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The total edge irregularity strength of hexagonal grid graphs

by
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Dedicated to Professor Dr. Tudor Zamfirescu on the occasion of his 80th birthday

Abstract

For a graph $G = (V, E)$, a labeling $\partial \colon V \cup E \to \{1, 2, \ldots, k\}$ is called an edge irregular total k-labeling of G if the weights of any two different edges are distinct, where the weight of the edge xy under ∂ is defined to be $wt(xy) = \partial(x) + \partial(xy) + \partial(y)$. The total edge irregularity strength $\mathrm{tes}(G)$ of G is the minimum k for which G has an edge irregular total k-labeling. Al-Mushayt et al. "prove" that $\mathrm{tes}(H_n^m) = \left\lceil \frac{3mn+2(m+n)+1}{3} \right\rceil$ for the hexagonal grid graph H_n^m, but the labeling they constructed is actually not a total $\left\lceil \frac{3mn+2(m+n)+1}{3} \right\rceil$-labeling. In this paper, we first describe a correct edge irregular total $\left\lceil \frac{3mn+2(m+n)+1}{3} \right\rceil$-labeling of H_n^m for any $m, n \geq 1$, and so show that $\mathrm{tes}(H_n^m) = \left\lceil \frac{3mn+2(m+n)+1}{3} \right\rceil$. Moreover, we determine the exact value of the total edge irregularity strength for a more general hexagonal grid graph $H_n^{m_1,m_2,\ldots,m_n}$ by giving an edge irregular total $\mathrm{tes}(H_n^{m_1,m_2,\ldots,m_n})$-labeling, where $H_n^{m_1,m_2,\ldots,m_n}$ consists of n columns of hexagons and has m_i hexagons in the i-th column, $n \geq 2$, and $m_1, \ldots, m_n \geq 1$.

Key Words: Graph labelings, edge irregular total labelings, the total edge irregularity strength, hexagonal grid graphs.
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1 Introduction

Al-Mushayt et al. [3] claimed to have obtained the exact value of the total edge irregularity strength of the hexagonal grid graph H_n^m by describing an edge irregular total $\left\lceil \frac{3mn+2(m+n)+1}{3} \right\rceil$-labeling of H_n^m. However, it will be seen that this labeling is not a total $\left\lceil \frac{3mn+2(m+n)+1}{3} \right\rceil$-labeling. This paper is devoted to determining the precise values of the total edge irregularity strength of the hexagonal grid graph H_n^m for any $m, n \geq 1$, and a more general hexagonal grid graph $H_n^{m_1,m_2,\ldots,m_n}$ with n columns of hexagons having m_i hexagons in the i-th column, where $n \geq 2, m_i \geq 1, 1 \leq i \leq n$.

Here and throughout the paper, the graphs will be finite unless otherwise stated. A labeling of a graph $G = (V, E)$ is defined as a map from the graph elements to the numbers (usually the positive or non-negative integers), which are called labels. If the domain of a labeling is taken as V, E, and $V \cup E$, respectively, then the corresponding labeling is called a vertex labeling, an edge labeling, and a total labeling.

Graph labelings have various applications in network-related areas, such as communication network addressing, software testing, information security, and coding theory problems.

Considering the sum of all labels related to a graph element, which is called the weight of the element, is very interesting. For example, for a total labeling ∂ of a graph $G = (V, E)$, the weight of the edge xy is defined by

$$wt(xy) = \partial(x) + \partial(xy) + \partial(y), \tag{1}$$

and the weight of the vertex x is given by

$$wt(x) = \partial(x) + \sum_{xy \in E} \partial(xy).$$

In 1988, Chartrand et al. [9] defined an irregular labeling of a connected graph G with order at least 3 as an edge labeling of G such that the weights at each vertex (the sum of the labels of all edges incident with the vertex) are distinct. The minimum of the largest label of an edge over all such irregular labelings is called the irregularity strength of G, denoted by $\mathrm{s}(G)$. This parameter has been extensively studied ever since, see, for example, [2, 4, 8, 18]. However, it seems difficult to determine the irregularity strength of a graph even for graphs with a simple structure, see the survey paper [15].

Inspired by the notion of the irregularity strength of a graph, the work on total labelings and an excellent book by Wallis [22], Bača et al. [6] introduced the total edge irregularity strength of a graph.

More precisely, for a graph $G = (V, E)$, a total k-labeling $\partial\colon V \cup E \to \{1, 2, \dots, k\}$ is called an edge irregular total k-labeling of G, if for any two distinct edges e and f, $wt(e) \neq wt(f)$. The total edge irregularity strength of G, denoted by $\mathrm{tes}(G)$, is defined to be the minimum k for which the graph G has an edge irregular total k-labeling.

Bača et al. [6] gave a lower bound on the total edge irregularity strength of a graph G:

$$\mathrm{tes}(G) \geq \max\left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\}, \tag{2}$$

where $\Delta(G)$ is the maximum degree of G. They also determined the exact values of the total edge irregularity strength of paths, cycles, stars, wheels and friendship graphs, which implies that the above lower bound is tight.

Ivančo and Jendrol' [12] showed that the equality of (2) holds for trees. Therefore, they posed the following conjecture:

Conjecture 1.1. [12] *Let G be an arbitrary graph different from K_5. Then*

$$\mathrm{tes}(G) = \max\left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\},$$

where $\Delta(G)$ denotes the maximum degree of G.

Notice that Bača et al. [6, Theorem 7] proved that $\mathrm{tes}(K_5) = 5$, while the lower bound in (2) is equal to 4. Conjecture 1.1 has been verified for complete graphs and complete bipartite graphs [13, 14], for zigzag graphs [1], for generalized Petersen graphs [11], for generalized prisms [7], for octagonal grid graphs [21], for polar grid graphs [20], and for ladder-related graphs [17], etc. We refer the readers to the survey [10] for further details.

In the study of magic labelings of graphs, Bača [5] introduced the hexagonal grid graph, also called the honeycomb, which is widely used in computer graphics [16], cellular phone base stations [19], image processing, and in chemistry as the representation of benzenoid hydrocarbons. For $m, n \geq 1$, let H_n^m be the hexagonal grid graph with m rows and n columns of hexagons, and $V(H_n^m)$ and $E(H_n^m)$ denote the vertex set and the edge set of H_n^m, respectively. It is easy to see that

$$|V(H_n^m)| = 2mn + 2(m+n) \text{ and } |E(H_n^m)| = 3mn + 2(m+n) - 1.$$

Al-Mushayt et al. [3] claimed to have determined the explicit value of the total edge irregularity strength for H_n^m. The idea of their proof is as follows. Let

$$\tau = \left\lceil \frac{3mn + 2(m+n) + 1}{3} \right\rceil.$$

Since $2 \leq \Delta(H_n^m) \leq 3$ for any $m, n \geq 1$, we have

$$\left\lceil \frac{|E(H_n^m)| + 2}{3} \right\rceil = \tau > 2 = \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil,$$

which, together with (2), gives that

$$\mathrm{tes}(H_n^m) \geq \tau. \tag{3}$$

Therefore, in order to prove $\mathrm{tes}(H_n^m) = \tau$, it suffices to find an edge irregular total τ-labeling of H_n^m.

Al-Mushayt et al. [3] first described an edge irregular total $\lceil \frac{8n+5}{3} \rceil$-labeling ϕ_1 of the graph H_n^2. However, the total labeling given in [3, Theorem 1] is not an edge irregular total $\lceil \frac{8n+5}{3} \rceil$-labeling, since the label of some vertex under ϕ_1 may exceed $\lceil \frac{8n+5}{3} \rceil$. For example, the label of the vertex $a_{n-1,6}$ under ϕ_1: $\phi_1(a_{n-1,6}) = 3n - 5 > \lceil \frac{8n+5}{3} \rceil$ for $n \geq 23$. In the same vein, we claim that the total labeling ϕ_2 described in [3, Theorem 2] is not an edge irregular total τ-labeling of the graph H_n^m for $n \geq 3$ and $m \neq 2$, either. Taking $i = n-1$ and $j = 2m+2$, we have that for any $n > 8m + 9$, the label of the vertex $a_{i,j}$ with respect to ϕ_2:

$$\phi_2(a_{i,j}) = mn - 2m + n - 2 > \tau.$$

In this paper, we renumber the vertices and the edges of the hexagonal grid graph H_n^m for all $m, n \geq 1$, and obtain the following theorem.

Theorem 1.2. *For any $m, n \geq 1$, the total edge irregularity strength of H_n^m is equal to*

$$\left\lceil \frac{3mn + 2(m+n) + 1}{3} \right\rceil.$$

Therefore, it is immediate from Theorem 1.2 that Conjecture 1.1 also holds for the hexagonal grid graphs.

Motivated by the definition of honeycombs, we introduce more general hexagonal grid graphs $H_n^{m_1, m_2, \ldots, m_n}$, and show that Conjecture 1.1 also holds for $H_n^{m_1, m_2, \ldots, m_n}$. To be specific, we determine the exact value of the total edge irregularity strength of $H_n^{m_1, m_2, \ldots, m_n}$.

Theorem 1.3. *Let $m_1, m_2, \ldots, m_n \geq 1$, and $n \geq 2$. Then*

$$\operatorname{tes}(H_n^{m_1,m_2,\ldots,m_n})$$
$$= \left\lceil \frac{2m_1 + 2m_n + \sum_{t=1}^{n} m_t + n + 2\sum_{t=1}^{n-1} \max\{m_t, m_{t+1}\} + \sum_{t=1}^{n-1} 1^{(t)} + 2}{3} \right\rceil,$$

where for any $q \geq 1$,

$$1^{(2q-1)} = \begin{cases} 1, & \text{if } m_{2q-1} \leq m_{2q}; \\ 0, & \text{if } m_{2q-1} > m_{2q}, \end{cases} \qquad 1^{(2q)} = \begin{cases} 1, & \text{if } m_{2q} \geq m_{2q+1}; \\ 0, & \text{if } m_{2q} < m_{2q+1}. \end{cases}$$

This paper is organized as follows. In Section 2, we prove Theorem 1.2 by determining the precise values of the total edge irregularity strength of H_n^m for $n = 1$ and $n \geq 2$, respectively. In Section 3, we give a proof of Theorem 1.3 by describing an edge irregular total $\operatorname{tes}(H_n^{m_1,\ldots,m_n})$-labeling of the hexagonal grid graph $H_n^{m_1,\ldots,m_n}$.

2 The total edge irregularity strength of the hexagonal grid graphs

In this section, we give a proof of Theorem 1.2 by describing an edge irregular total $\left\lceil \frac{3mn+2(m+n)+1}{3} \right\rceil$-labeling of H_n^m for any $m, n \geq 1$.

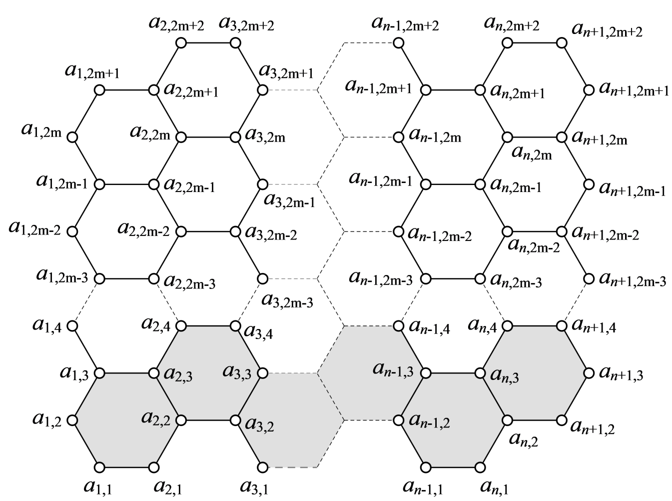

Figure 1: The hexagonal grid graph H_n^m.

For the sake of convenience, we divide the vertex set of H_n^m into disjoint subsets V_i (see

Figure 1):

$$V(H_n^m) = \bigcup_{i=1}^{n+1} V_i, \tag{4}$$

where

$$\begin{aligned} V_1 &= \{a_{1,j} \colon 1 \le j \le 2m+1\}, \\ V_i &= \{a_{i,j} \colon 1 \le j \le 2m+2\}, \text{ for } 2 \le i \le n, \\ V_{n+1} &= \begin{cases} \{a_{n+1,j} \colon 1 \le j \le 2m+1\}, \text{ if } n \text{ is odd;} \\ \{a_{n+1,j} \colon 2 \le j \le 2m+2\}, \text{ if } n \text{ is even,} \end{cases} \end{aligned}$$

and the edge set of H_n^m is cut into mutually disjoint subsets A_i and B_i:

$$E(H_n^m) = \bigcup_{i=1}^{n+1} A_i \cup \bigcup_{i=1}^{n} B_i, \tag{5}$$

where

$$\begin{aligned} A_1 &= \{a_{1,j}a_{1,j+1} \colon 1 \le j \le 2m\}, \\ A_i &= \{a_{i,j}a_{i,j+1} \colon 1 \le j \le 2m+1\}, \text{ for } 2 \le i \le n, \\ A_{n+1} &= \begin{cases} \{a_{n+1,j}a_{n+1,j+1} \colon 1 \le j \le 2m\}, & \text{if } n \text{ is odd;} \\ \{a_{n+1,j}a_{n+1,j+1} \colon 2 \le j \le 2m+1\}, & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

and for $1 \le i \le n$,

$$B_i = \{a_{i,j}a_{i+1,j} \colon 1 \le j \le 2m+2, i \text{ and } j \text{ have the same parity}\}.$$

In order to obtain a unique labeling, we need the Dirichlet character of modulus 2, which is a function $\chi \colon \mathbb{Z} \to \{0,1\}$ defined by

$$\chi(z) = \begin{cases} 1, & \text{if } 2 \nmid z; \\ 0, & \text{if } 2 \mid z. \end{cases}$$

By means of the above notation, we derive the precise value of the total edge irregularity strength of H_1^m.

Theorem 2.1. *For any integer $m \ge 1$, we have* $\operatorname{tes}(H_1^m) = \lceil \frac{5m}{3} \rceil + 1$.

Proof. It follows from (4) and (5) that

$$V(H_1^m) = V_1 \cup V_2 \quad \text{and} \quad E(H_1^m) = A_1 \cup A_2 \cup B_1,$$

see Figure 2, where

$$\begin{aligned} V_i &= \{a_{i,j} \colon 1 \le j \le 2m+1\}, \text{for } i = 1,2, \\ A_i &= \{a_{i,j}a_{i,j+1} \colon 1 \le j \le 2m\}, \text{for } i = 1,2, \\ B_1 &= \{a_{1,j}a_{2,j} \colon 1 \le j \le 2m+1, j \text{ is odd}\}. \end{aligned}$$

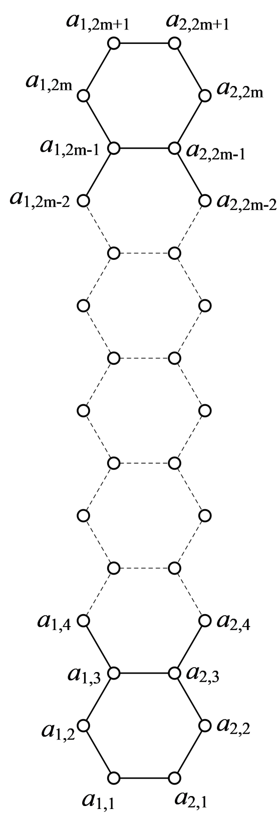

Figure 2: The hexagonal grid graph H_1^m.

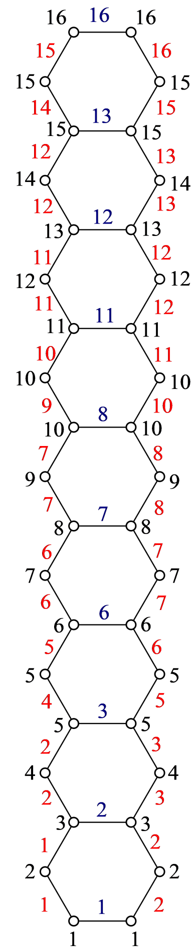

Figure 3: An edge irregular total 16-labeling of H_1^9.

Let $k = \lceil \frac{5m}{3} \rceil + 1$. As mentioned in the introduction, to show that $\text{tes}(H_1^m) = k$, we only need to find an edge irregular total k-labeling ϕ_1 of H_1^m.

Because of the edge irregularity of ϕ_1, we require that the weights of the edges satisfy that for $a_{i,j}a_{i,j+1} \in A_i$ $(i = 1, 2)$, $wt(a_{i,j}a_{i,j+1}) = \frac{5j+2i+\chi(j)}{2}$, and for $a_{1,j}a_{2,j} \in B_1$, $wt(a_{1,j}a_{2,j}) = \frac{5j+1}{2}$. One can easily check that the weights of all edges of H_1^m are pairwise distinct. Next we define the desired edge irregular total k-labeling ϕ_1.

(I) For any $a_{i,j} \in V_i$, we define

$$\phi_1(a_{i,j}) = \left\lceil \frac{5j + \chi(j)}{6} \right\rceil.$$

(II) In view of (1), for the edge $a_{i,j}a_{i,j+1} \in A_i$ and $i = 1, 2$, we have

$$wt(a_{i,j}a_{i,j+1}) = \phi_1(a_{i,j}) + \phi_1(a_{i,j}a_{i,j+1}) + \phi_1(a_{i,j+1}).$$

Therefore, the label of the edge $a_{i,j}a_{i,j+1}$ is given by

$$\phi_1(a_{i,j}a_{i,j+1}) = i - 1 + \frac{5j + \chi(j)}{2} - \left\lceil \frac{5j + \chi(j)}{6} \right\rceil - \left\lceil \frac{5j - \chi(j)}{6} \right\rceil,$$

and here we have used the fact that $\chi(j + 1) = 1 - \chi(j)$.
(III) Similar to (II), we deduce that the label of the edge $a_{1,j}a_{2,j} \in B_1$ is

$$\phi_1(a_{1,j}a_{2,j}) = \frac{5j + 1}{2} - 2\left\lceil \frac{5j + 1}{6} \right\rceil,$$

since the condition that $a_{1,j}a_{2,j} \in B_1$ implies that j is odd.

One can verify that all labels of the vertices and the edges under ϕ_1 are at least 1 and at most k. Thus the resulting labeling ϕ_1 is an edge irregular total k-labeling of the graph H_1^m. Hence, we reach that $\mathrm{tes}(H_1^m) = \lceil \frac{5m}{3} \rceil + 1$. □

For example, when $m = 9$, by Theorem 2.1, we have $\mathrm{tes}(H_1^9) = 16$, and an edge irregular total 16-labeling of H_1^9 is illustrated in Figure 3.

Theorem 2.2. *For any integers $m \geq 1$ and $n \geq 2$, we have* $\mathrm{tes}(H_n^m) = \left\lceil \frac{3mn+2(m+n)+1}{3} \right\rceil$.

Proof. The dissections of the vertex set and the edge set of H_n^m are shown in (4) and (5). Recall that

$$\tau = \left\lceil \frac{3mn + 2(m + n) + 1}{3} \right\rceil.$$

It suffices to construct an edge irregular total τ-labeling ϕ_2 of H_n^m to prove that $\mathrm{tes}(H_n^m) = \tau$.

To this end, we first assign the weights to all edges with respect to ϕ_2. For the edges $a_{i,j}a_{i,j+1} \in A_i (1 \leq i \leq n + 1)$, let

$$wt(a_{i,j}a_{i,j+1}) = \begin{cases} 2 + j, & \text{if } i = 1; \\ (3m + 2)i - 3m + j - 1, & \text{if } 2 \leq i \leq n; \\ (3m + 2)n + j + \chi(n), & \text{if } i = n + 1. \end{cases}$$

For the edges $a_{i,j}a_{i+1,j} \in B_i$, let

$$wt(a_{i,j}a_{i+1,j}) = (3m + 2)i - m + \left\lfloor \frac{j + 1}{2} \right\rfloor,$$

where $1 \leq i \leq n$. It can be shown that the weights of all edges of H_n^m are pairwise distinct, in fact, they successively attain values $3, 4, \ldots, 3mn + 2(m + n) + 1$. Now we give the labels of the vertices and the edges under the edge irregular total τ-labeling ϕ_2.

(I) We define the labels of the vertices as follows:

$$\phi_2(a_{i,j}) = \begin{cases} \left\lfloor \frac{j+1}{2} \right\rfloor, & \text{if } i = 1; \\ \left\lceil \frac{3mi-m+2i}{3} \right\rceil - 1 + \frac{-2m+j-2+\chi(j)}{2} \cdot \chi(i+1), & \text{if } 2 \le i \le n; \\ \tau, & \text{if } i = n+1. \end{cases}$$

(II) Using the fact that

$$wt(a_{i,j}a_{i,j+1}) = \phi_2(a_{i,j}) + \phi_2(a_{i,j}a_{i,j+1}) + \phi_2(a_{i,j+1}),$$

we label the edge $a_{i,j}a_{i,j+1} \in A_i$ by

$$\phi_2(a_{i,j}a_{i,j+1}) = \begin{cases} 1, & \text{if } i = 1; \\ (m+2)i - m + 2 - 2\left\lceil \frac{-m+2i}{3} \right\rceil + (-2m+j-1) \cdot \chi(i), & \text{if } 2 \le i \le n; \\ (3m+2)n + j + \chi(n) - 2\tau, & \text{if } i = n+1. \end{cases}$$

(III) For the edge $a_{i,j}a_{i+1,j} \in B_i$, we define

$$\phi_2(a_{i,j}a_{i+1,j}) = \begin{cases} 2m - \frac{j}{2} + \frac{5}{2} - \left\lceil \frac{2m+1}{3} \right\rceil, & \text{if } i = 1; \\ (m+2)i - \left\lceil \frac{-m+2i}{3} \right\rceil - \left\lceil \frac{2(m+i+1)}{3} \right\rceil + 3, & \text{if } 2 \le i \le n-1; \\ 2n(m+1) - \left\lceil \frac{-m+2n}{3} \right\rceil - \tau + 2 + \frac{-2m+j-1}{2} \cdot \chi(n), & \text{if } i = n. \end{cases}$$

One can check that all vertex and edge labels are between 1 and τ. Therefore, the constructed labeling ϕ_2 is the desired edge irregular total τ-labeling of the graph H_n^m. Then we have $\text{tes}(H_n^m) \le \tau$, which, together with (3), gives that $\text{tes}(H_n^m) = \tau$. $\square$

By Theorem 2.2, we obtain that the total edge irregularity strengths of H_5^5 and H_6^5 are 32 and 38, respectively. We give an edge irregular total 32-labeling of H_5^5 in Figure 4, and an edge irregular total 38-labeling of H_6^5 is shown in Figure 5.

Combining Theorem 2.1 and Theorem 2.2 yields Theorem 1.2.

3 The total edge irregularity strength of the hexagonal grid graph $H_n^{m_1,m_2,\ldots,m_n}$

This section provides a proof of Theorem 1.3. For $n, m_1, \ldots, m_n \ge 1$, let $H_n^{m_1,m_2,\ldots,m_n}$ be the hexagonal grid graph with n columns of hexagons where the i-th column has m_i hexagons. Notice that, when $m_1 = m_2 = \cdots = m_n = m$, we find that H_n^m is a special case of $H_n^{m_1,m_2,\ldots,m_n}$. For example, when $n = 4, m_1 = 1, m_2 = 4, m_3 = 2, m_4 = 1$, the hexagonal grid graph $H_4^{1,4,2,1}$ is shown in Figure 6. Thus, we have $|E(H_4^{1,4,2,1})| = 38$ and $|V(H_4^{1,4,2,1})| = 31$.

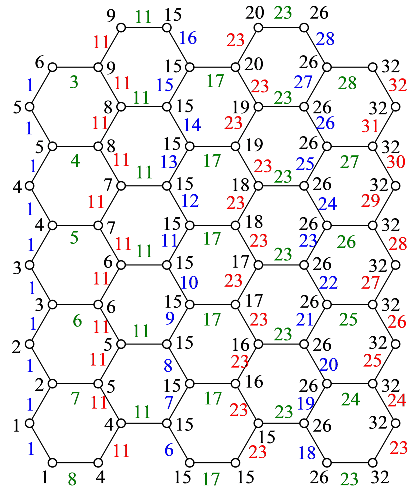

Figure 4: An edge irregular total 32-labeling of H_5^5.

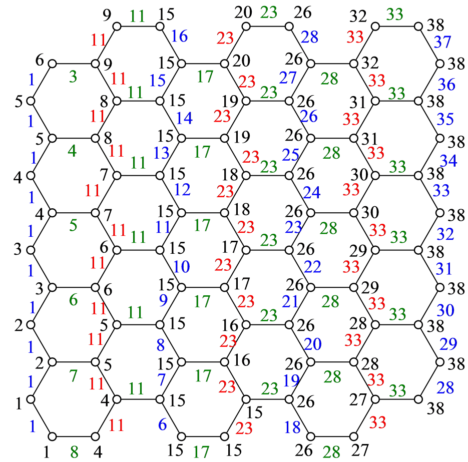

Figure 5: An edge irregular total 38-labeling of H_6^5.

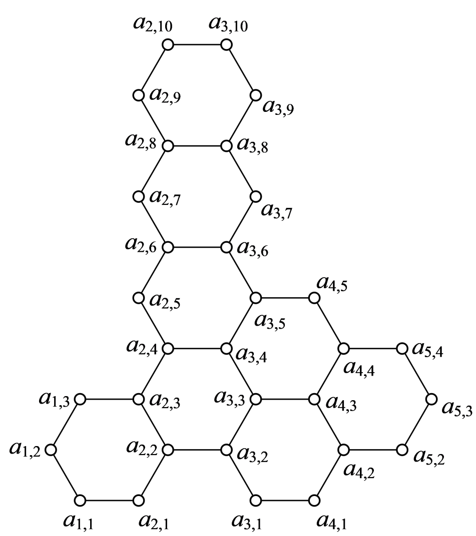

Figure 6: The hexagonal grid graph $H_4^{1,4,2,1}$.

The following function plays an important role in this section. For $q \geq 1$, let

$$1^{(2q-1)} = \begin{cases} 1, & \text{if } m_{2q-1} \leq m_{2q}; \\ 0, & \text{if } m_{2q-1} > m_{2q}, \end{cases} \qquad 1^{(2q)} = \begin{cases} 1, & \text{if } m_{2q} \geq m_{2q+1}; \\ 0, & \text{if } m_{2q} < m_{2q+1}. \end{cases}$$

For the sake of convenience, we split the vertex set of $H_n^{m_1,m_2,\ldots,m_n}$ into mutually disjoint subsets V_i:

$$V(H_n^{m_1,m_2,\ldots,m_n}) = \bigcup_{i=1}^{n+1} V_i, \tag{6}$$

where

$$V_1 = \{a_{1,j} \colon 1 \leq j \leq 2m_1 + 1\},$$

for $2 \leq i \leq n$,

$$V_i = \{a_{i,j} \colon 1 \leq j \leq 2\max\{m_{i-1}, m_i\} + 1^{(i-1)} + 1\},$$

$$V_{n+1} = \begin{cases} \{a_{n+1,j} \colon 1 \leq j \leq 2m_n + 1\}, & \text{if } n \text{ is odd;} \\ \{a_{n+1,j} \colon 2 \leq j \leq 2m_n + 2\}, & \text{if } n \text{ is even,} \end{cases}$$

and the edge set of $H_n^{m_1,m_2,\ldots,m_n}$ is divided into mutually disjoint subsets A_i and B_i:

$$E(H_n^{m_1,m_2,\ldots,m_n}) = \bigcup_{i=1}^{n+1} A_i \cup \bigcup_{i=1}^{n} B_i, \tag{7}$$

where

$$A_1 = \{a_{1,j}a_{1,j+1} \colon 1 \leq j \leq 2m_1\},$$

for $2 \leq i \leq n$,

$$A_i = \{a_{i,j}a_{i,j+1} \colon 1 \leq j \leq 2\max\{m_{i-1}, m_i\} + 1^{(i-1)}\},$$

$$A_{n+1} = \begin{cases} \{a_{n+1,j}a_{n+1,j+1} \colon 1 \leq j \leq 2m_n\}, & \text{if } n \text{ is odd;} \\ \{a_{n+1,j}a_{n+1,j+1} \colon 2 \leq j \leq 2m_n + 1\}, & \text{if } n \text{ is even,} \end{cases}$$

and for $1 \leq i \leq n$,

$$B_i = \{a_{i,j}a_{i+1,j} \colon 1 \leq j \leq 2m_i + 2, i \text{ and } j \text{ have the same parity}\}.$$

Therefore, we have

$$|E(H_n^{m_1,m_2,\ldots,m_n})| = 2m_1 + 2m_n + \sum_{t=1}^{n} m_t + n + 2\sum_{t=1}^{n-1} \max\{m_t, m_{t+1}\} + \sum_{t=1}^{n-1} 1^{(t)}.$$

Now we proceed to prove Theorem 1.3.

The proof of Theorem 1.3. Let

$$k = \left\lceil \frac{2m_1 + 2m_n + \sum_{t=1}^{n} m_t + n + 2\sum_{t=1}^{n-1} \max\{m_t, m_{t+1}\} + \sum_{t=1}^{n-1} 1^{(t)} + 2}{3} \right\rceil .$$

Since for $m_1, m_2, \ldots, m_n \geq 1, n \geq 2$, we have $\Delta(H_n^{m_1,m_2,\ldots,m_n}) = 3$ and

$$\left\lceil \frac{|E(H_n^{m_1,m_2,\ldots,m_n})| + 2}{3} \right\rceil = k \geq 2 = \left\lceil \frac{\Delta(H_n^{m_1,m_2,\ldots,m_n}) + 1}{2} \right\rceil ,$$

(2) gives that $\mathrm{tes}(H_n^{m_1,m_2,\ldots,m_n}) \geq k$. Thus, to show Theorem 1.3, it suffices to construct an edge irregular total k-labeling of the graph $H_n^{m_1,m_2,\ldots,m_n}$ implying that $\mathrm{tes}(H_n^{m_1,m_2,\ldots,m_n}) \leq k$.

For $2 \leq i \leq n+1$, let

$$f(i) = 2m_1 + \sum_{t=1}^{i-1} m_t + (i-1) + 2\sum_{t=1}^{i-2} \max\{m_t, m_{t+1}\} + \sum_{t=1}^{i-2} 1^{(t)}. \tag{8}$$

Then

$$k = \left\lceil \frac{f(n+1) + 2m_n + 2}{3} \right\rceil .$$

We consider the edge irregular total k-labelings of $H_n^{m_1,m_2,\ldots,m_n}$ from the following two cases.

Case 1. If for each $2 \leq i \leq n$,

$$\begin{aligned} 2m_{i-1} \leq m_i + 2m_n + \sum_{t=i}^{n} m_t + n - i - 3 \\ + 2\sum_{t=i}^{n-1} \max\{m_t, m_{t+1}\} + \sum_{t=i}^{n-1} 1^{(t)} - 1^{(i-1)}, \end{aligned} \tag{9}$$

then we want to find an edge irregular total k-labeling ψ_1 of $H_n^{m_1,m_2,\ldots,m_n}$ such that the weight of the edge $a_{i,j}a_{i,j+1}$ from A_i under ψ_1 is given by

$$wt(a_{i,j}a_{i,j+1}) = \begin{cases} j+2, & \text{if } i = 1; \\ f(i) + j + 2, & \text{if } 2 \leq i \leq n; \\ f(n+1) + j + 1 + \chi(n), & \text{if } i = n+1, \end{cases}$$

and the weight of the edge $a_{i,j}a_{i+1,j}$ from B_i under ψ_1 is equal to

$$f(i+1) + 1 + \left\lfloor \frac{j+1}{2} \right\rfloor - m_i,$$

where $1 \leq i \leq n$.

For any $a_{i,j}a_{i,j+1} \in A_i$ and $a_{i',j'}a_{i'+1,j'} \in B_i'$ with $i < i'$ and $2 \le i \le n$, we derive that

$$\begin{aligned}
&wt(a_{i',j'}a_{i'+1,j'}) - wt(a_{i,j}a_{i,j+1}) \\
&= f(i'+1) + 1 + \left\lfloor \frac{j'+1}{2} \right\rfloor - m_{i'} - (f(i) + j + 2) \\
&= \sum_{t=i}^{i'-1} m_t + i' - i + 2\sum_{t=i-1}^{i'-1} \max\{m_t, m_{t+1}\} + \sum_{t=i-1}^{i'-1} 1^{(t)} + \left\lfloor \frac{j'+1}{2} \right\rfloor - j.
\end{aligned}$$

By the definition of A_i and B_i, we know that $1 \le j \le 2\max\{m_{i-1}, m_i\} + 1^{(i-1)}$ and $1 \le j' \le 2m_{i'} + 2$. Thus,

$$\begin{aligned}
&wt(a_{i',j'}a_{i'+1,j'}) - wt(a_{i,j}a_{i,j+1}) \\
&\ge \sum_{t=i}^{i'-1} m_t + i' - i + 2\sum_{t=i-1}^{i'-1} \max\{m_t, m_{t+1}\} + \sum_{t=i-1}^{i'-1} 1^{(t)} + 1 - 2\max\{m_{i-1}, m_i\} - 1^{(i-1)} \\
&> 0,
\end{aligned}$$

which implies that $wt(a_{i,j}a_{i,j+1})$ and $wt(a_{i',j'}a_{i'+1,j'})$ are distinct. Analogously, one can verify that the weights of all edges from the sets A_i and B_i under ψ_1 are pairwise distinct.

Next we explicitly give the total k-labeling ψ_1.

(I) We define the vertex labels as follows: For any $a_{i,j} \in V_i (1 \le i \le n+1)$,

$$\psi_1(a_{i,j}) = \begin{cases} \left\lceil \frac{j}{3} \right\rceil, & \text{if } i = 1; \\ \left\lceil \frac{f(i)+j}{3} \right\rceil, & \text{if } 2 \le i \le n; \\ \left\lceil \frac{f(n+1)+j+\chi(n)}{3} \right\rceil, & \text{if } i = n+1. \end{cases}$$

(II) From the definition of the weight of the edges

$$wt(a_{i,j}a_{i,j+1}) = \psi_1(a_{i,j}) + \psi_1(a_{i,j}a_{i,j+1}) + \psi_1(a_{i,j+1}),$$

we define the edge labels of the edge $a_{i,j}a_{i,j+1} \in A_i$:

$$\psi_1(a_{i,j}a_{i,j+1}) = \begin{cases} 2 + j - \left\lceil \frac{j}{3} \right\rceil - \left\lceil \frac{j+1}{3} \right\rceil, & \text{if } i = 1; \\ f(i) + 2 + j - \left\lceil \frac{f(i)+j}{3} \right\rceil - \left\lceil \frac{f(i)+j+1}{3} \right\rceil, & \text{if } 2 \le i \le n; \\ f(n+1) + 1 + j + \chi(n) & \\ \quad - \left\lceil \frac{f(n+1)+j+\chi(n)}{3} \right\rceil - \left\lceil \frac{f(n+1)+1+j+\chi(n)}{3} \right\rceil, & \text{if } i = n+1. \end{cases}$$

(III) Similar to (II), we define the labels of the edge $a_{i,j}a_{i+1,j} \in B_i (1 \le i \le n)$:

$$\psi_1(a_{i,j}a_{i+1,j}) = \begin{cases} m_1 + 2 + \left\lfloor \frac{j+1}{2} \right\rfloor - \left\lceil \frac{j}{3} \right\rceil - \left\lceil \frac{j+1}{3} \right\rceil, & \text{if } i = 1; \\ f(i+1) + 1 + \left\lfloor \frac{j+1}{2} \right\rfloor - m_i \\ \quad - \left\lceil \frac{f(i)+j}{3} \right\rceil - \left\lceil \frac{f(i+1)+j}{3} \right\rceil, & \text{if } 2 \le i \le n-1; \\ f(n+1) + 1 + \left\lfloor \frac{j+1}{2} \right\rfloor - m_n \\ \quad - \left\lceil \frac{f(n)+j}{3} \right\rceil - \left\lceil \frac{f(n+1)+j+\chi(n)}{3} \right\rceil, & \text{if } i = n. \end{cases}$$

We claim that all vertex and edge labels lie in between 1 and k. We illustrate our method of proving this claim with an example of showing that the label of the edge $a_{i,j}a_{i+1,j} \in B_i$ under ψ_1 satisfies that $1 \le \psi_1(a_{i,j}a_{i+1,j}) \le k$, where $2 \le i \le n-1$. We first prove $\psi_1(a_{i,j}a_{i+1,j}) \ge 1$. Since for any integer s,

$$\left\lfloor \frac{s+1}{2} \right\rfloor \ge \frac{s}{2}, \quad \text{and} \quad \left\lceil \frac{s}{3} \right\rceil \le \frac{s+2}{3}, \tag{10}$$

the label of $a_{i,j}a_{i+1,j}$ is

$$\begin{aligned} \psi_1(a_{i,j}a_{i+1,j}) &= f(i+1) + 1 + \left\lfloor \frac{j+1}{2} \right\rfloor - m_i - \left\lceil \frac{f(i)+j}{3} \right\rceil - \left\lceil \frac{f(i+1)+j}{3} \right\rceil \\ &\ge \frac{2f(i+1) - f(i) - 1}{3} - \frac{j}{6} - m_i. \end{aligned}$$

By the definition of B_i, we know that $1 \le j \le 2m_i + 2$. Thus,

$$\psi_1(a_{i,j}a_{i+1,j}) \ge \frac{2f(i+1) - f(i) - 4m_i - 2}{3}.$$

Substituting (8) into the above relation yields that

$$\begin{aligned} \psi_1(a_{i,j}a_{i+1,j}) \ge \frac{1}{3} \Bigg(& 2m_1 + \sum_{t=1}^{i-1} m_t - 2m_i + i + 2\sum_{t=1}^{i-1} \max\{m_t, m_{t+1}\} \\ & + 2\max\{m_{i-1}, m_i\} + \sum_{t=1}^{i-1} 1^{(t)} + 1^{(i-1)} - 1 \Bigg). \end{aligned}$$

Because $\max\{m_{i-1}, m_i\} \ge m_i$, $i \ge 2$, and $m_1, \ldots, m_n \ge 1$, we obtain that $\psi_1(a_{i,j}a_{i+1,j}) \ge 1$.

It remains to show that for $2 \le i \le n-1$, $\psi_1(a_{i,j}a_{i+1,j}) \le k$. It follows from the definition of the label of the edge $a_{i,j}a_{i+1,j}$ that

$$\begin{aligned} \psi_1(a_{i,j}a_{i+1,j}) - k = {} & f(i+1) + 1 + \left\lfloor \frac{j+1}{2} \right\rfloor - m_i \\ & - \left\lceil \frac{f(i)+j}{3} \right\rceil - \left\lceil \frac{f(i+1)+j}{3} \right\rceil - \left\lceil \frac{f(n+1)+2m_n+2}{3} \right\rceil. \end{aligned}$$

Using the fact that for any real number s,

$$\lfloor s \rfloor \leq s, \quad \lceil s \rceil \geq s, \tag{11}$$

we have

$$\psi_1(a_{i,j}a_{i+1,j}) - k \leq \frac{1}{3}(2f(i+1) - f(i) - f(n+1) - 2m_n) - m_i + \frac{5}{6} - \frac{j}{6}.$$

Since $1 \leq j \leq 2m_i + 2$, we obtain that

$$\psi_1(a_{i,j}a_{i+1,j}) - k \leq \frac{1}{3}(2f(i+1) - f(i) - f(n+1) - 2m_n) - m_i + \frac{2}{3}.$$

Substituting (8) into the right hand side of the above relation gives that

$$\psi_1(a_{i,j}a_{i+1,j}) - k \leq \frac{1}{3}(-m_i - \sum_{t=i}^{n} m_t + i - n + 2\max\{m_{i-1}, m_i\}$$

$$- 2\sum_{t=i}^{n-1} \max\{m_t, m_{t+1}\} + 1^{(i-1)} - \sum_{t=i}^{n-1} 1^{(t)} - 2m_n + 3).$$

Next we show that $\psi_1(a_{i,j}a_{i+1,j}) - k \leq 0$ from the two cases: $m_{i-1} \leq m_i$ and $m_{i-1} > m_i$.

Subcase 1. If $m_{i-1} \leq m_i$, then $\max\{m_{i-1}, m_i\} = m_i$. Thus,

$$\psi_1(a_{i,j}a_{i+1,j}) - k \leq \frac{1}{3}\left(- \sum_{t=i+1}^{n} m_t + i - n - 2\sum_{t=i}^{n-1} \max\{m_t, m_{t+1}\} + 1^{(i-1)}\right.$$

$$\left. - \sum_{t=i}^{n-1} 1^{(t)} - 2m_n + 3\right).$$

Due to the condition that $2 \leq i \leq n-1$, $m_1, \ldots, m_n \geq 1$ and $1^{(i-1)} \leq 1$, we arrive at $\psi_1(a_{i,j}a_{i+1,j}) - k \leq 0$.

Subcase 2. If $m_{i-1} > m_i$, then $\max\{m_{i-1}, m_i\} = m_{i-1}$. Hence, we have

$$\psi_1(a_{i,j}a_{i+1,j}) - k \leq \frac{1}{3}(2m_{i-1} - m_i - 2m_n - \sum_{t=i}^{n} m_t - n + i + 3$$

$$- 2\sum_{t=i}^{n-1} \max\{m_t, m_{t+1}\} - \sum_{t=i}^{n-1} 1^{(t)} - 1^{(i-1)}) \leq 0,$$

where the last inequality is exactly the condition (9).

In this case, the above defined labeling ψ_1 is the desired edge irregular total k-labeling of $H_n^{m_1,m_2,\ldots,m_n}$.

Case 2. If there exists an integer i^* with $2 \leq i^* \leq n$ such that

$$2m_{i^*-1} > m_{i^*} + 2m_n + \sum_{t=i^*}^{n} m_t + n - i^* - 3$$

$$+ 2\sum_{t=i^*}^{n-1} \max\{m_t, m_{t+1}\} + \sum_{t=i^*}^{n-1} 1^{(t)} - 1^{(i^*-1)}, \tag{12}$$

then we look for an edge irregular total k-labeling ψ_2 of $H_n^{m_1,\ldots,m_n}$ satisfying that the weights of the edges under ψ_2 are the same as the weights under ψ_1. Therefore, the weights of all edges of $H_n^{m_1,\ldots,m_n}$ under ψ_2 are pairwise distinct.

Let S^* be the set of all integers satisfying (12), that is,

$$S^* = \Bigg\{ 2 \le i^* \le n \colon 2m_{i^*-1} > m_{i^*} + 2m_n + \sum_{t=i^*}^{n} m_t + n - i^* - 3 \\ + 2\sum_{t=i^*}^{n-1} \max\{m_t, m_{t+1}\} + \sum_{t=i^*}^{n-1} 1^{(t)} - 1^{(i^*-1)} \Bigg\},$$

and let $S = \{2, 3, \ldots, n\} \setminus S^*$.

We claim that for any $i \in S^*$, $m_{i-1} \ge m_i$, and the equality holds if and only if $m_{n-1} = m_n = 1$. In fact, for any $2 \le i \le n-1$, it is immediate from $m_1, \ldots, m_n \ge 1$ and $1^{(i-1)} \le 1$ that

$$2m_n + \sum_{t=i+1}^{n} m_t + n - i - 3 + 2\sum_{t=i}^{n-1} \max\{m_t, m_{t+1}\} + \sum_{t=i}^{n-1} 1^{(t)} - 1^{(i-1)} > 0. \tag{13}$$

Then it follows from (12) and (13) that $m_{i-1} > m_i$ for any $2 \le i \le n-1$. When $i = n$, (12) is reduced to

$$2m_{n-1} > 4m_n - 3 - 1^{(n-1)},$$

which implies that $m_{n-1} > m_n$ if $m_n \ge 2$. When $m_n = 1$, it is clear that $m_{n-1} \ge 1 = m_n$ as $m_i \ge 1$ for all i.

Now we describe the edge irregular total k-labeling ψ_2.

(I) We define the vertex labels under ψ_2 as follows: For any $a_{i,j} \in V_i (1 \le i \le n+1)$,

$$\psi_2(a_{i,j}) = \begin{cases} \left\lceil \frac{j}{3} \right\rceil, & \text{if } i = 1; \\ \left\lceil \frac{f(i)+j}{3} \right\rceil, & \text{if } i \in S; \text{or } i \in S^* \text{ and } i+j \text{ is odd}; \\ \left\lceil \frac{f(i+1)-m_i-1}{3} \right\rceil, & \text{if } i \in S^* \text{ and } i+j \text{ is even}; \\ \left\lceil \frac{f(n+1)+j+\chi(n)}{3} \right\rceil, & \text{if } i = n+1. \end{cases}$$

(II) For the edge $a_{i,j}a_{i,j+1} \in A_i$, where $1 \le i \le n+1$, recall that

$$wt(a_{i,j}a_{i,j+1}) = \psi_2(a_{i,j}) + \psi_2(a_{i,j}a_{i,j+1}) + \psi_2(a_{i,j+1}).$$

Hence, the label of the edge $a_{i,j}a_{i,j+1} \in A_i$ is defined by

$$\psi_2(a_{i,j}a_{i,j+1}) = \begin{cases} 2 + j - \left\lceil \frac{j}{3} \right\rceil - \left\lceil \frac{j+1}{3} \right\rceil, & \text{if } i = 1; \\ f(i) + 2 + j - \left\lceil \frac{f(i)+j}{3} \right\rceil - \left\lceil \frac{f(i)+j+1}{3} \right\rceil, & \text{if } i \in S; \\ f(i) + 2 + j - \left\lceil \frac{f(i)+j+\chi(i+j+1)}{3} \right\rceil & \\ \quad - \left\lceil \frac{f(i+1)-m_i-1}{3} \right\rceil, & \text{if } i \in S^*; \\ f(n+1) + 1 + j + \chi(n) - \left\lceil \frac{f(n+1)+j+\chi(n)}{3} \right\rceil & \\ \quad - \left\lceil \frac{f(n+1)+j+1+\chi(n)}{3} \right\rceil, & \text{if } i = n+1. \end{cases}$$

(III) Using the fact that

$$wt(a_{i,j}a_{i+1,j}) = \psi_2(a_{i,j}) + \psi_2(a_{i,j}a_{i+1,j}) + \psi_2(a_{i+1,j}),$$

we give the label of the edge $a_{i,j}a_{i+1,j} \in B_i$ as follows:

$$\psi_2(a_{i,j}a_{i+1,j}) = \begin{cases} m_1 + 2 + \left\lfloor \frac{j+1}{2} \right\rfloor - \left\lceil \frac{j}{3} \right\rceil - \left\lceil \frac{j+1}{3} \right\rceil, & \text{if } i = 1; \\ f(i+1) + 1 + \left\lfloor \frac{j+1}{2} \right\rfloor - m_i & \\ - \left\lceil \frac{f(i)+j}{3} \right\rceil - \left\lceil \frac{f(i+1)+j+1_n(i)}{3} \right\rceil, & \text{if } i \in S; \\ f(i+1) + 1 + \left\lfloor \frac{j+1}{2} \right\rfloor - m_i & \\ - \left\lceil \frac{f(i+1)-m_i-1}{3} \right\rceil - \left\lceil \frac{f(i+1)+j+1_n(i)}{3} \right\rceil, & \text{if } i \in S^*, \end{cases}$$

where the function $1_n \colon \mathbb{Z} \to \{0, 1\}$ is given by

$$1_n(z) = \begin{cases} \chi(n), & \text{if } z = n; \\ 0, & \text{otherwise.} \end{cases}$$

We can verify that all vertex and edge labels are at least 1 and at most k. We only prove that the label of the edge $a_{i,j}a_{i,j+1} \in A_i$ under ψ_2 is between 1 and k when $i \in S^*$, i is even and j is odd. The other cases can be deduced similarly.

Since $i \in S^*$, we have that $i = n$ and $m_{n-1} = m_n = 1$, or $2 \le i \le n$ and $m_{i-1} > m_i$. We show that $1 \le \psi_2(a_{i,j}a_{i,j+1}) \le k$ from the following two cases.

Subcase 1. If $i = n$ and $m_{n-1} = m_n = 1$, then $1^{(n-1)} = 1$, and, by the definition of A_i, $1 \le j \le 3$. In this case, the label of $a_{i,j}a_{i,j+1}$ is given by

$$\psi_2(a_{i,j}a_{i,j+1}) = f(n) + j + 1 - \left\lceil \frac{f(n)+j}{3} \right\rceil - \left\lceil \frac{f(n)}{3} \right\rceil.$$

By the properties of the ceiling function, we obtain that

$$\frac{1}{3}(f(n)+1) \le \frac{1}{3}(f(n)+2j-1) \le \psi_2(a_{i,j}a_{i,j+1}) \le \frac{f(n)+2j}{3} + 1 \le \left\lceil \frac{f(n)}{3} \right\rceil + 3 = k.$$

Since $n \geq 2$, we deduce that

$$f(n) = 2m_1 + \sum_{t=1}^{n-1} m_t + (n-1) + 2\sum_{t=1}^{n-2} \max\{m_t, m_{t+1}\} + \sum_{t=1}^{n-2} 1^{(t)} \geq 3.$$

Therefore,

$$1 \leq \psi_2(a_{i,j}a_{i,j+1}) \leq k.$$

Subcase 2. If $2 \leq i \leq n$ and $m_{i-1} > m_i$, then $1^{(i-1)} = 0$ and so $1 \leq j \leq 2m_{i-1} - 1$. By the definition of $f(i)$, we have

$$f(i+1) = f(i) + m_i + 2m_{i-1} + 1.$$

Hence, the label of $a_{i,j}a_{i,j+1}$ is

$$\begin{aligned}\psi_2(a_{i,j}a_{i,j+1}) &= f(i) + 2 + j - \left\lceil \frac{f(i)+j}{3} \right\rceil - \left\lceil \frac{f(i+1) - m_i - 1}{3} \right\rceil \\ &= f(i) + 2 + j - \left\lceil \frac{f(i)+j}{3} \right\rceil - \left\lceil \frac{f(i) + 2m_{i-1}}{3} \right\rceil. \qquad (14)\end{aligned}$$

In view of (10), we derive that for any $1 \leq j \leq 2m_{i-1} - 1$,

$$\psi_2(a_{i,j}a_{i,j+1}) \geq \frac{1}{3}(f(i) - 2m_{i-1} + 2 + 2j) \geq \frac{1}{3}(f(i) - 2m_{i-1} + 4). \qquad (15)$$

Since $i \geq 2$, and $\max\{m_{i-2}, m_{i-1}\} \geq m_{i-1}$ for $i > 2$, we obtain that

$$\begin{aligned}&f(i) - 2m_{i-1} \\ &= 2m_1 + \sum_{t=1}^{i-1} m_t + (i-1) + 2\sum_{t=1}^{i-2} \max\{m_t, m_{t+1}\} + \sum_{t=1}^{i-2} 1^{(t)} - 2m_{i-1} > 0. \qquad (16)\end{aligned}$$

Combining (15) and (16) gives that $\psi_2(a_{i,j}a_{i,j+1}) \geq 1$.

Next we show that $\psi_2(a_{i,j}a_{i,j+1}) \leq k$, that is, $\psi_2(a_{i,j}a_{i,j+1}) - k \leq 0$. It follows from (11) and (14) that

$$\psi_2(a_{i,j}a_{i,j+1}) - k \leq \frac{1}{3}(f(i) + 2j - 2m_{i-1} - f(n+1) - 2m_n + 4).$$

Since $1 \leq j \leq 2m_{i-1} - 1$,

$$\psi_2(a_{i,j}a_{i,j+1}) - k \leq \frac{1}{3}(f(i) + 2m_{i-1} - f(n+1) - 2m_n + 2).$$

Notice that

$$\begin{aligned}&f(i) + 2m_{i-1} - f(n+1) - 2m_n + 2 \\ &= -\sum_{t=i}^{n} m_t + i - n - 2m_n - 2\sum_{t=i}^{n-1} \max\{m_t, m_{t+1}\} - \sum_{t=i-1}^{n-1} 1^{(t)} + 1 \leq 0.\end{aligned}$$

Hence, $\psi_2(a_{i,j}a_{i,j+1}) \leq k$.

Thus, the resulting labeling ψ_2 is the desired edge irregular total k-labeling of $H_n^{m_1,m_2,\ldots,m_n}$.

Thus, the proof is complete. $\square$

For example, for the graph $H_8^{4,6,7,2,4,1,4,5}$, we have $\mathrm{tes}(H_8^{4,6,7,2,4,1,4,5}) = 46$ from Theorem 1.3, and Figure 7 gives an edge irregular total 46-labeling of $H_8^{4,6,7,2,4,1,4,5}$. Theorem 1.3 gives that $\mathrm{tes}(H_4^{1,1,13,1}) = 27$, and an edge irregular total 27-labeling of $H_4^{1,1,13,1}$ is shown in Figure 8.

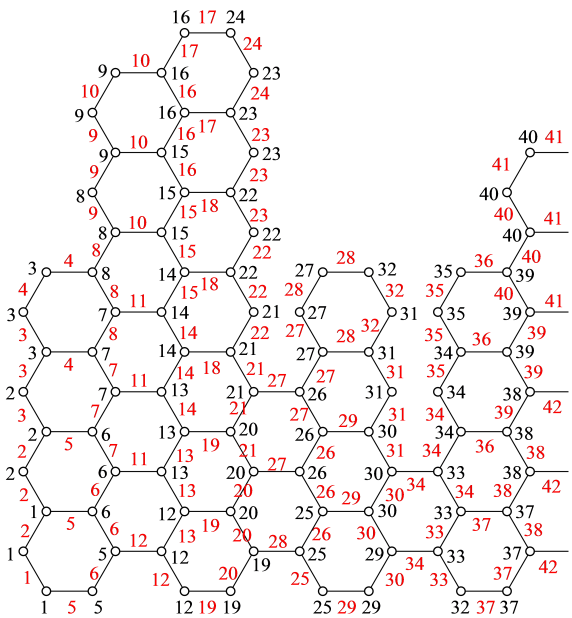

Figure 7: An edge irregular total 46-labeling of $H_8^{4,6,7,2,4,1,4,5}$.

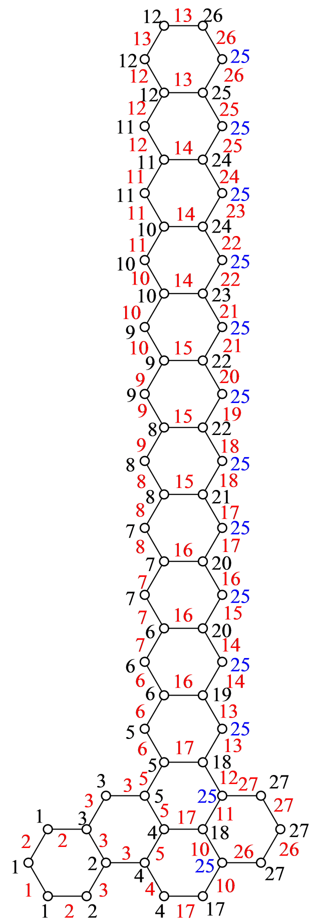

Figure 8: An edge irregular total 27-labeling of $H_4^{1,1,13,1}$.

It is worth mentioning that although Theorem 1.3 implies Theorem 1.2 (i.e., Theorem 2.2) when $n \geq 2$, due to the fact that H_n^m is a special case of $H_n^{m_1,\ldots,m_n}$, the labels given by

Theorem 2.2 are different from that given by Theorem 1.3. For example, the edge irregular total 32-labeling of H_5^5 and the edge irregular total 38-labeling of H_6^5 defined in the proof of Theorem 1.3 are illustrated in Figure 9 and Figure 10, respectively, which are different from the labelings given by Figure 4 and Figure 5.

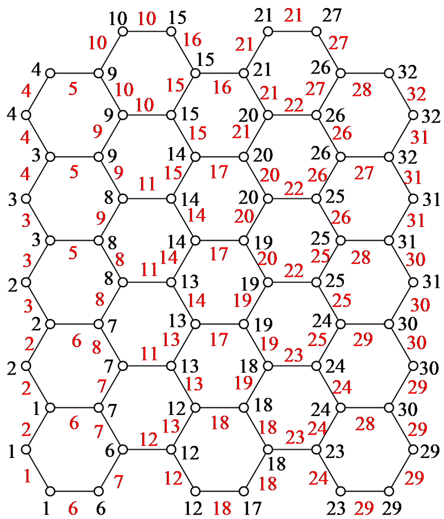

Figure 9: An edge irregular total 32-labeling of H_5^5.

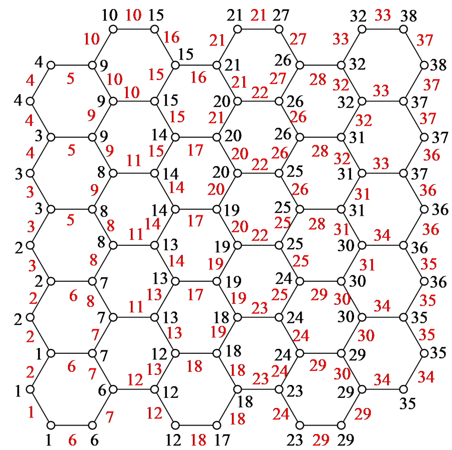

Figure 10: An edge irregular total 38-labeling of H_6^5.

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