path $(n-2,n-1,n)$. By Remark 1, this hamiltonian cycle in $T_{t+1}\langle 1,3,5;t\rangle$ can be extended to a hamiltonian cycle in $T_{2t=t+1+(t-1)}\langle 1,3,5;t\rangle$, which preserves the same property. Thus, by using the technique of Remark 1, $T_n\langle 1,3,5;t\rangle$ is hamiltonian for all even $n \cong 2\, mod\,(t-1)$.

Case 3. $n \cong 4\, mod\,(t-1)$.

The smallest n is $t+3$. If $t \cong 1\, mod\, 4$, then a hamiltonian cycle in $T_{n=t+3}\langle 1,3,5;t\rangle$ is $(1,2) \cup D_{2\to 6} \cup D_{6\to 10} \cup \cdots \cup D_{n-10\to n-6} \cup (n-6, n-1, n, 3, 4) \cup D_{4\to 8} \cup D_{8\to 11} \cup \cdots \cup D_{n-8\to n-4} \cup (n-4, n-3, n-2 = t+1, 1)$, see Figure 12. If $t \cong 3\, mod\, 4$, then a hamiltonian cycle in $T_{n=t+3}\langle 1,3,5;t\rangle$ is $(1,2) \cup D_{2\to 6} \cup D_{6\to 10} \cup \cdots \cup D_{n-4\to n} \cup (n,3,4) \cup D_{4\to 8} \cup D_{8\to 12} \cup \cdots \cup D_{n-6\to n-2} \cup (n-2 = t+1, 1)$, see Figure 13. These hamiltonian cycles

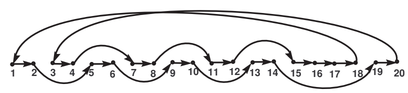

Figure 12: A hamiltonian cycles in $T_{20}\langle 1,3,5;17\rangle$

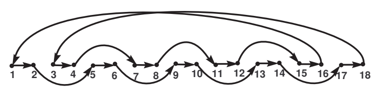

Figure 13: A hamiltonian cycles in $T_{18}\langle 1,3,5;15\rangle$

in $T_{t+3}\langle 1,3,5;t\rangle$ do not contain the path $(n-2,n-1,n)$. Now, the next representative in this class is $n = t+3+(t-1) = 2t+2$. If $t \cong 1\, mod\, 4$, then a hamiltonian cycle in $T_{n=2t+2}\langle 1,3,5;t\rangle$ is $D_{1\to 5} \cup D_{5\to 9} \cup \cdots \cup D_{t\to t+4} \cup (t+4, t+5, \ldots, n-2, n-1, n, n-t = t+2, 2, 3 \cup D_{3\to 7} \cup D_{7\to 11} \cup \cdots \cup D_{t-6\to t-2} \cup (t-2, t+1, 1)$, see Figure 14. If $t \cong 3\, mod\, 4$,

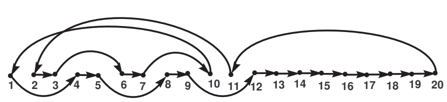

Figure 14: A hamiltonian cycle in $T_{20}\langle 1,3,5;9\rangle$

then a hamiltonian cycle in $T_{2t+2}\langle 1,3,5;t\rangle$ is $D_{1\to 5} \cup D_{5\to 9} \cup \cdots \cup D_{t-6\to t-2} \cup (t-2, t+3, t+4, \ldots, n-2, n-1, n, n-t = t+2, 2, 3) \cup D_{3\to 7} \cup D_{7\to 11} \cup \cdots \cup D_{t-4\to t} \cup (t, t+1, 1)$, see Figure 15. Since in both these cases, $(n-2, n-1, n)$ is a path in the hamiltonian cycles of $T_{2t+2}\langle 1,3,5;t\rangle$, by using the technique of Remark 1, $T_n\langle 1,3,5;t\rangle$ is hamiltonian for all even $n \cong 4\, mod\,(t-1)$.

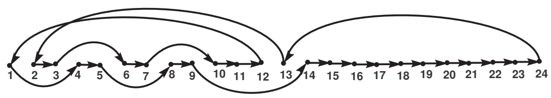

Figure 15: A hamiltonian cycle in $T_{24}\langle 1,3,5;11\rangle$

Case 3. $n \cong 6,8,\dots,t-3\,mod\,(t-1)$.
The smallest n is each class are $t+5, t+7, \dots, 2t-4$, in order. Clearly here $t \geq 9$, as $t+5 \leq 2t-4$. Let $n = t+(4u+1) \in \{t+5, t+7, \dots, 2t-4\}$, where $u \in \{1, 2\dots, \lfloor\frac{t-5}{4}\rfloor\}$ (as $t+(4u+1) \leq 2t-4$, so $u \leq \frac{t-5}{4}$). If $t \cong 1\,mod\,4$, then a hamiltonian cycle in $T_n\langle 1,3,5;t\rangle$ is $(1,2,3,\dots,n-t-1) \cup D_{n-t-1\to n-t+3} \cup D_{n-t+3\to n-t+7} \cup \dots \cup D_{t-1\to t+3} \cup (t+3, t+4, \dots, n-2, n-1, n, n-t, n-t+1) \cup D_{n-t+1\to n-t+5} \cup D_{n-t+5\to n-t+9} \cup \dots \cup D_{t-3\to t+1} \cup (t+1, 1)$, see Figure 16. If $t \cong 3\,mod\,4$, then a hamiltonian cycle in $T_{n=t+(4u+1)}\langle 1,3,5;t\rangle$ is $(1,2,3,\dots,n-t-1) \cup D_{n-t-1\to n-t+3} \cup D_{n-t+3\to n-t+7} \cup \dots \cup D_{t-7\to t-3} \cup (t-3, t+2, t+3, \dots, n-2, n-1, n, n-t, n-t+1) \cup D_{n-t+1\to n-t+5} \cup D_{n-t+5\to n-t+9} \cup \dots \cup D_{t-5\to t-1} \cup (t-1, t, t+1, 1)$, see Figure 17. Now, let $n = t+(4v+3) \in \{t+5, t+7, \dots, 2t-4\}$, where $v \in \{1, 2\dots, \lfloor\frac{t-7}{4}\rfloor\}$

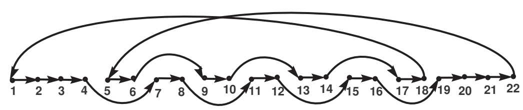

Figure 16: A hamiltonian cycle in $T_{22}\langle 1,3,5;17\rangle$

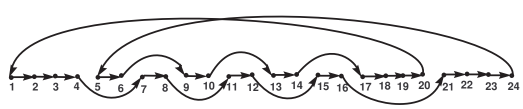

Figure 17: A hamiltonian cycle in $T_{24}\langle 1,3,5;19\rangle$

(as $t+(4v+3) \leq 2t-4$, so $v \leq \frac{t-7}{4}$). Clearly, here $t \geq 11$, because for $t = 9$, we have $n > 2t-4$ (as $12+4v > 14$). For $t \cong 1\,mod\,4$ and $t \cong 3\,mod\,4$, hamiltonian cycles in $T_{t+4v+3}\langle 1,3,5;t\rangle$ is same as ones in $T_{t+4u+1}\langle 1,3,5;t\rangle$ for $t \cong 3\,mod\,4$ and $t \cong 1\,mod\,4$, respectively. Since each of these hamiltonian cycles in $T_{n\in\{t+5,t+7,\dots,2t-4\}}\langle 1,3,5;t\rangle$ contains the path $(n-2,\, n-1,\, n)$, by using the technique of Remark 1, $T_n\langle 1,3,5;t\rangle$ is hamiltonian for all even $n \cong 6,8,\dots,t-3\,mod\,(t-1)$.

Conversely, since t is odd, $T_n\langle 1,3,5;t\rangle$ is bipartite and, being hamiltonian, n must be even. $\square$

3 Toeplitz graphs $T_n\langle 1,3,5;t\rangle$ for even t

Theorem 3.1. *$T_n\langle 1,3,5;2\rangle$ is hamiltonian for $n = 6$, 7, and all $n \cong 3\, mod\, 5$.*

Proof. For $n = 6$ and 7, it is easily seen that unique hamiltonian cycles in $T_6\langle 1,3,5;2\rangle$ and $T_7\langle 1,3,5;2\rangle$ are $(1,6,4,2,5,3,1)$ and $(1,6,4,2,7,5,3,1)$, respectively.

For $n \cong 3\, mod\, 5$, the smallest n is 8. The unique hamiltonian cycle in $T_8\langle 1,3,5;2\rangle$ is $(1,2,7,8,6,4,5,3,1)$ which contains the edge $(n-1 = 7, n = 8)$. We can extend this hamiltonian cycle in $T_8\langle 1,3,5;2\rangle$ to a hamiltonian cycle in $T_{13}\langle 1,3,5;2\rangle$, by replacing the edge $(7,8)$ with the path $(7,12,13,11,9,10,8)$, which preserves the same property. Suppose, for some non-negative integer r, $T_{n=8+5r}\langle 1,3,5;2\rangle$ has a hamiltonian cycle containing the edge $(n-1,n)$, then $T_{n+5}\langle 1,3,5;2\rangle$ enjoys the same property. Thus $T_n\langle 1,3,5;2\rangle$ is hamiltonian for all $n \cong 3\, mod\, 5$. □

For all $n \ncong 3\, mod\, 5$ and $n \neq 6,7$, the hamiltonicity of $T_n\langle 1,3,5;2\rangle$ remains undecided.

Theorem 3.2. *$T_n\langle 1,3,5;4\rangle$ is hamiltonian for all n.*

Proof. For $n = 6$ and $n = 7$, it is easily seen that unique hamiltonian cycles in $T_n\langle 1,3,5;4\rangle$ are $(1,6,2,3,4,5,1)$ and $(1,4,7,3,6,2,5,1)$, in order.

Now for $n \geq 8$. Hamiltonian cycles in $T_8\langle 1,3,5;4\rangle$, $T_9\langle 1,3,5;4\rangle$, and $T_{10}\langle 1,3,5;4\rangle$ are $(1,2,3,6,7,8,4,5,1)$, $(1,6,2,3,4,7,8,9,5,1)$ and $(1,4,7,3,8,9,10,6,2,5,1)$, respectively, which contain the path $(n-2,n-1,n)$. By Remark 1, these hamiltonian cycles in $T_n\langle 1,3,5;4\rangle$ can be extended to hamiltonian cycles in $T_{n+3}\langle 1,3,5;4\rangle$ which enjoy the same property. Suppose, for some non-negative integer r, $T_{n=n_o+3r}\langle 1,3,5;4\rangle$ has a hamiltonian cycle containing the path $(n-2,n-1,n)$ then $T_{n+3}\langle 1,3,5;4\rangle$ enjoys the same property and thus $T_n\langle 1,3,5;4\rangle$ is hamiltonian for all $n \geq 8$. This finishes the proof. □

Now, for even $t \geq 6$ and we will see that $T_n\langle 1,3,5;t\rangle$ is hamiltonian for all n.

Theorem 3.3. *For even $t \geq 6$, $T_n\langle 1,3,5;t\rangle$ is hamiltonian for all n.*

Proof. *Case 1.* $n \cong 1\, mod\,(t-1)$.
The smallest n, different from t, is $2t-1$. If $t \cong 0\, mod\, 4$, then a hamiltonian cycle in $T_{n=2t-1}\langle 1,3,5;t\rangle$ is $D_{1\to 5} \cup D_{5\to 9} \cup \cdots \cup D_{t-7\to t-3} \cup (t-3, t, t+3, t+4, \ldots, n-2, n-1, n, n-t = t-1, t+2, 2, 3) \cup D_{3\to 7} \cup D_{7\to 11} \cup \cdots \cup D_{t-9\to t-5} \cup (t-5, t-2, t+1, 1)$, see Figure 18. If $t \cong 2\, mod\, 4$, then a hamiltonian cycle in $T_{n=2t-1}\langle 1,3,5;t\rangle$ is $D_{1\to 5} \cup D_{5\to 9} \cup \cdots \cup$

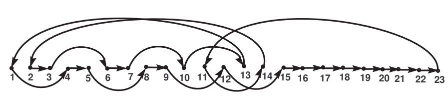

Figure 18: A hamiltonian cycle in $T_{23}\langle 1,3,5;12\rangle$

$D_{t-9\to t-5} \cup (t-5, t-2, t+3, t+4, \ldots, n-2, n-1, n, n-t = t-1, t+2, 2, 3) \cup D_{3\to 7} \cup D_{7\to 11} \cup \cdots \cup D_{t-3\to t+1} \cup (t+1, 1)$, see Figure 19. These hamiltonian cycles contain the path $(n-2, n-1, n)$. Suppose, for some non-negative integer r, $T_{n=(2t-1)+r(t-1)}\langle 1,3,5;t\rangle$ has a

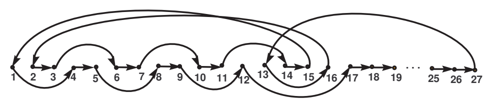

Figure 19: A hamiltonian cycle in $T_{27}\langle 1,3,5;14\rangle$

hamiltonian cycle containing the path $(n-2,n-1,n)$ then, by Remark 1, $T_{n+t-1}\langle 1,3,5;t\rangle$ enjoys the same property. Thus $T_n\langle 1,3,5;t\rangle$ is hamiltonian for all $n \cong 1\, mod\,(t-1)$.

Case 2. $n \cong 2\, mod\,(t-1)$.
The smallest n is $t+1$. A hamiltonian cycle in $T_{n=t+1}\langle 1,3,5;t\rangle$ is $(1,2,3,\dots,n-2,n-1,n,1)$, which contains the path $(n-2,n-1,n)$. By using the technique of Remark 1, $T_n\langle 1,3,5;t\rangle$ is hamiltonian for all $n \cong 2\, mod\,(t-1)$.

Case 3. $n \cong 3\, mod\,(t-1)$.
The smallest n is $t+2$. If $t \cong 0\, mod\, 4$, then a hamiltonian cycle in $T_{n=t+2}\langle 1,3,5;t\rangle$ is $D_{1\to 5} \cup D_{5\to 9} \cup \dots \cup D_{n-9\to n-5} \cup (n-5,n,2,3) \cup D_{3\to 7} \cup D_{7\to 11} \cup \dots \cup D_{n-7\to n-3} \cup (n-3,n-2,n-1=t+1,1)$, see Figure 20. If $t \cong 2\, mod\, 4$, then a hamiltonian cycle in $T_{n=t+2}\langle 1,3,5;t\rangle$ is $D_{1\to 5} \cup D_{5\to 9} \cup \dots \cup D_{n-7\to n-3} \cup (n-3,n,2,3) \cup D_{3\to 7} \cup D_{7\to 11} \cup \dots \cup D_{n-5\to n-1} \cup (n-1=t+1,1)$, see Figure 21. These hamiltonian cycles in $T_{n=t+2}\langle 1,3,5;t\rangle$

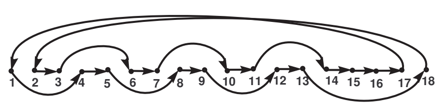

Figure 20: A hamiltonian cycle in $T_{18}\langle 1,3,5;16\rangle$

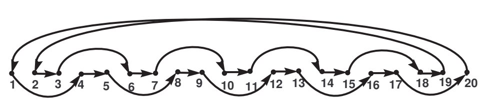

Figure 21: A hamiltonian cycle in $T_{20}\langle 1,3,5;18\rangle$

do not contain the path $(n-2,n-1,n)$. Now, the next representative in this class is $n=t+2+(t-1)=2t+1$. If $t \cong 0\, mod\, 4$, then a hamiltonian cycle in $T_{n=2t+1}\langle 1,3,5;t\rangle$ is $D_{1\to 5} \cup D_{5\to 9} \cup \dots \cup D_{t-7\to t-3} \cup (t-3,t+2,2,3) \cup D_{3\to 7} \cup D_{7\to 11} \cup \dots \cup D_{t-5\to t-1} \cup (t-1,t,t+3,t+4,\dots,n-2,n-1,n,n-t=t+1,1)$, see Figure 22. If $t \cong 2\, mod\, 4$, then a hamiltonian cycle in $T_{n=2t+1}\langle 1,3,5;t\rangle$ is $D_{1\to 5} \cup D_{5\to 9} \cup \dots \cup D_{t-5\to t-1} \cup (t-1,t+2,2,3) \cup D_{3\to 7} \cup D_{7\to 11} \cup \dots \cup D_{t-7\to t-3} \cup (t-3,t,t+3,t+4,\dots,n-2,n-1,n,n-t=t+1,1)$, see Figure 23. These hamiltonian cycles in $T_{2t+1}\langle 1,3,5;t\rangle$ contain the path $(n-2,n-1,n)$. By using the technique of Remark 1, $T_n\langle 1,3,5;t\rangle$ is hamiltonian for $n \cong 3\, mod\,(t-1)$.

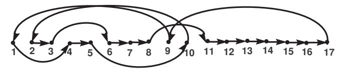

Figure 22: A hamiltonian cycle in $T_{17}\langle 1,3,5;8\rangle$

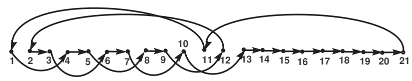

Figure 23: A hamiltonian cycle in $T_{21}\langle 1,3,5;10\rangle$

Case 4. $n \cong 4\, mod\,(t-1)$.
The smallest n is $t+3$. For even t. If $t \cong 0\, mod\, 3$, then a hamiltonian cycle in $T_{n=t+3}\langle 1,3,5;t\rangle$ is $(1,4,\ldots,n-5,n,3,6,\ldots,n-6,n-1,2,5,\ldots,n-4,n-3,n-2=t+1,1)$, see Figure 24. If $t \cong 1\, mod\, 3$, then a hamiltonian cycle in $T_{n=t+3}\langle 1,3,5;t\rangle$ is $(1,4,\ldots,n,3,6,\ldots,n-1,2,5,\ldots,n-2=t+1,1)$, see Figure 25. If $t \cong 2\, mod\, 3$, then a hamiltonian cycle in $T_n\langle 1,3,5;t\rangle$ is $(1,4,\ldots,n-1,2,5,\ldots,n,3,6,\ldots,n-2=t+1,1)$, see Figure 26. These

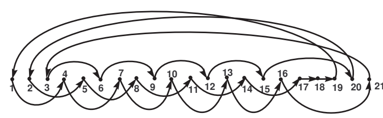

Figure 24: A hamiltonian cycle in $T_{21}\langle 1,3,5;18\rangle$

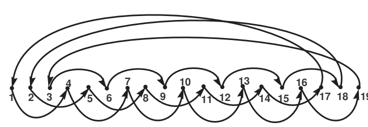

Figure 25: A hamiltonian cycle in $T_{19}\langle 1,3,5;16\rangle$

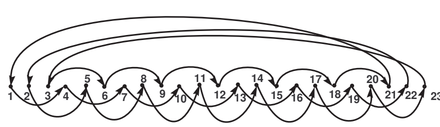

Figure 26: A hamiltonian cycle in $T_{23}\langle 1,3,5;20\rangle$

hamiltonian cycles in $T_{t+3}\langle 1,3,5;t\rangle$ do not contain the path $(n-2,n-1,n)$. Now, the next representative in this class is $n=2t+2$. If $t \cong 0\, mod\, 3$ and $t \neq 6$, then a hamiltonian cycle in $T_{n=2t+2}\langle 1,3,5;t\rangle$ is $(1,4,7,8,11,14,\dots,t-1,t+4,t+5,5,10,13,16,\dots,t-2,t+3,t+6,t+7,\dots,n-2,n-1,n,n-t=t+2,2,3,6,9,\dots,t,t+1,1)$, see Figure 27. And for $t=6$, a hamiltonian cycle in $T_{14}\langle 1,3,5;6\rangle$ is $(1,4,5,10,11,14,8,2,3,6,9,12,13,7,1)$. If $t\cong 1\, mod\, 3$, then a hamiltonian cycle in $T_{n=2t+2}\langle 1,3,5;t\rangle$ is $(1,6,9,12,\dots,t-1,t+4,t+5,5,8,11,\dots,t-2,t+3,t+6,t+7,\dots,n-2,n-1,n,n-t=t+2,2,3,4,7,10,\dots,t,t+1,1)$, see Figure 28. If $t\cong 2\, mod\, 3$, then a hamiltonian cycle in $T_{n=2t+2}\langle 1,3,5;t\rangle$ is $(1,4,7,10,\dots,t-1,t+4,t+5,5,8,11,\dots,t+6,t+7,\dots,n-2,n-1,n,n-t=t+2,2,3,6,9,\dots,t+1,1)$, see Figure 29. Since, in all these cases, $(n-2,n-1,n)$ is a path in the hamiltonian cycles of $T_{2t+2}\langle 1,3,5;t\rangle$, by using the technique in Remark 1, $T_n\langle 1,3,5;t\rangle$ is hamiltonian for all $n\cong 4\, mod\,(t-1)$.

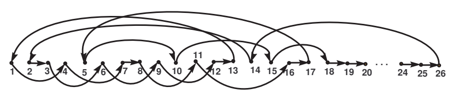

Figure 27: A hamiltonian cycle in $T_{26}\langle 1,3,5;12\rangle$

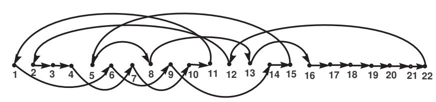

Figure 28: A hamiltonian cycle in $T_{22}\langle 1,3,5;10\rangle$

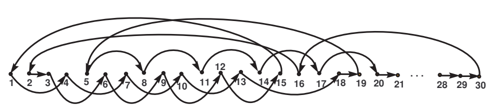

Figure 29: A hamiltonian cycle in $T_{30}\langle 1,3,5;14\rangle$

Case 5. $n\cong 5,7,9,\dots,t-3\, mod\,(t-1)$.
The smallest n in each class are $t+4,t+6,t+8,\dots,2t-4$. Clearly here $t\geq 8$ as $t+4\leq 2t-4$. For $n\in\{t+4,t+6,t+8,\dots,2t-4\}$. Let $n=t+4u$, where $u\in\{1,2\dots,\lfloor\frac{t-4}{4}\rfloor\}$ (as $t+4u\leq 2t-4$, so $u\leq\frac{t-4}{4}$). If $t\cong 0\, mod\, 4$, then a hamiltonian cycle in $T_{n=t+4u}\langle 1,3,5;t\rangle$ is $(1,2,3,\dots,n-t-1)\cup D_{n-t-1\to n-t+3}\cup D_{n-t+3\to n-t+7}\cup\dots\cup D_{t-1\to t+3}\cup(t+3,t+4,\dots,n-2,n-1,n,n-t,n-t+1)\cup D_{n-t+1\to n-t+5}\cup D_{n-t+5\to n-t+9}\cup\dots\cup D_{t-3\to t+1}\cup\dots\cup(t+1,1)$, see Figure 30. If $t\cong 2\, mod\, 4$, then a hamiltonian cycle in $T_{n=t+4u}\langle 1,3,5;t\rangle$ is $(1,2,3,\dots,n-$

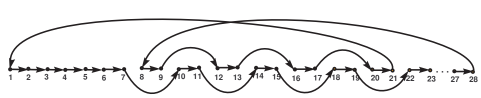

Figure 30: A hamiltonian cycle in $T_{28}\langle 1,3,5;20\rangle$

$t-1) \cup D_{n-t-1\to n-t+3} \cup D_{n-t+3\to n-t+7} \cup \cdots \cup D_{t-7\to t-3} \cup (t-3, t+2, t+3, \ldots, n-2, n-1, n, n-t, n-t+1) \cup D_{n-t+1\to n-t+5} \cup D_{n-t+5\to n-t+9} \cup \cdots \cup D_{t-5\to t-1} \cup (t-1, t, t+1, 1)$, see Figure 31. Now, let $n = t+4v+2$, where $v \in \{1, 2\ldots, \lfloor \frac{t-6}{4} \rfloor\}$ (as $t+4v+2 \le 2t-4$,

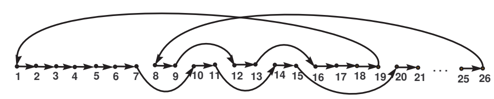

Figure 31: A hamiltonian cycle in $T_{26}\langle 1, 3, 5; 18\rangle$

so $v \le \frac{t-6}{4}$). For $t \cong 0\, mod\, 4$ and $t \cong 2\, mod\, 4$, $T_{n=t+4v+2}\langle 1, 3, 5; t\rangle$ has hamiltonian cycles similar to ones in $T_{n=t+4u}\langle 1, 3, 5; t\rangle$ for $t \cong 2\, mod\, 4$ and $t \cong 0\, mod\, 4$, respectively. Since these hamiltonian cycles in $T_{n\in\{t+4,t+6,\ldots,2t-4\}}\langle 1, 3, 5; t\rangle$ contain the path $(n-2,\ n-1,\ n)$, by using the technique of Remark 1, $T_n\langle 1, 3, 5; t\rangle$ is hamiltonian for $n \cong 5, 7, 9, \ldots, t-3\, mod\, (t-1)$.

Case 6. $n \cong 6, 8, 10, \ldots, t-2\, mod\, (t-1)$.

The smallest n in each class are $t+7, t+9, \ldots, 2t-3$. Clearly here $t \ge 8$ and $2(t+1)-n$ is an odd number which is in fact the number of consecutive vertices between $n-t$ and $t+1$, including both.

(i) Let $2(t+1)-n \cong 0\, mod\, 3$. Clearly $n-t$ is odd, as t is even and $n \in \{t+7, t+9, \ldots, 2t-3\}$ is odd. If $n-t \cong 1\, mod\, 4$, then a hamiltonian cycle in $T_n\langle 1, 3, 5; t\rangle$ is $D_{1\to 5} \cup D_{5\to 9} \cup \cdots \cup D_{n-t-8\to n-t-4} \cup (n-t-4, n-t+1, n-t+4, \ldots, t+3, t+4, \ldots, n-2, n-1, n, n-t, n-t+3, \ldots, t+2, 2, 3) \cup D_{3\to 7} \cup D_{7\to 11} \cup \cdots \cup D_{n-t-6\to n-t-2} \cup (n-t-2, n-t-1, n-t+2\ldots, t+1, 1)$, see Figure 32. If $n-t \cong 3\, mod\, 4$, then a hamiltonian cycle in $T_n\langle 1, 3, 5; t\rangle$ is $D_{1\to 5} \cup D_{5\to 9} \cup \cdots \cup D_{n-t-6\to n-t-2} \cup (n-t-2, n-t+1, \ldots, t+3, t+4, \ldots, n-2, n-1, n, n-t, n-t+3, \ldots, t+2, 2, 3) \cup D_{3\to 7} \cup D_{7\to 11} \cup \cdots \cup D_{n-t-8\to n-t-4} \cup (n-t-4, n-t-1, \ldots, t+1, 1)$, see Figure 33.

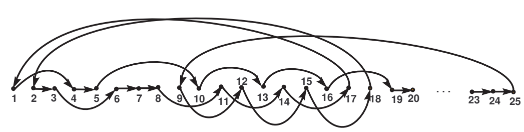

Figure 32: A hamiltonian cycle in $T_{25}\langle 1, 3, 5; 16\rangle$

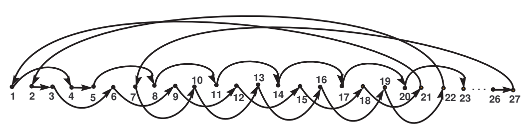

Figure 33: A hamiltonian cycle in $T_{27}\langle 1, 3, 5; 20\rangle$

(ii) Let $2(t+1)-n \cong 1\, mod\, 3$. If $n-t \cong 1\, mod\, 4$, then a hamiltonian cycle in $T_n\langle 1,3,5;t\rangle$ is $D_{1\to 5}\cup D_{5\to 9}\cup\cdots\cup D_{n-t-8\to n-t-4}\cup(n-t-4, n-t+1, n-t+4, \ldots, t+2, 2, 3)\cup D_{3\to 7}\cup D_{7\to 11}\cup\cdots\cup D_{n-t-6\to n-t-2}\cup(n-t-2, n-t-1, n-t+2, \ldots, t+3, t+4, \ldots, n-2, n-1, n, n-t, n-t+3, \ldots, t+1, 1)$, see Figure 34. If $n-t \cong 3\, mod\, 4$, then a hamiltonian cycle in $T_n\langle 1,3,5;t\rangle$ is $D_{1\to 5}\cup D_{5\to 9}\cup\cdots\cup D_{n-t-6\to n-t-2}\cup(n-t-2, n-t+1, \ldots, t+2, 2, 3)\cup D_{3\to 7}\cup D_{7\to 11}\cup\cdots\cup D_{n-t-8\to n-t-4}\cup(n-t-4, n-t-1, \ldots, t+3, t+4, \ldots, n-2, n-1, n, n-t, n-t+3, \ldots, t+1, 1)$, see Figure 35.

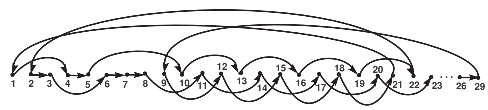

Figure 34: A hamiltonian cycle in $T_{29}\langle 1,3,5;20\rangle$

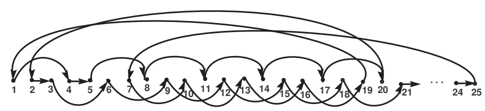

Figure 35: A hamiltonian cycle in $T_{25}\langle 1,3,5;18\rangle$

(iii) Let $2(t+1)-n \cong 2\, mod\, 3$. If $n-t \cong 1\, mod\, 4$, then a hamiltonian cycle in $T_n\langle 1,3,5;t\rangle$ is $D_{1\to 5}\cup D_{5\to 9}\cup\cdots\cup D_{n-t-8\to n-t-4}\cup(n-t-4, n-t-1, \ldots, t-1, t, t+3, t+4, \ldots, n-2, n-1, n, n-t, n-t+3, \ldots, t-3, t+2, 2, 3)\cup D_{3\to 7}\cup D_{7\to 11}\cup\cdots\cup D_{n-t-6\to n-t-2}\cup(n-t-2, n-t+1, \ldots, t+1, 1)$, see Figure 36. If $n-t \cong 3\, mod\, 4$, then a hamiltonian

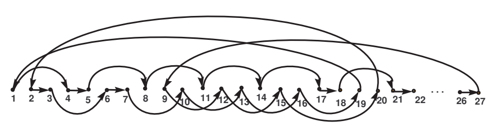

Figure 36: A hamiltonian cycle in $T_{27}\langle 1,3,5;18\rangle$

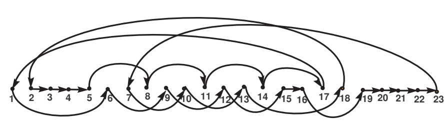

Figure 37: A hamiltonian cycle in $T_{23}\langle 1,3,5;16\rangle$

cycle in $T_n\langle 1,3,5;t\rangle$ is $D_{1\to 5}\cup D_{5\to 9}\cup\cdots\cup D_{n-t-10\to n-t-6}\cup(n-t-6, n-t-1, n-t+2, \ldots, t-1, t, t+3, t+4, \ldots, n-2, n-1, n, n-t, n-t+3, \ldots, t-3, t+2, 2, 3)\cup D_{3\to 7}\cup D_{7\to 11}\cup\cdots\cup D_{n-t-8\to n-t-4}\cup(n-t-4, n-t-3, n-t-2, n-t+1, \ldots, t+1, 1)$, see Figure 37. Since these hamiltonian cycles in $T_{n\in\{t+7,t+9,\ldots,2t-3\}}\langle 1,3,5;t\rangle$ contain the path $(n-2,\, n-1,\, n)$, by using the technique of Remark 1, $T_n\langle 1,3,5;t\rangle$ is hamiltonian for $n \cong 6,8,\ldots,t-2\, mod\,(t-1)$. This finishes the proof. □

Concluding remark: We state a conjecture that $T_n\langle 1,3,5;2\rangle$ is non hamiltonian for $n \cong 0,1,2,4\, mod\, 5$ different from $n=6$ and 7. The next task in our opinion is to complete the hamiltonicity investigation in Toeplitz graphs $T_n\langle 1,3,5,s_4,\ldots,s_k;t_1,t_2,\ldots,t_l\rangle$ by resolving this conjecture.

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