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Cohomology and deformations of n-Hom-Lie algebra morphisms

by
Anja Arfa[(1)], Nizar Ben Fraj[(2)], Abdenacer Makhlouf[(3)]

Abstract

The main purpose of this paper is to define a cohomology complex of n-Hom-Lie algebra morphisms and consider their deformation theory. In particular, we discuss infinitesimal deformations, equivalent deformations and obstructions. Moreover, we study $(n+1)$-Hom-Lie algebra morphisms induced by n-Hom-Lie algebra morphisms and provide examples.

Key Words: n-Hom-Lie algebra, n-Hom-Lie algebra morphism, cohomology, deformation.
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1 Introduction

Filippov [8] introduced n-Lie algebras which are a generalization of Lie algebras. The binary bracket is replaced by a n-ary multilinear operation which is skew-symmetric and satisfies the n-Jacobi identity or Filippov identity, for $n > 2$. The motivation for ternary Lie algebras came first from Nambu mechanics [15], generalizing classical mechanics and allowing more than one hamiltonian. The algebraic formulation of this theory is due to Takhtajan [18]. Ternary operations appeared also is String Theory and were used to construct solutions of the Yang-Baxter equation [17]. Hom-type generalizations of n-Lie algebras called n-Hom-Lie algebras were introduced by Ataguema, Silvestrov and the last author in [2]. These type of algebras were motivated by q-deformations of algebras of vector fields like Witt and Virasoro algebras. Their main feature is that usual identities are twisted by linear maps. Structure, representations and extensions of n-Hom-Lie algebras were studied in [1, 7]. Methods to construct n-Hom-Lie algebras from n-Lie algebra have been discussed in [2]. Furthermore, 3-Lie or 3-Hom-Lie algebras can be obtained from Lie or Hom-Lie algebras, respectively, using a so-called trace maps, see [4, 5]. The construction provides similarly $(n+1)$-(Hom-)Lie algebras from n-(Hom-)Lie algebras. These $(n+1)$-ary algebras are called $(n+1)$-(Hom-)Lie algebras induced by n-(Hom-)Lie algebras. The relationships between their properties have been studied in [6, 13].

Deformation theory is based on formal power series and is closely related to a suitable cohomology. The approach was introduced first by Gerstenhaber for rings and associative algebras using Hochschild cohomology [10] and then extended to Lie algebras, using Chevalley-Eilenberg cohomology, by Nijenhuis and Richardson. They considered deformations of Lie algebras morphisms in [16], that were also studied by Frégier in [9]. Generalizations for n-Lie algebras have been considered in various papers see [14] for a review and [3] for n-Lie algebra morphisms. Cohomology of multiplicative n-Hom-Lie algebras were provided in [1].

This aim of this paper is to deal with n-Hom-Lie algebra morphisms, construct a cohomology complex and study their deformations. For that, we define a cohomology structure of n-Hom-Lie algebras with values in a module compatible with that of n-Hom-Lie algebra morphisms. The major line of this paper consists on deformations of n-Hom-Lie algebras morphisms. We discuss concepts of infinitesimal deformations , equivalence and obstruction. We denote by $\mathcal{N}$ and $\mathcal{N}'$ two n-Hom-Lie algebras. Equivalence classes of infinitesimal deformations of n-Hom-Lie algebras are characterized by the cohomology groups $H^2(\mathcal{N},\mathcal{N})$ and by $H^1(\mathcal{N},\mathcal{N}')$ for that of the morphism $\phi : \mathcal{N} \to \mathcal{N}'$. Furthermore, we study $(n+1)$-Hom-Lie algebra morphisms induced by n-Hom-Lie algebra morphisms and compare their corresponding cohomologies.

The paper is organized as follows: In Section 1, we review the basics about n-Hom-Lie algebras and their representation theory. In Section 2, we define the cohomology of n-Hom-Lie algebras with values in an adjoint module. Thus, we define coboundary operator and the n-cochains module $C^n(\phi,\phi)$ in the cohomology of n-Hom-Lie algebras morphisms. Section 3 deals with deformations of n-Hom-Lie algebras morphisms. We study infinitesimal deformations and equivalent deformations, as well as obstructions. We show that the obstruction to extend a deformation of order N to a deformation of order $N+1$ is a coboundary. In the last Section, we study $(n+1)$-Hom-Lie algebra morphisms induced by n-Hom-Lie algebra morphisms. We restrict ourselves to 3-Hom-Lie algebras induced by Hom-Lie algebras and provide examples.

2 Basics

In this section, we summarize the definitions and basic properties of n-Lie algebras and n-Hom-Lie algebras. We recall as well their representation theory.

Definition 2.1. *A n-ary Hom-Nambu algebra is a triple $(\mathcal{N}, [\cdot,\dots,\cdot], \widetilde{\alpha})$ consisting of a vector space $\mathcal{N}$, a n-linear map $[\cdot,\dots,\cdot] : \mathcal{N}^n \to \mathcal{N}$ and a family $\widetilde{\alpha} = (\alpha_i)_{1\leq i\leq n-1}$ of linear maps $\alpha_i : \mathcal{N} \to \mathcal{N}$, satisfying*

$$\begin{aligned}[\alpha_1(x_1),\dots,\alpha_{n-1}(x_{n-1}),[y_1,\dots,y_n]] = \\ \sum_{i=1}^{n}[\alpha_1(y_1),\dots,\alpha_{i-1}(y_{i-1}),[x_1,\dots,x_{n-1},y_i],\alpha_i(y_{i+1}),\dots,\alpha_{n-1}(y_n)]\end{aligned} \tag{2.1}$$

for all $(x_1,\dots,x_{n-1}) \in \mathcal{N}^{n-1}$, $(y_1,\dots,y_n) \in \mathcal{N}^n$. The identity (2.1) *is called Hom-Nambu identity, it is also called fundamental identity or Filippov-Jacobi identity.*

Let $x = (x_1,\dots,x_{n-1}) \in \mathcal{N}^{n-1}$, $\widetilde{\alpha}(x) = (\alpha_1(x_1),\dots,\alpha_{n-1}(x_{n-1})) \in \mathcal{N}^{n-1}$ and let $(y_1,\dots,y_n) \in \mathcal{N}^n$. The Hom-Nambu identity (2.1) may be written in terms of adjoint map as

$$ad(\widetilde{\alpha}(x))([y_1,\dots,y_n]) = \sum_{i=1}^{n}[\alpha_1(y_1),\dots,\alpha_{i-1}(y_{i-1}),ad(x)(y_i),\alpha_i(y_{i+1}),\dots,\alpha_{n-1}(y_n)].$$

Definition 2.2. *A n-ary Hom Nambu algebra $(\mathcal{N}, [\cdot,\dots,\cdot], \widetilde{\alpha})$ where $\widetilde{\alpha} = (\alpha_i)_{1\leq i\leq n-1}$ is called n-Hom-Lie algebra (n-ary Hom-Nambu-Lie algebra) if the bracket is skew-symmetric that is*

$$[x_{\sigma(1)},\dots,x_{\sigma(n)}] = Sgn(\sigma)[x_1,\dots,x_n] \quad \forall \sigma \in S_n \quad and \quad x_1,\dots,x_n \in \mathcal{N}.$$

Remark 2.3. *When the maps $(\alpha_i)_{1\leq i\leq n-1}$ are all identity maps, one recovers the classical n-Lie algebras. The Hom-Nambu identity (2.1), for $n = 2$ corresponds to Hom-Jacobi identity, which reduces to Jacobi identity when $\alpha_1 = id$.*

Definition 2.4. *Let $(\mathcal{N}, [\cdot, \dots, \cdot], \widetilde{\alpha})$ and $(\mathcal{N}', [\cdot, \dots, \cdot]', \widetilde{\alpha}')$ be two n-Hom-Lie algebras where $\widetilde{\alpha} = (\alpha_i)_{i=1,\dots,n-1}$ and $\widetilde{\alpha}' = (\alpha'_i)_{i=1,\dots,n-1}$. A linear map $f : \mathcal{N} \to \mathcal{N}'$ is a n-Hom-Lie algebra morphism if it satisfies*

$$\begin{aligned} f([x_1, \dots, x_n]) &= [f(x_1), \dots, f(x_n)]' \\ f \circ \alpha_i &= \alpha'_i \circ f \quad \forall i = 1, \dots, n-1. \end{aligned}$$

Definition 2.5. *A multiplicative n-Hom-Lie algebra is a n-Hom-Lie algebra $(\mathcal{N}, [\cdot, \dots, \cdot], \widetilde{\alpha})$, where $\widetilde{\alpha} = (\alpha_i)_{1\leq i\leq n-1})$ with $\alpha_1 = \dots = \alpha_{n-1} = \alpha$, satisfying*

$$\alpha([x_1, \dots, x_n]) = [\alpha(x_1), \dots, \alpha(x_n)], \forall x_1, \dots, x_n \in \mathcal{N}.$$

We denote a multiplicative n-Hom-Lie algebra by $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$, where $\alpha : \mathcal{N} \to \mathcal{N}$ is a linear map.

Remark 2.6. *Let $(\mathcal{N}, [\cdot, \dots, \cdot])$ be a n-Lie algebra and let $\rho : \mathcal{N} \to \mathcal{N}$ be a n-Lie algebra endomorphism. Then $(\mathcal{N}, \rho \circ [\cdot, \dots, \cdot], \rho)$ is a multiplicative n-Hom-Lie algebra.*

The concept of representation of n-Lie algebras is generalized to n-Hom-Lie algebras in a natural was as follows.

Definition 2.7. *Let $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ be a multiplicative n-Hom-Lie algebra. A representation ρ of $\mathcal{N}$ on a vector space V is a linear map $\rho : \mathcal{N}^{n-1} \to End(V)$ such that for $x = (x_1, \dots, x_{n-1})$, $y = (y_1, \dots, y_{n-1}) \in \mathcal{N}^{n-1}$ and $y_n \in \mathcal{N}$, we have*

$$\rho(\alpha(x)) \circ \rho(y) = \rho(\alpha(y)) \circ \rho(x) + \rho[x, y]_\alpha \circ \upsilon$$

$$\rho(\alpha(x_1), \dots, \alpha(x_{n-2}), [y_1, \dots, y_n]) \circ \upsilon = \sum_{i=1}^{n} (-1)^{n-i} \rho(\alpha(y_1), \dots, \widehat{\alpha(y_i)}, \dots, \alpha(y_n)) \circ \rho(x_1, \dots, x_{n-2}, y_i),$$

where $\upsilon \in End(V)$ and $[x, y]_\alpha = \sum_{i=1}^{n-1} (\alpha(y_1), \dots, ad(x)(y_i), \dots, \alpha(y_{n-1}))$. The representation space (V, υ) is said to be a $\mathcal{N}$-module.

Let $(\mathcal{N}, [., \dots, .], \alpha)$ and $(\mathcal{N}', [., \dots, .]', \alpha')$ be two n-Hom-Lie algebras and $\phi : \mathcal{N} \to \mathcal{N}'$ be a n-Hom-Lie algebra morphism. Let $\wedge^{n-1}\mathcal{N}$ be the set of elements $x_1 \wedge \dots \wedge x_{n-1}$ that are skew-symmetric in their arguments. On $\wedge^{n-1}\mathcal{N}$, for $x = x_1 \wedge \dots \wedge x_{n-1} \in \wedge^{n-1}\mathcal{N}$, $y = y_1 \wedge \dots \wedge y_{n-1} \in \wedge^{n-1}\mathcal{N}$, $z \in \mathcal{N}'$, we define

- a linear map $L' : \wedge^{n-1}\mathcal{N} \wedge \mathcal{N}' \to \mathcal{N}', L'(x) \cdot z = [\phi(x_1), \dots, \dots, \phi(x_{n-1}), z]'$. for $z \in \mathcal{N}'$.
- a bilinear map $[\ ,\ ]_\alpha : \wedge^{n-1}\mathcal{N} \times \wedge^{n-1}\mathcal{N} \to \wedge^{n-1}\mathcal{N}$
  by $[x, y]_\alpha = L(x) \bullet_\alpha y = \sum_{i=0}^{n-1} (\alpha(y_1), \dots, L(x).y_i, \dots, \alpha(y_{n-1}))$.

- The map $\bar{\phi} : \wedge^{n-1}\mathcal{N} \to \wedge^{n-1}\mathcal{N}'$ by $\bar{\phi}(x) = \phi(x_1) \wedge \ldots \wedge \phi(x_{n-1})$.

We denote by $\mathcal{L}(\mathcal{N})$ the space $\wedge^{n-1}\mathcal{N}$ and we call it the fundamental set.

Lemma 2.8. *Let $(\mathcal{N}, [., \ldots, .], \alpha)$ and $(\mathcal{N}', [., \ldots, .]', \alpha')$ be two multiplicative n-Hom-Lie algebras and $\phi : \mathcal{N} \to \mathcal{N}'$ be a n-Hom-Lie algebra morphism.*
For $x, y \in \mathcal{L}(\mathcal{N})$ and $z \in \mathcal{N}'$, we have

$$L'([x, y]_\alpha) \cdot \alpha'(z) = L'(\alpha(x)) \cdot L'(y) \cdot z - L'(\alpha(y)) \cdot L'(x) \cdot z.$$

Proof.

$$\begin{aligned}
& L'(\alpha(x_1), \ldots, \alpha(x_{n-1})) \cdot L'(y_1, \ldots, y_{n-1}) \cdot \alpha'(y_n) \\
= \; & L(\alpha(x_1), \ldots, \alpha(x_{n-1})) \cdot ([\phi(y_1), \ldots, \phi(y_{n-1}), \alpha'(y_n)]') \\
= \; & [\phi(\alpha(x_1)), \ldots, \phi(\alpha(x_{n-1})), [\phi(y_1), \ldots, \phi(y_{n-1}), \alpha'(y_n)]']' \\
= \; & \sum_{i=1}^{n-1} [\phi(\alpha(y_1)), \ldots, \phi(\alpha(y_{i-1})), [\phi(x_1), \ldots, \phi(x_{n-1}), \phi(y_i)]', \phi(\alpha(y_{i+1})), \ldots, \phi(\alpha(y_{n-1})), \alpha'(y_n)]' \\
+ \; & [\phi(\alpha(y_1)), \ldots, \phi(\alpha(y_{n-1})), [\phi(x_1), \ldots, \phi(x_{n-1}), \alpha'(y_n)]']' \\
= \; & \sum_{i=1}^{n-1} [\phi(\alpha(y_1)), \ldots, \phi(\alpha(y_{i-1})), \phi \circ ad(x_1, \ldots, x_{n-1})(y_i), \ldots, \phi(\alpha(y_{n-1})), \alpha'(y_n)]' \\
+ \; & [\phi(\alpha(y_1)), \ldots, \phi(\alpha(y_{n-1})), [\phi(x_1), \ldots, \phi(x_{n-1}), \alpha'(y_n)]']' \\
= \; & \sum_{i=1}^{n-1} L'(\alpha(y_1), \ldots, ad(x)(y_i), \ldots, \alpha(y_{n-1})).\alpha'(y_n) \\
+ \; & [\phi(\alpha(y_1)), \ldots, \phi(\alpha(y_{n-1})), [\phi(x_1), \ldots, \phi(x_{n-1}), \alpha'(y_n)]']'
\end{aligned}$$

On the other hand,

$$L'([x, y]_\alpha) \cdot \alpha'(y_n) = L'(\sum_{i=1}^{n-1} (\alpha(y_1), \ldots, ad(x)(y_i), \ldots, \alpha(y_{n-1}))) \cdot \alpha'(y_n).$$

Thus, the result holds. ∎

Example 2.9. *Let $(\mathcal{N}, [., \ldots, .], \alpha)$ be a multiplicative n-Hom-Lie algebra. The map ad is a representation, where the operator υ is the twist map α.*

Corollary 2.10. *Let $(\mathcal{N}, [., \ldots, .], \alpha)$ and $(\mathcal{N}', [., \ldots, .]', \alpha')$ be two n-Hom-Lie algebras and $\phi : \mathcal{N} \to \mathcal{N}'$ be a n-Hom-Lie algebra morphism.*
The map L' defined above is an adjoint representation of the n-Hom-Lie algebra $(\mathcal{N}, [., \ldots, .], \alpha)$ via ϕ, where the operator υ is the twist map α'. Thus $M = (\mathcal{N}', L', \alpha')$ is a $\mathcal{N}$-module.

Moreover, we have the following fundamental result, providing a representation of a n-Hom-Lie algebra by a Hom-Leibniz algebra. Recall that a Hom-Leibniz algebra is a triple $(V, [-, -], \alpha)$, consisting of a vector space, a binary bracket and a linear map satisfying the following identity :

$$[[X, Y], \alpha(Z)] = [[X, Z], \alpha(Y)] + [\alpha(X), [Y, Z]].$$

Remark 2.11. *The triple* $(\mathcal{L}(N), [\ ,\ ]_\alpha, \alpha)$ *is a Hom-Leibniz algebra.*

Notice that $\wedge^{n-1}\mathcal{N}$ merely reflects that the fundamental object $X = (x_1, \ldots, x_n) \in \wedge^{n-1}\mathcal{N}$ is antisymmetric in its arguments; it does not imply that X is a $(n-1)$-multivector obtained by the associative wedge product of vectors.

3 Cohomology of multiplicative n-Hom-Lie algebras with values in an adjoint module

The algebra valued cohomology theory was studied for multiplicative n-Hom-Lie algebras in [1]. The purpose of this section is to construct a cochain complex $C^*_{\alpha,\alpha'}(\mathcal{N},\mathcal{N}')$ that defines a Chevalley-Eilenberg cohomology for multiplicative n-Hom-Lie algebras with values in an adjoint module.

Definition 3.1. *Let* $(\mathcal{N}, [\cdot,\ldots,\cdot], \alpha)$ *and* $(\mathcal{N}', [\cdot,\ldots,\cdot]', \alpha')$ *be two multiplicative n-Hom-Lie algebras and* $\phi : \mathcal{N} \to \mathcal{N}'$ *be a n-Hom-Lie algebra morphism. Regard* $\mathcal{N}'$ *as a representation of* $\mathcal{N}$ *via* ϕ *wherever appropriate. An* $(m+1)$*-cochain is a* $(m+1)$*-linear map* $f : \otimes^m \mathcal{L}(\mathcal{N}) \wedge \mathcal{N} \to \mathcal{N}'$ *such that*

$$\alpha' \circ f(x_1, x_2, \ldots, x_m, z) = f(\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_m), \alpha(z))$$

for all $x_1, x_2, \ldots, x_m \in \mathcal{L}(\mathcal{N})$ *and* $z \in \mathcal{N}$*. We denote the set of* $(m+1)$*-cochain by* $C^m_{\alpha,\alpha'}(\mathcal{N},\mathcal{N}')$*. For* $m \geq 1$*, the coboundary operator is the linear map* $\delta^{m+1} : C^m_{\alpha,\alpha'}(\mathcal{N},\mathcal{N}') \to C^{m+1}_{\alpha,\alpha'}(\mathcal{N},\mathcal{N}')$ *defined by*

$$\begin{aligned} &\delta^{m+1} f(x_1, \ldots, x_m, x_{m+1}, z) \\ &= \sum_{1\leq i\leq j} (-1)^i f(\alpha(x_1), \ldots, \widehat{\alpha(x_i)}, \ldots, \alpha(x_{j-1}), [x_i, x_j], \ldots, \alpha(x_{m+1}), \alpha(z)) \\ &\quad + \sum_{i=1}^{m+1} (-1)^i f(\alpha(x_1), \ldots, \widehat{\alpha(x_i)}, \ldots, \alpha(x_{m+1}), ad(x_i)(z)) \\ &\quad + \sum_{i=1}^{m+1} (-1)^{i+1} L'(\alpha^m(x_i)).f(x_1, \ldots, \widehat{x}_i, \ldots, x_{m+1}, z) \\ &\quad + \sum_{i=1}^{n-1} (-1)^m [\phi(\alpha^m(x^1_{m+1})), \ldots, f(x_1, \ldots, x_m, x^i_{m+1}), \ldots, \phi(\alpha^m(x^{n-1}_{m+1})), \phi(\alpha^m(z))]'. \end{aligned} \tag{3.1}$$

Theorem 1. *The pair* $(\mathcal{C}^*(\mathcal{N},\mathcal{N}'), \delta)$ *defines a cochain complex. The corresponding cohomology, denoted by* $H^*(\mathcal{N},\mathcal{N}')$*, is called the cohomology of the n-Hom-Lie algebra* $\mathcal{N}$ *with coefficients in the representation* $\mathcal{N}'$*.*

Proof. The operator is well defined since $\delta^{m+1}(f) \circ (\overline{\alpha}^{\otimes(m+1)} \wedge \alpha) = \alpha' \circ \delta^{m+1}(f)$. A Straightforward computation based on the property of multiplicative algebra and the compatibility condition of the morphism ϕ with the morphisms α and α' that is $\phi \circ \alpha = \alpha' \circ \phi$ and requires some simplification using mainly Leibniz structure on $\mathcal{L}(\mathcal{N})$, leads to $\delta^{m+2} \circ \delta^{m+1} = 0$. □

Remark 3.2. *In the particular case where $\mathcal{N}' = \mathcal{N}$ and $L' = ad$, the n-Hom-Lie algebra is a $\mathcal{N}$-module over itself. We recover the coboundary operator defined in [1]. One considers the previous definition with $L' = ad$ and the last sum without ϕ and denote $C^n_{\alpha,\alpha'}(\mathcal{N},\mathcal{N}')$ by $C^n_\alpha(\mathcal{N},\mathcal{N})$.*

3.1 Cohomology of multiplicative n-Hom-Lie algebra morphisms

The original cohomology theory associated to deformation of Lie algebra morphisms was developed by Frégier in [9]. The aim of this part is to define explicitly a cochain complex with a coboundary operator and the n-cochains module $\mathcal{C}^m(\phi,\phi)$ providing a cohomology of n-Hom-Lie algebra morphisms.

Let $\phi : \mathcal{N} \to \mathcal{N}'$ be a multiplicative n-Hom-Lie algebra morphism. Regard $\mathcal{N}'$ as a representation of $\mathcal{N}$ via ϕ wherever appropriate. We define the module of $(m+1)$-cochains of the morphism ϕ to be

$$\mathcal{C}^m(\phi,\phi) = \mathcal{C}^m_\alpha(\mathcal{N},\mathcal{N}) \otimes \mathcal{C}^m_{\alpha'}(\mathcal{N}',\mathcal{N}') \otimes \mathcal{C}^{m-1}_{\alpha,\alpha'}(\mathcal{N},\mathcal{N}'),$$

where $\mathcal{C}^m_\alpha(\mathcal{N},\mathcal{N})$ is defined in Remark 3.2 and $\mathcal{C}^{m-1}_{\alpha,\alpha'}(\mathcal{N},\mathcal{N}')$ is given in Definition 3.1. The coboundary operator $\delta^{m+1} : \mathcal{C}^m(\phi,\phi) \to \mathcal{C}^{m+1}(\phi,\phi)$ is defined by

$$\delta^{m+1}(\varphi_1,\varphi_2,\varphi_3) = (\delta^{m+1}\varphi_1, \delta^{m+1}\varphi_2, \delta^m\varphi_3 + (-1)^m(\phi\circ\varphi_1 - \varphi_2\circ(\bar{\phi}^{\otimes m}\wedge\phi))),$$

where $\delta^{m+1}\varphi_1$ and $\delta^{m+1}\varphi_2$ are defined in [1] and $\delta^m\varphi_3$ by (3.1).

Proposition 3.3. *We have $\delta^{m+2}\circ\delta^{m+1} = 0$. Hence $(C^*(\phi,\phi),\delta)$ is a cochain complex. The corresponding cohomology is denoted by $H^*(\phi,\phi)$.*

4 Deformations of n-Hom-Lie algebra morphisms

In this section, we aim to study one parameter formal deformations of n-Hom-Lie algebra morphisms. Deformations of n-Hom-Lie algebras have been discussed in terms of Chevalley-Eilenberg cohomology, see [1]. Recall that the main idea is to change the scalar field $\mathbb{K}$ to a formal power series ring $\mathbb{K}[\![t]\!]$, in one variable t. The main results provide cohomological interpretations.

Let $\mathcal{N}[\![t]\!]$ be the set of formal power series whose coefficients are elements of the vector space $\mathcal{N}$, ($\mathcal{N}[\![t]\!]$ is obtained by extending the coefficients domain of $\mathcal{N}$ from $\mathbb{K}$ to $\mathbb{K}[\![t]\!]$). Given a $\mathbb{K}$-n-linear map $\varphi : \mathcal{N}\times\ldots\times\mathcal{N} \to \mathcal{N}$, it admits naturally an extension to a $\mathbb{K}[\![t]\!]$-n-linear map $\varphi : \mathcal{N}[\![t]\!]\times\ldots\times\mathcal{N}[\![t]\!] \to \mathcal{N}[\![t]\!]$, that is, if $x_i = \sum_{j\geq 0} a_i^j t^j$, $1\leq i\leq n$ then $\varphi(x_1,\ldots,x_n) = \sum\limits_{j_1,\ldots,j_n\geq 0} t^{j_1+\cdots j_n}\varphi(a_1^{j_1},\ldots,a_n^{j_n})$.

Definition 4.1. *A deformation of a multiplicative n-Hom-Lie algebra $(\mathcal{N},[.,\ldots,.],\alpha)$ is given by a $\mathbb{K}[\![t]\!]$-n-linear map $[\cdot,\ldots,\cdot]_t : \mathcal{N}[\![t]\!]\times\cdots\times\mathcal{N}[\![t]\!] \to \mathcal{N}[\![t]\!]$ of the form $[\cdot,\ldots,\cdot]_t = \sum\limits_{i\geq 0} t^i[\cdot,\ldots,\cdot]_i$, where each $[\cdot,\ldots,\cdot]_i$ is a $\mathbb{K}$-n-linear $[\cdot,\ldots,\cdot]_i : \mathcal{N}\times\ldots\times\mathcal{N} \to \mathcal{N}$ and*

$[\cdot,\ldots,\cdot]_0 = [\cdot,\ldots,\cdot]$ *such that*

$$\begin{aligned}&[\alpha(x_1,\ldots,\alpha(x_{n-1}),[y_1,\ldots,y_n]_t]_t\\ &=\sum_{i=1}^{n-1}[\alpha(y_1),\ldots,\alpha(y_{i-1}),[x_1,\ldots,x_{n-1},y_i]_t,\alpha(y_{i+1}),\ldots,\alpha(y_n)]_t\end{aligned} \tag{4.1}$$

Let $\phi:\mathcal{N}\to\mathcal{N}'$ be a n-Hom-Lie algebra morphism. Define a deformation of ϕ to be a triple $\Theta_t = ([.,\ldots,.]_{\mathcal{N},t},[.,\ldots,.]_{\mathcal{N}',t},\phi_t)$ in which

- $[\cdot,\ldots,\cdot]_{\mathcal{N},t} = \sum_{i=0}^{\infty}[\cdot,\ldots,\cdot]_{\mathcal{N},i}t^i$ *is a deformation of $\mathcal{N}$,*
- $[\cdot,\ldots,\cdot]_{\mathcal{N}',t} = \sum_{i=0}^{\infty}[\cdot,\ldots,\cdot]_{\mathcal{N}',i}t^i$ *is a deformation of $\mathcal{N}'$,*
- $\phi_t:\mathcal{N}[\![t]\!]\to\mathcal{N}'[\![t]\!]$ *is a deformation of the n-Hom-Lie algebra morphism of the form $\phi_t = \sum_{i=0}^{\infty}\phi_i t^i$ where each $\phi_i:\mathcal{N}\to\mathcal{N}'$ is a $\mathbb{K}$-linear map and $\phi_0=\phi$, such that ϕ_t satisfies the following equations*

$$\phi_t([x_1,\ldots,x_n]_{\mathcal{N},t}) = [\phi_t(x_1),\ldots,\phi_t(x_n)]_{\mathcal{N}',t} \quad \text{and} \quad \phi_t\circ\alpha = \alpha'\circ\phi_t. \tag{4.2}$$

The deformation is said of order N if the sums run from 0 to N.

Remark 4.2. *Equation (4.1) can be expressed as*

$$L_t([x,y]_\alpha)\cdot\alpha(y_n) = L_t(\alpha(x))\cdot(L_t(y)\cdot y_n) - L_t(\alpha(y))\cdot(L_t(x)\cdot y_n),$$

where $x=(x_1,\ldots,x_{n-1}), y=(y_1,\ldots,y_{n-1})$ and $L_t(x)\cdot y_n = [x_1,\ldots,x_{n-1},y_n]_t$.

Proposition 4.3. *The linear coefficient, $\theta_1 = ([.,.]_{\mathcal{N},1},[.,.]_{\mathcal{N}',1},\phi_1)$, which is called the infinitesimal of the deformation Θ_t of ϕ, is a 2-cocycle in $C^2(\phi,\phi)$.*

Definition 4.4. (1) *Let $(\mathcal{N},[\cdot,\ldots,\cdot],\alpha)$ be a n-Hom-Lie algebra. Let $\mathcal{N}_t = (\mathcal{N}[\![t]\!],[.,\ldots,.]_t,\alpha)$ and $\mathcal{N}'_t = (\mathcal{N}[\![t]\!],[\cdot,\ldots,\cdot]'_t,\alpha)$ be two deformations of $\mathcal{N}$. We say that $\mathcal{N}_t$ and $\mathcal{N}'_t$ are equivalent if there exists a formal automorphism $\psi_t:\mathcal{N}[\![t]\!]\to\mathcal{N}[\![t]\!]$ that may be written in the form $\psi_t = \sum_{i\geq 0}\psi_i t^i$, where $\psi_i\in End(\mathcal{N})$ and $\psi_0 = Id$ and such that*

$$\psi_t([x_1,\ldots,x_n]_t) = [\psi_t(x_1),\ldots,\psi_t(x_n)]'_t \quad \text{and} \quad \psi_t\circ\alpha = \alpha\circ\psi_t.$$

(2) *Let $\Theta_t = ([\cdot,\ldots,\cdot]_{\mathcal{N},t},[\cdot,\ldots,\cdot]_{\mathcal{L},t},\phi_t)$ and $\widetilde{\Theta}_t = ([\cdot,\ldots,\cdot]'_{\mathcal{N},t},[\cdot,\ldots,\cdot]'_{\mathcal{L},t},\widetilde{\phi}_t)$ be two deformations of a n-Hom-Lie algebra morphism $\phi:\mathcal{N}\to\mathcal{L}$. A formal automorphism $\phi_t:\Theta_t\to\widetilde{\Theta}_t$ is a pair $(\psi_{\mathcal{N},t},\psi_{\mathcal{L},t})$, where $\psi_{\mathcal{N},t}:\mathcal{N}[\![t]\!]\to\mathcal{N}[\![t]\!]$ and $\psi_{\mathcal{L},t}:\mathcal{L}[\![t]\!]\to\mathcal{L}[\![t]\!]$ are formal automorphisms, such that $\widetilde{\phi}_t = \psi_{\mathcal{L},t}\ \phi_t\ \psi^{-1}_{\mathcal{N},t}$. Two deformations Θ_t and $\widetilde{\Theta}_t$ are equivalent if and only if there exists a formal automorphism $\Theta_t\to\widetilde{\Theta}_t$.*

Theorem 2. *The infinitesimal of a deformation Θ_t of ϕ is a 2-cocycle in $C^2(\phi,\phi)$ whose cohomology class is determined by the equivalence class of the first term of Θ_t.*

Theorem 3. *Let $(\mathcal{N}, [.,\dots,.]_{\mathcal{N}})$ and $(\mathcal{N}', [.,\dots,.]_{\mathcal{N}'})$ be two n-Hom-Lie algebras. Let $\Theta_t = ([.,\dots,.]_{\mathcal{N},t}, [.,\dots,.]_{\mathcal{N}',t}, \phi_t)$ be a deformation of a n-Hom-Lie algebra morphism $\phi : \mathcal{N} \to \mathcal{N}'$. Then, there exists an equivalent deformation $\widetilde{\Theta}_t = ([.,\dots,.]'_{\mathcal{N},t}, [.,\dots,.]'_{\mathcal{N}',t}, \widetilde{\phi}_t)$ such that $\widetilde{\theta}_1 \in Z^2(\phi,\phi)$ and $\widetilde{\theta}_1 \notin B^2(\phi,\phi)$. Hence, if $H^2(\phi,\phi) = 0$ then every formal deformation is equivalent to a trivial deformation.*

Let $(\mathcal{N}, [\cdot,\dots,\cdot], \alpha)$ and $(\mathcal{N}', [\cdot,\dots,\cdot]', \alpha')$ be two n-Hom-Lie algebras and let ϕ be a n-Hom-Lie algebra morphism. A deformation of order N of ϕ is a triple

$$\Theta_t = ([\cdot,\dots,\cdot]_t; [\cdot,\dots,\cdot]'_t; \phi_t),$$

where $[\cdot,\dots,\cdot]_t = \sum_{i=0}^{N} [\cdot,\dots,\cdot]_i t^i$, $[\cdot,\dots,\cdot]'_t = \sum_{i=0}^{N} [\cdot,\dots,\cdot]'_i t^i$ and $\psi_t = \sum_{i=0}^{N} \psi_i t^i$, satisfying $\phi_t([x_1,\dots,x_n]_t) = [\phi_t(x_1),\dots,\phi_t(x_n)]'_t$. Given a deformation Θ_t of order N, it extends to a deformation of order $N+1$ if and only if there exists a 2-cochain θ_{N+1} such that $\overline{\Theta}_t = \Theta_t + t^{N+1}\theta_{N+1}$ is a deformation of order $N+1$. The deformation $\overline{\Theta}_t$ is called an order $N+1$ extension of Θ_t.
Set $\mathcal{O}b_{\mathcal{N}}$ (resp. $\mathcal{O}b_{\mathcal{N}'}$) be the obstruction of a deformation of a n-Hom-Lie algebra $\mathcal{N}$ (resp. $\mathcal{N}'$):

$$\begin{aligned}
\mathcal{O}b_{\mathcal{N}} &= -\sum_{\substack{k+l=N+1\\ k,l>0}} [\alpha(x_1^1),\dots,\alpha(x_1^{n-1}), [x_2^1,\dots,x_2^{n-1},z]_k]_l \\
&+ \sum_{\substack{k+l=N+1\\ k,l>0}} \sum_{i=1}^{n-1} [\alpha(x_2^1),\dots,\alpha(x_2^{i-1}), [x_1^1,\dots,x_1^{n-1},x_2^i]_k, \alpha(x_2^{i+1}),\dots,\alpha(x_2^{n-1}),\alpha(z)]_l \\
&+ \sum_{\substack{k+l=N+1\\ k,l>0}} [\alpha(x_2^1),\dots,\alpha(x_2^{n-1}), [x_1^1,\dots,x_1^{n-1},z]_k]_l.
\end{aligned}$$

Let $\mathcal{O}b_\phi$ be the obstruction of the extension of the n-Hom-Lie algebra morphism ϕ:

$$\mathcal{O}b_\phi = \sum_{\substack{i+j=N+1\\ i,j>0}} \phi_i \circ [x_1,\dots,x_n]_j - {\sum}' [\phi_{l_1}(x_1),\cdots,\phi_{l_i}(x_i),\cdots,\phi_{i_n}(x_n)]'_j.$$

with

$${\sum}' = \sum_{j=1}^{N} \sum_{\substack{l_i>0\\ 1\le i\le n}} + \sum_{j=1}^{N} \sum_{\substack{l_1+\dots+\widehat{l_i}+\dots+l_n>0\\ l_i>0\\ 1\le i\le n}} + \sum_{i=1}^{n} \sum_{\substack{l_i+\dots+\widehat{l_i}+\dots+l_n=N+1-l_i\\ l_i>0, j=0\\ 1\le i\le n}} .$$

Theorem 4.5. *Let $(\mathcal{N}, [.,\dots,.])$ and $(\mathcal{N}', [.,\dots,.]')$ be two n-Hom-Lie algebras and ϕ be a n-Hom-Lie algebra morphism. Let $\Theta_t = ([.,\dots,.]_t, [.,\dots,.]'_t, \phi_t)$ be an order N one-parameter formal deformation of ϕ. Then $\mathcal{O}b = (\mathcal{O}b_{\mathcal{N}}, \mathcal{O}b_{\mathcal{N}'}, \mathcal{O}b_\phi) \in Z^3(\phi,\phi)$. Therefore the deformation extends to a deformation of order $N+1$ if and only if $\mathcal{O}b$ is a coboundary.*

5 Morphisms of ternary Hom-Lie algebras induced by morphisms of Hom-Lie algebras

In [4] and [5], the authors introduced a construction of a 3-Hom-Lie algebra (ternary Hom-Lie algebras) from a Hom-Lie algebra along a linear form, and more generally a $(n+1)$-Hom-Lie algebra from a n-Hom-Lie algebra, called $(n+1)$-Hom-Lie algebra induced by n-Hom-Lie algebra. In this section we will investigate morphisms of 3-Hom-Lie algebras induced by morphisms of Hom-Lie algebras.

Definition 5.1. *Let $\varphi_\tau : \mathcal{N}^n \to \mathcal{N}$ be a n-linear map and $\tau : \mathcal{N} \to \mathbb{K}$ be a linear form. Define $\varphi_\tau : \mathcal{N}^{n+1} \to \mathcal{N}$ by*

$$\varphi_\tau(x_1, \ldots, x_n) = \sum_{k=1}^{n+1} (-1)^{k-1} \tau(x_k) \varphi(x_1, \ldots, \hat{x}_k, \ldots, x_{n+1})$$

where the hat over $\hat{x}_k$ on the right hand side means that x_k is excluded, that is φ is calculated on $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1})$.

Definition 5.2. *For $\varphi : \mathcal{N}^n \to \mathcal{N}$, we call a linear map $\tau : \mathcal{N} \to \mathbb{K}$ a φ-trace or trace map if*
$\tau(\varphi(x_1, \ldots, x_n)) = 0$ for all $x_1, \ldots, x_n \in \mathcal{N}$.

Theorem 5.3. *[4, 5, 13] Let $(\mathcal{N}, \varphi, \alpha_1, \ldots, \alpha_n)$ be a n-Hom-Lie algebra and τ a φ-trace. If $\tau \circ \alpha_i = \tau$ for $i = 1, \ldots, n$ then $(\mathcal{N}, \varphi_\tau, \alpha_1, \ldots, \alpha_{n+1})$ is a $(n+1)$-Hom-Lie algebra. Moreover, if (A, φ, α) is a multiplicative n-Hom-Lie algebra, then, under the same condition, $(A, \varphi_\tau, \alpha)$ is a multiplicative $(n+1)$-Hom-Lie algebra.*

Let $(\mathcal{N}_1, [.,.]_1, \alpha_1)$ and $(\mathcal{N}_2, [.,.]_2, \alpha_2)$ be two Hom-Lie algebras. let τ_1 be a $[.,.]_1$-trace and τ_2 be a $[.,.]_2$-trace. Let $(\mathcal{N}_{\tau,1}, [.,.,.]_{\tau_1}, \alpha_1)$ and $(\mathcal{N}_{\tau,2}, [.,.,.]_{\tau_2}, \alpha_2)$ be two 3-Hom-Lie algebras induced respectively by $(\mathcal{N}_1, [.,.]_1, \alpha_2)$ and $(\mathcal{N}_2, [.,.]_2, \alpha_2)$. Let $\phi : \mathcal{N}_1 \to \mathcal{N}_2$ be a Hom-Lie algebra morphism between $(\mathcal{N}_1, [.,.,.]_1, \alpha_1)$ and $(\mathcal{N}_2, [.,.,.]_1, \alpha_2)$, i.e. $\phi([x,y]_1) = [\phi(x), \phi(y)]_2$. We want to extend this morphism to induced ternary Hom-Lie algebras. We should have

$$\phi([x,y,z]_{\tau_1}) = [\phi(x), \phi(y), \phi(z)]_{\tau_2}$$

according to the definition of the ternary bracket

$$\phi([x,y,z]_{\tau_1}) = \circlearrowleft_{x,y,z} \tau_1(x)\phi([y,z]_1) = \circlearrowleft_{x,y,z} \tau_1(x)[\phi(y), \phi(z)]_2$$

In the other hand,

$$[\phi(x), \phi(y), \phi(z)]_{\tau_2} = \circlearrowleft_{\phi(x),\phi(y),\phi(z)} \tau_2(\phi(x))[\phi(y), \phi(z)]_2$$

A theorem for constructing 3-Hom-Lie algebra morphism induced by Hom-Lie algebra can be formulated as follows:

Theorem 5.4. *The map ϕ is a morphism of 3-Hom-Lie algebras induced by binary Hom-Lie algebras morphism if $\tau_2(\phi) = \tau_1$.*

Remark 5.5. *A necessary and sufficient condition for the construction of 3-Hom-Lie algebra morphism induced by Hom-Lie algebra morphism can be written as*

$$(\tau_1(x)-\tau_2(\phi(x))[\phi(y),\phi(z)]+(\tau_1(y)-\tau_2(\phi(y))[\phi(z),\phi(x)]+(\tau_1(z)-\tau_2(\phi(z))[\phi(x),(y)]=0,$$

for all $x,y,z\in\mathcal{N}_1$.

The previous results can easily and similarly stated for general situation of $(n+1)$-Hom-Lie algebras induced by n-Hom-Lie algebras.

5.1 Cohomology

In this section, we study the connections between the cohomology of a given n-Hom-Lie algebra morphism and the cohomology of the induced $(n+1)$-Hom-Lie algebra morphism.

Proposition 5.6. *[13] Let* $(\mathcal{N},[\cdot,\ldots,\cdot],\alpha)$ *be a multiplicative* n*-Hom-Lie algebra,* τ *be a trace map and* $(\mathcal{N},[\cdot,\ldots,\cdot]_{\tau_1},\alpha_1)$ *be the induced multiplicative* $(n+1)$*-Hom-Lie algebra. Let* $\varphi\in Z^2(\mathcal{N},\mathcal{N})$ *such that:*

1. $\sum\limits_{i=1}^{n}\sum\limits_{k=1,k\neq i}^{n}(-1)^{k+n-1}\tau(y_i)\tau(y_k)\varphi(y_1,\cdots,\hat{y}_k,\ldots,y_{i-1},X_n\cdot x_n,y_{i+1},\ldots,y_n,z),$
2. $\sum\limits_{i=1}^{n}\sum\limits_{k=1,k\neq i}^{n}(-1)^{k+n-1}\tau(y_i)\tau(y_k)[y_1,\cdots,\hat{y}_k,\ldots,y_{i-1},\varphi(X_n,x_n),y_{i+1},\ldots,y_n,z],$
3. $\tau\circ\varphi=0.$

Then $\varphi_\tau(X,z)=\sum\limits_{i=1}^{n}(-1)^{i-1}\tau(x_i)\varphi(X_i,z)+(-1)^n\tau(z)\varphi(X_n,x_n)$ *is a 2-cocyle of the induced* $(n+1)$*-Hom Lie algebra for* $X=x_1\wedge\ldots\wedge x_n\in\wedge^n\mathcal{N},X_i=x_1\wedge\ldots\wedge x_{i-1}\wedge x_{i+1}\wedge\ldots\wedge x_n\in\wedge^{n-1}\mathcal{N}$.

Theorem 5.7. *Let* $(\mathcal{N}_1,[\cdot,\ldots,\cdot],\alpha_1)$ *(resp.* $\mathcal{N}_2,[\cdot,\ldots,\cdot],\alpha_2)$*) be a multiplicative* n*-Hom Lie algebra,* τ_1 *(resp.* τ_2 *) be a trace map and* $(\mathcal{N}_{\tau_1},[\cdot,\ldots,\cdot]_{\tau_1},\alpha_1)$ *(resp.* $(\mathcal{N}_{\tau_2},[\cdot,\ldots,\cdot]_{\tau_2},\alpha_2)$*) be the induced multiplicative* $(n+1)$*-Hom-Lie algebra. Let* ϕ *be a morphism of* $(n+1)$*-Hom-Lie algebra induced by a morphism of n-Hom-Lie algebra.*
Let $\varphi_{\tau_1}(X,z)$ *be a 2-cocyle of the induced* $(n+1)$*-Hom Lie algebra* $(\mathcal{N}_{\tau,1},[.,\ldots,.]_{\tau_1},\alpha_1)$ *(resp.* $\varphi_{\tau_2}(X,z)$ *a 2-cocyle of the induced* $(n+1)$*-Hom Lie algebra* $(\mathcal{N}_{\tau_2},[.,\ldots,.]_{\tau_2},\alpha_2)$ *defined in the pervious proposition. Let* $\rho\in Z^1(\mathcal{N}_1,\mathcal{N}_2)$*. Then* $\rho_\tau(x_j)=\sum\limits_{i=1 i\neq j}^{n}(-1)^{i-1}\tau_1(x_i)\rho(x_i)+(-1)^n\tau_1(z)\rho(x_n)$ *is a 1-cocyle of the induced* $(n+1)$*-Hom-Lie algebra morphism. Hence* $(\varphi_{\tau_1},\varphi_{\tau_2},\rho_\tau)$ *is a 2-cocycle in* $Z^2(\phi,\phi)$*.*

Proof. Let $\varphi_{\tau_1}(X,z)\in Z^2(\mathcal{N}_{\tau_1},\mathcal{N}_{\tau_1})$ and $\varphi_{\tau_2}(X,z)\in Z^2(\mathcal{N}_{\tau_2},\mathcal{N}_{\tau_2})$ satisfying the condition above, then

$$\begin{aligned}\delta^2\rho_\tau(X,z)&=\phi\circ\varphi_{1,\tau}(X,z)-\varphi_{2,\tau}(\phi,\phi)(X,z)-\delta^1\rho_\tau(X,z)\\&=\sum_{i=1}^{n}(-1)^{i-1}\tau_1(x_i)\phi\circ\varphi_1(X_i,z)+(-1)^n\tau_1(z)\phi\circ\varphi_1(X_n,x_n)\end{aligned}$$

$$- \sum_{i=1}^{n}(-1)^{i-1}\tau_2(\phi(x_i))\varphi_2(\phi(X_i),\phi(z)) - (-1)^n\tau_2(z)\varphi_2(\phi(X_n),\phi(x_n))$$
$$- \sum_{j=1}^{n}\sum_{i=1}^{n}(-1)^{i-1}\tau_1(x_i)[\phi(x_1),\ldots,\rho(x_i),\ldots,\phi(x_n),\phi(z)]$$
$$- (-1)^n\tau(z)[\phi(x_1),\ldots,\rho(x_i),\ldots,\phi(x_{n-1}),\phi(x_n)]$$
$$+ \sum_{i=1}^{n}(-1)^{i-1}\tau_1(x_i)\rho([x_1,\ldots,x_{i-1},x_{i+1},x_n,z]) + (-1)^n\tau_1(z)\rho([x_1,\ldots,x_{n-1},x_n])$$
$$= \sum_{i=1}^{n}(-1)^{i-1}\tau_1(x_i)\delta^2\rho(X_i,z) + (-1)^n\tau_1(z)\delta^2\rho(X_n,x_n) = 0+0=0$$

□

5.2 Deformations

Let $(\mathcal{N}_1,[\cdot,\ldots,\cdot],\alpha_1)$ (resp. $\mathcal{N}_2,[\cdot,\ldots,\cdot],\alpha_2)$) be a multiplicative n-Hom Lie algebra, τ_1 (resp. τ_2) be a trace and $(\mathcal{N}_{\tau_1},[\cdot,\ldots,\cdot]_{\tau_1},\alpha_1)$ (resp. $(\mathcal{N}_{\tau_2},[\cdot,\ldots,\cdot]_{\tau_2},\alpha_2)$) be the induced multiplicative $(n+1)$-Hom-Lie algebra. Let ϕ be the morphism of $(n+1)$-Hom-Lie algebra induced by a morphism of a n-Hom-Lie algebra.

Now, let $[\cdot,\cdot]_{1,t} = \sum\limits_{i=0}^{\infty}[\cdot,\cdot]_{1,i}t^i$ be a one-parameter formal deformation of $\mathcal{N}_1$ and $[\cdot,\cdot]_{2,t} = \sum\limits_{i=0}^{\infty}[\cdot,\cdot]_{2,i}t^i$ be a one-parameter formal deformation of $\mathcal{N}_2$. Let $\phi_t : \mathcal{N}_1[\![t]\!] \to \mathcal{N}_2[\![t]\!]$ be a deformation of the Hom-Lie algebra morphism $\phi : \mathcal{N}_1 \to \mathcal{N}_2$ of the form $\phi_t = \sum\limits_{n=0}^{\infty}\phi_n t^n$ such that ϕ_t satisfies the following equation

$$\phi_t([x_1,x_2]_{1,t}) = [\phi_t(x_1),\phi_t(x_2)]_{2,t} \quad \text{and} \quad \phi_t\circ\alpha_1 = \alpha_2\circ\phi_t.$$

Assume that τ_1 satisfies $\tau_1([x,y]_{1,t}) = 0$, then $[.,.,.]_{\tau_1,t}$ is a one parameter formal deformation of the induced 3-Hom Lie algebra $(\mathcal{N}_1,[.,.,.]_{\tau_1,t},\alpha_1)$ if

$$[x,y,z]_{\tau_1,t} = \circlearrowleft_{x,y,z} \tau_1(x)[y,z]_{1,t} = \sum_{i=0}^{k} t^i \circlearrowleft_{x,y,z} \tau_1(x)[y,z]_{1,i}.$$

Also, assume that τ_2 satisfies $\tau_1([x,y]_{2,t}) = 0$. Then $[.,.,.]_{\tau_2}$ is a one parameter formal deformation of the induced 3-Hom-Lie algebra $(\mathcal{N}_2,[.,.,.]_{\tau_1},\alpha_2)$ if

$$[x,y,z]_{\tau_2,t} = \circlearrowleft_{x,y,z} \tau_2(x)[y,z]_{2,t} = \sum_{i=0}^{k} t^i \circlearrowleft_{x,y,z} \tau_1(x)[y,z]_{2,i}.$$

Futhermore, ϕ_t is a deformation of the induced morphism ϕ, if

$$\phi_t([x,y,z]_{\tau_1,t}) = \circlearrowleft_{x,y,z} \tau_1(x)\phi_t([y,z]_{1,t}) = \circlearrowleft_{x,y,z} \tau_1(x)[\phi(y),\phi(z)]_{2,t}.$$

In the other hand

$$[\phi_t(x),\phi_t(y),\phi_t(z)]_{\tau_2,t} = \circlearrowleft_{\phi_t(x),\phi_t(y),\phi_t(z)} \tau_2(\phi_t(x))[\phi_t(y),\phi_t(z)]_{\tau_2,t}.$$

Then ϕ_t is a deformation of the induced $(n+1)$-Hom-Lie algebra morphism if $\tau_1(x) = \tau_2(\phi_t(x))$.

Example 1. *Consider the table below, which gives an example of construction of two induced multiplicative 3-Hom-Lie algebras (given in [13]).*

Hom-Lie algebra	*Trace*	*induced 3-Hom-Lie algebra*
$[e_1,e_2]_1 = e_4$ $[e_3,e_4]_1 = e_2$ $\alpha_1(e_1) = e_3+e_4;\ \alpha_1(e_2) = e_4$ $\alpha_1(e_3) = e_1+e_2;\ \alpha_1(e_4) = e_2$	$\tau_1(x) = x_1 + x_3$	$[e_1,e_2,e_3]_{1,\tau} = e_4$ $[e_1,e_3,e_4]_{1,\tau} = e_2$ $\alpha_1(e_1) = e_3+e_4;\ \alpha_1(e_2) = e_4$ $\alpha_1(e_3) = e_1+e_2;\ \alpha_1(e_4) = e_2$
$[f_1,f_2]_2 = f_4;$ $\alpha_2(f_1) = f_1+f_2+f_3+f_4$ $\alpha_2(f_2) = f_4$ $\alpha_2(f_3) = 0;\ \alpha_2(f_4) = 0$	$\tau_2(x) = x_1$	$[f_1,f_2,f_3]_{2,\tau} = f_4$ $\alpha_2(f_1) = f_1+f_2+f_3+f_4$ $\alpha_2(f_2) = f_4;$ $\alpha_2(f_3) = 0;\ \alpha_2(f_4) = 0$
Morphism of Hom-Lie algebra		
$\phi(e_1) = \lambda_{1,1}f_1 + \lambda_{1,1}f_2 + \lambda_{1,1}f_3 + 2\lambda_{1,1}f_4;\quad \phi(e_2) = 0$ $\phi(e_3) = \lambda_{1,1}f_1 + \lambda_{1,1}f_2 + \lambda_{1,1}f_3 + 2\lambda_{1,1}f_4;\quad \phi(e_4) = 0$		
Morphism of 3-Hom-Lie algebra induced by morphism of Hom-Lie algebra		
$\phi(e_1) = f_1+f_2+f_3+2f_4;\quad \phi(e_2) = 0$ $\phi(e_3) = f_1+f_2+f_3+2f_4;\quad \phi(e_4) = 0$		

Let $(A_1,[\ ,\ ]_1,\alpha_1)$ be the first Hom-Lie algebra and $(A_2,[\ ,\ ]_2,\alpha_2)$ be the second Hom-Lie algebra. We denote by $(A_{\tau_1},[.,.,.]_{\tau_1},\alpha)$ the multiplicative 3-Hom-Lie algebra induced by the first Hom-Lie algebra and $(B_{\tau_2},[.,.,.]_{\tau_1},\alpha)$ the multiplicative 3-Hom-Lie algebra induced by the second Hom-Lie algebra. We construct the morphisms ϕ of 3-Hom-Lie algebras induced by the morphisms of Hom-Lie algebras, satisfying the condition $\tau_2(\phi) = \tau_1$, where τ_1 is a $[\ ,\ ]_1$-trace and τ_2 is a $[\ ,\ ]_2$-trace.

Denote the structure constants of a Hom-Lie algebra $(A,[.,.],\alpha)$ of dimension n with respect to a basis $B = \{e_1,\dots,e_n\}$, by $(c^k_{i,j})_{1\le i,j,k\le n}$ and by $(C^q_{i,j,k})_{1\le i,j,k,q\le n}$ those of the induced 3-Hom-Lie algebra $(A,[.,.,.]_\tau)$. A linear map $\alpha : A \to A$ will be represented by a $n\times n$ matrix, $b = (b^j_i)_{1\le i,j\le n}$. A bilinear map $\varphi : A\otimes A \to A$ (2-cochain) will be represented by $n\times n$ matrix, $p = (p^k_{i,j})_{1\le i,j,k\le n}$. The condition for φ (represented by the matrix p) to be a 2-cocycle for a Hom-Lie algebra is written as follows

$$\sum_{s=1}^{n}(\sum_{v=1}^{n} -c^s_{jk}b^i_v a^o_{sv} + c^s_{ik}b^j_v a^o_{sv} - c^s_{ij}b^k_v a^o_{sv} + b^i_s a^v_{jk}c^o_{sv} - b^j_s a^v_{ik}c^o_{sv} + b^k_s a^v_{ij}c^o_{sv}) = 0. \tag{5.1}$$

A trilinear map $\psi : A\otimes A\otimes A \to A$ (2-cochain) will be represented by a $n\times n$ matrix, $a = (a^v_{i,j,k})_{1\le i,j,k,v\le n}$. The condition for ψ (represented by the matrix a) to be a 2-cocycle for a 3-Hom-Lie algebra is written as follows

$$\begin{aligned}\sum_{s=1}^{n}(\sum_{t=1}^{n}(\sum_{v=1}^{n} & -C^s_{i,j,k}b^q_t b^p_v a^o_{s,t,v} - b^k_z C^t_{i,j,q}b^p_v a^o_{s,t,v} - b^k_s b^q_t C^v_{i,j,p}a^o_{s,t,v} + b^i_s b^j_t C^v_{k,q,p}a^o_{s,t,v} \\ & + b^i_s b^j_t a^v_{k,q,p}C^o_{s,t,v} - b^k_s b^q_t a^v_{i,j,p}C^o_{s,t,v} - a^s_{i,j,k}b^q_t b^p_v C^o_{s,t,v} - b^k_s a^t_{i,j,q}b^p_v C^o_{s,t,v})) = 0\end{aligned} \tag{5.2}$$

Solving the equations (5.1) *for the first Hom-Lie algebra, we obtain the necessary conditions applied respectively to* $p = (p^k_{i,j})_{1\le i,j,k\le n}$ *and* $b = (b_{ij})_{1\le i,j\le n}$*. We get*

$$\begin{array}{lll} \varphi_1(e_1,e_3) = p_1e_2 - p_1e_4; & \varphi_1(e_1,e_4) = p_2e_2 + p_3; & \varphi_1(e_1,e_2) = -p_2e_2 + p_4e_4; \\ \varphi_1(e_2,e_3) = -p_2e_2 - p_3e_4; & \varphi_1(e_2,e_4) = 0; & \varphi_1(e_3,e_4) = (p_4 - p_3)e_4, \end{array}$$

where p_1, p_2, p_3, p_4 *are parameters. Now, for a 2-cocycle* $\varphi_1 \in Z^2(A_1, A_1)$*, let us consider* $\varphi_{1,\tau} \in Z^2(A_{\tau_2}, A_{\tau_1})$ *defined as in Proposition 5.6 [13]. We get*

$$\left\{\begin{array}{l} \varphi_{\tau_1}(e_1,e_2,e_3) = \varphi_1(e_2,e_3) + \varphi_1(e_1,e_2) = -2p_2e_2 + (-p_3 + p_4)e_4 \\ \varphi_{\tau_1}(e_1,e_2,e_4) = \varphi_1(e_2,e_4) = 0 \\ \varphi_{\tau_1}(e_1,e_3,e_4) = \varphi_1(e_3,e_4) - \varphi_1(e_1,e_4) = -p_2 + (p_4 - 2p_3)e_4 \\ \varphi_{\tau_1}(e_2,e_3,e_4) = -\varphi_1(e_2,e_4) = 0. \end{array}\right.$$

All the 2*-cocycles of* A_{τ_1} *are induced by* 2*-cocycles of* A_1*. We eliminate all constants underlying coboundaries and we deduce that* $\dim H^2(A_{\tau_1}, A_{\tau_1}) = 0$*. In a similar way, we determine the* 2*-cocycles of the second Hom-Lie algebra* A_2*. We get*

$$\begin{array}{ll} \varphi_2(f_1,f_2) = k_1f_2 + k_2e_3 + k_3e_4; & \varphi_2(f_1,f_3) = k_4e_3 + k_5e_4; \\ \varphi_2(f_1,f_4) = k_9e_3 + k_{10}e_4; & \varphi_2(f_2,f_3) = k_5e_3 + k_6e_4; \\ \varphi_2(f_2,f_4) = k_7e_3 + k_8e_4; & \varphi_2(f_3,f_4) = (-k_9 - k_7)e_3 + (k_1 - k_{10} - k_8)e_4, \end{array}$$

where $k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_9, k_{10}$ *are parameters. For a* 2*-cocycle* $\varphi_2 \in Z^2(A_2, A_2)$*, let us consider* $\varphi_{2,\tau} \in Z^2(A_{\tau_2}, A_{\tau_2})$*.*

$$\left\{\begin{array}{l} \varphi_{\tau_2}(e_1,e_2,e_3) = \varphi_2(e_2,e_3) = k_5f_3 + k_6f_4 \\ \varphi_{\tau_2}(e_1,e_2,e_4) = \varphi_2(e_2,e_4) = k_7f_3 + k_8f_4 \\ \varphi_{\tau_2}(e_1,e_3,e_4) = \varphi_2(e_3,e_4) = (-k_9 - k_7)f_3 + (k_1 - k_{10} - k_8)f_4 \\ \varphi_{\tau_2}(e_2,e_3,e_4) = -\varphi_2(e_2,e_4) = 0. \end{array}\right.$$

Solving the equations (5.2) *for the second* 3*-Hom-Lie algebra, we obtain the necessary conditions applied respectively to* $p = (p^k_{i,j})_{1\le i,j,k\le n}$ *and* $b = (b_{ij})_{1\le i,j\le n}$*. We get, the space of* 2*-cocycles of* A_{τ_2} *is generated by*

$$\left\{\begin{array}{l} \psi_{\tau_2}(e_1,e_2,e_3) = c_1f_3 + c_2f_4 \\ \psi_{\tau_2}(e_1,e_2,e_4) = c_3f_3 + c_4f_4 \\ \psi_{\tau_2}(e_1,e_3,e_4) = c_5f_3 + c_6f_4 \\ \psi_{\tau_2}(e_2,e_3,e_4) = c_7f_3 + c_8f_4, \end{array}\right. \tag{5.3}$$

There exist 2*-cocycles of* A_{τ_2} *which are not induced by a* 2*-cocycle of* A_2*. We eliminate all constants underlying coboundaries. Gluing these bits of information together we deduce that* $\dim H^2(A_{\tau_2}, A_{\tau_2})$ *is equal to the number of independent constants remaining in the expression of the 2-cocycle (5.3). Thus, we can see that* $\dim H^2(A_{\tau_2}, A_{\tau_2}) = 5$ *and spanned by the following 2-cocycles*

$$\left\{\begin{array}{l} \psi_{2,1,\tau}(f_1,f_2,f_3) = 0 \\ \psi_{2,1,\tau}(f_1,f_2,f_4) = f_3 \\ \psi_{2,1,\tau}(f_1,f_3,f_4) = 0 \\ \psi_{2,1,\tau}(f_2,f_3,f_4) = 0, \end{array}\right. \quad \left\{\begin{array}{l} \psi_{2,2,\tau}(f_1,f_2,f_3) = 0 \\ \psi_{2,2,\tau}(f_1,f_2,f_4) = 0 \\ \psi_{2,2,\tau}(f_1,f_3,f_4) = f_3 \\ \psi_{2,2,\tau}(f_2,f_3,f_4) = 0, \end{array}\right. \quad \left\{\begin{array}{l} \psi_{2,3,\tau}(f_1,f_2,f_3) = 0 \\ \psi_{2,3,\tau}(f_1,f_2,f_4) = 0 \\ \psi_{2,3,\tau}(f_1,f_3,f_4) = f_4 \\ \psi_{2,3,\tau}(f_2,f_3,f_4) = 0. \end{array}\right.$$

$$\begin{cases} \psi_{2,4,\tau}(f_1,f_2,f_3)=0\\ \psi_{2,4,\tau}(f_1,f_2,f_4)=0\\ \psi_{2,4,\tau}(f_1,f_3,f_4)=0\\ \psi_{2,4,\tau}(f_2,f_3,f_4)=f_3. \end{cases} \qquad \begin{cases} \psi_{2,5,\tau}(f_1,f_2,f_3)=0\\ \psi_{2,5,\tau}(f_1,f_2,f_4)=0\\ \psi_{2,5,\tau}(f_1,f_3,f_4)=0\\ \psi_{2,5,\tau}(f_2,f_3,f_4)=f_4. \end{cases}$$

By a direct computation, using a computer algebra system, we deduce that the first space of cocycles $Z^1(A_1,A_2)$ of the Hom-Lie algebra morphism ϕ is generated by

$$\phi_1(e_1)=pf_1+pf_2+pf_3+2pf_4=\phi_1(e_3);\quad \phi_1(e_2)=\phi_1(e_4)=0,$$

where p is parameter.
Now, for a 2-cocycle $\phi_1\in Z^2(A_1,A_2)$, let us consider $\phi_{1,\tau}\in Z^2(A_{\tau_1},A_{\tau_2})$ defined as in Theorem 5.7.

$$\begin{cases} \phi_{1,\tau}(e_1)=-\tau_1(e_2)\phi_1(e_2)+\tau_1(e_3)\phi_1(e_3)-\tau_1(e_4)\phi_1(e_4)=\phi_1(e_3)\\ \phi_{1,\tau}(e_2)=\tau_1(e_1)\phi_1(e_1)-\tau_1(e_3)\phi_1(e_3)-\tau_1(e_4)\phi_1(e_4)=0\\ \phi_{1,\tau}(e_3)=\tau_1(e_1)\phi_1(e_1)-\tau_1(e_2)\phi_1(e_2)-\tau_1(e_4)\phi_1(e_4)=\phi_1(e_1)\\ \phi_{1,\tau}(e_4)=\tau_1(e_1)\phi_1(e_1)-\tau_1(e_2)\phi_1(e_2)-\tau_1(e_3)\phi_1(e_3)=0 \end{cases}$$

By a direct computation, we see that all the 2-cocycles $\phi_{1,\tau}$ are induced by a 2-cocycle ϕ_1.

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(1) Jouf University, Department of Mathematics, College of Sciences and Arts in Gurayat, Sakakah, Saudi Arabia
and
University of Sfax, Faculty of Sciences, BP 1171, 3000 Sfax, Tunisia
E-mail: arfaanja.mail@gmail.com

(2) University of Carthage, Preparatory Institute for Engineering Studies of Nabeul, Tunisia
E-mail: benfraj_nizar@yahoo.fr

(3) University of Haute Alsace, IRIMAS - Département de Mathématiques, F-68093 Mulhouse, France
E-mail: Abdenacer.Makhlouf@uha.fr