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Regularity for subelliptic double obstacle problems with discontinuous coefficients in Carnot group

by

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Abstract

In this paper, we consider the double obstacle problem for a subelliptic equation of p-Laplace type with VMO coefficients in Carnot groups. The interior Hölder regularity for solutions to the double obstacle problem with subcritical growth is established while p is close to 2.

Key Words: Subelliptic equation, p-Laplace type, double obstacle problem, Hölder regularity.
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1 Introduction

Let $\mathbb{G} = (\mathbb{R}^N, \circ)$ be the Carnot group of step $r \geq 2$ and let X_i, $i = 1, \dots, m$, be the left-invariant smooth vector fields on $\mathbb{G}$ associated with a fixed orthonormal basis of the bracket-generating layer of its Lie algebra $\mathfrak{g}$. Let $Xu = (X_1 u, \dots, X_m u)$ and let Ω be a bounded domain in $\mathbb{G}$. In this paper we consider the double obstacle problem for the following subelliptic equation of p-Laplace type

$$X^*(\langle A(x)Xu, Xu\rangle^{\frac{p-2}{2}} A(x)Xu) = B(x, u, Xu), \tag{1.1}$$

where $A = (a_{ij}(x))_{m\times m}$ is a symmetric positive-definite matrix with measurable coefficients and $B(x, u, Xu)$ satisfies a subcritical growth. Here $X_i^* = -X_i$ is the formal adjoint to X_i:

$$\int_\Omega (X_i^* u)\phi dx = \int_\Omega u X_i \phi dx, \ \forall u, \phi \in C_0^\infty(\Omega).$$

Obstacle problems appear in various branches of the theoretical and applied sciences, such as nonlinear potential theory and free boundary problems, control theory and optimal stopping, mechanical engineering and robotics, financial mathematics, fluid filtration in porous media, see [20, 5, 24]. In the Euclidean case (when $X = \nabla$, $A(x) = \mathbb{I}$), regularity of solutions to single and double obstacle problems for elliptic and degenerate elliptic equations has been extensively studied, see for example [22, 7, 8, 9, 17, 28, 32]. Lieberman [28] proved that the solutions to some degenerate double obstacle problems are as regular as the obstacles ψ_1 and ψ_2 when B satisfies a natural growth condition

$$|B(x, u, \nabla u)| \leq c(1 + |\nabla u|^p).$$

In [32], Mu and Ziemer proved the Hölder regularity for double obstacle problems of p-Laplace type with controllable growth condition

$$|B(x,u,\nabla u)| \le c(1+|\nabla u|^{p-1}).$$

Independently, relying on a perturbation argument, Choe [8] proved that the solutions of double obstacle problems (when B satisfies natural growth condition) have $C^{0,\alpha}$ or $C^{1,\alpha}$ regularity under various regularity assumptions on the obstacles. Here we also mention that the authors in [36] got the Hölder continuity for weak solutions to degenerate elliptic equation

$$\operatorname{div}(\langle A(x)\nabla u,\nabla u\rangle^{\frac{p-2}{2}}A(x)\nabla u) = B(x,u,\nabla u), \quad x\in\Omega,$$

where $A(x)$ belongs to $\mathrm{VMO}(\Omega)$ and B satisfies natural growth condition. Recently, the local $C^{1,\alpha}$ regularity of solutions to the fully nonlinear equation with variable exponents

$$\left[|Du|^{p(x)}+a(x)|Du|^{q(x)}\right]F(D^2u)=f(x),\ x\in\Omega$$

was obtained by Fang et al. [19]. For more regularity results about the equations with p-Laplacians, please refer to [18, 6] and the references therein.

Based on Hörmander's fundamental work [25], there has been tremendous work on subelliptic PDEs and corresponding obstacle problems arising from non-commuting vector fields, see, for example [4, 3, 29, 34, 33, 12, 35, 13, 10] for subelliptic equations or systems, and [11, 21, 31, 1, 14, 15, 16] for subelliptic obstacle problems. Dong and Niu [13] obtained the Morrey regularity and Hölder continuity for weak solutions to nondiagonal quasilinear subelliptic systems with bounded VMO coefficients for $p=2$. Zheng and Feng [35] studied the $C^{1,\alpha}_X$ regularity for weak solutions to subelliptic p-harmonic systems with subcritical growth in Carnot group. In [31], Marchi proved the $C^{1,\alpha}_X$ regularity of solutions to a class of double obstacle problem on Heisenberg group. In [16], Du and his collaborators got the $C^{0,\alpha}_X$ and $C^{1,\alpha}_X$ regularity of solutions to the single obstacle problem for (1.1) with controllable growth under different restrictions on the coefficients.

Motivated by the above works, we try to generalize the results in [8] of double obstacle problem for p-Laplace equation (i.e. $A(x)=\mathbb{I}$) to the double obstacle problem for a class of p-Laplace type subelliptic equation with *discontinuous coefficients in Carnot group*. Specifically, we consider the double obstacle problem for (1.1) with obstacles ψ_1 and ψ_2, i.e., the problem of finding a fuction $u\in\mathcal{K}(\Omega)$ satisfying the variational inequality

$$\int_\Omega \langle\langle AXu,Xu\rangle^{\frac{p-2}{2}}AXu, X(v-u)\rangle dx + \int_\Omega B(x,u,Xu)(v-u)dx \ge 0, \tag{1.2}$$

for all $v\in\mathcal{K}(\Omega)=\Big\{v\in HW^{1,p}(\Omega): v-\theta\in HW^{1,p}_0(\Omega), \psi_1\le v\le\psi_2 \text{ a.e. in } \Omega\Big\}$. Here $\theta\in HW^{1,p}(\Omega)$ is a boundary value function with $\psi_1\le\theta\le\psi_2$ a.e. in Ω. We make the following assumptions:

(H1) The matrix of coefficients $A(x)\in\mathrm{VMO}(\Omega)$ is symmetric, positive-definite and satisfies the uniform ellipticity condition that for some $\Lambda>0$,

$$\Lambda^{-1}|\xi|^2 \le \langle A(x)\xi,\xi\rangle \le \Lambda|\xi|^2, \quad \text{a.e. } x\in\Omega,\ \forall\xi\in\mathbb{R}^m. \tag{1.3}$$

(H2) For some $L > 0$, $B(x, u, \xi)$ satisfies the subcritical growth condition

$$|B(x, u, \xi)| \le L|\xi|^q, \ \forall 0 \le q < p, \tag{1.4}$$

when $x \in \Omega$ and $(u, \xi) \in \mathbb{R} \times \mathbb{R}^m$.

(H3) The obstacle functions $\psi_1, \psi_2 \in C^{1,\gamma}_{X,\mathrm{loc}}(\Omega)$, $0 < \gamma < 1$.

Let Q be the homogeneous dimension of $\mathbb{G}$. We prove the following result.

Theorem 1. *Suppose that (H1)-(H3) hold and $u \in \mathcal{K}(\Omega)$ is a solution to the double obstacle problem for* (1.1). *If p is close to 2, then for any $0 < \lambda < Q$, we have $Xu \in L^{p,\lambda}_{\mathrm{loc}}(\Omega)$. Moreover, there exists $0 < \alpha < 1$ such that $u \in C^{0,\alpha}_X(\Omega)$.*

The remainder of the paper is divided into two sections. In Section 2, we recall some basic facts of Carnot group and some preliminary results that will be used in our proof. In Section 3, Theorem 1 is proved by establishing a Morrey type estimate for solutions to the double obstacle problem for (1.1).

2 Some Preliminaries

Let us first recall some preliminary facts on Carnot groups. We refer the reader to [2] and references therein for further details.

A Carnot group $\mathbb{G} = (\mathbb{R}^N, \circ)$ of step r is a simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a decomposition $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$ such that (i) $\mathfrak{g}$ is stratified, i.e., $[V_1, V_j] = V_{j+1}$ for $1 \le j \le r-1$; (ii) $\mathfrak{g}$ is r-nilpotent, i.e., $[V_j, V_r] = 0$ for $1 \le j \le r$. The homogeneous dimension of $\mathbb{G}$ is

$$Q = \sum_{i=1}^{r} i\dim(V_i)$$

which is often larger than the topological dimension $\sum_{i=1}^{r} \dim(V_i)$ of the group. The trivial Carnot group with step one is $\mathbb{R}^N$. The most important prototype of Carnot group with step two is the Heisenberg group $\mathbb{H}^n$.

For a Carnot group $\mathbb{G}$ of step $r \ge 2$, the left invariant orthonormal basis $X_1, \dots, X_m$ for V_1 is called the horizontal directions and the left invariant vector fields of V_i $(2 \le i \le r)$ are called commutator directions in the sense that they are generated as commutators of order i of linear combinations of $X_1, \dots, X_m$. It is well known that the family of vector fields $\{X_1, ..., X_m\}$ satisfies the Hörmander finite rank condition: $\mathrm{rank}\,(\mathrm{Lie}\{X_1, ..., X_m\}) = N$. Let f be a function defined on a bounded domain of $\Omega \subset \mathbb{G}$, then $Xf = (X_1 f, ..., X_m f)$ denotes the horizontal gradient of f and hence $|Xf| = \left(\sum_{j=1}^m |X_j f|^2\right)^{\frac{1}{2}}$.

An absolutely continuous path $\gamma : [a, b] \to \mathbb{G}$ is said to be X-subunit if there exist functions $c_i(t)$, $a \le t \le b$ such that

$$\sum_{i=1}^{m} c_i(t)^2 \le 1 \text{ and } \gamma'(t) = \sum_{i=1}^{m} c_i(t) X_i(\gamma(t)) \text{ a.e. } t \in [a, b].$$

The Carnot-Carathéodory distance $d_X(x, y)$ is defined as the infimum of those $T > 0$ for which there exists a X-subunit path $\gamma : [0, T] \to \mathbb{G}$ with $\gamma(0) = x$ and $\gamma(T) = y$. This metric d_X is left invariant and 1-homogeneous with respect to the group dilations δ_λ, i.e.,

$$d_X(z \circ x, z \circ y) = d_X(x, y), \;\; d_X(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d_X(x, y)$$

for all $x, y, z \in \mathbb{G}$ and $\lambda > 0$. Using these properties, one infers that the Haar measure of $B_r(x) = \{y \in \mathbb{G} : d_X(x, y) < r\}$ is given by

$$|B_r(x)| = \omega_{\mathbb{G}} r^Q, \tag{2.1}$$

where $|B_r(x)|$ denotes the Lebesgue measure of $B_r(x)$, $\omega_{\mathbb{G}} = |B_1(e)|$ and e is the group identity.

For any $1 < p < \infty$, let $HW^{1,p}(\Omega)$ be the set of $L^p(\Omega)$ functions whose distributional horizontal gradient is $L^p(\Omega)$:

$$HW^{1,p}(\Omega) = \{u \in L^p(\Omega) : X_j u \in L^p(\Omega), j = 1, 2, \cdots, m\}.$$

$HW^{1,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{HW^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega)}.$$

Denoted by $HW_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ w.r.t. $\|\cdot\|_{HW^{1,p}}$. We will write $u \in HW_{\mathrm{loc}}^{1,p}(\Omega)$ to mean $u \in HW^{1,p}(K)$ for every compact set $K \subset \Omega$.

Lemma 1 (Sobolev Inequality [23, 29])**.** *For any $1 \le p < \infty$ and $u \in HW^{1,p}(B_R)$, there exists a constant $C > 0$ such that*

$$\left(\fint_{B_R} |u - u_R|^{\kappa p} dx \right)^{\frac{1}{\kappa p}} \le CR \left(\fint_{B_R} |Xu|^p dx \right)^{\frac{1}{p}}, \tag{2.2}$$

where $u_R = \fint_{B_R} u dx$ is the integral average of u on B_R, and $1 \le \kappa \le Q/(Q-p)$ if $1 \le p < Q$; $1 \le \kappa < \infty$ if $p \ge Q$. Moreover, for any $u \in HW_0^{1,p}(B_R)$,

$$\left(\fint_{B_R} |u|^{\kappa p} dx \right)^{\frac{1}{\kappa p}} \le CR \left(\fint_{B_R} |Xu|^p dx \right)^{\frac{1}{p}}. \tag{2.3}$$

Let us recall several function spaces in $\mathbb{G}$ ([30, 33]). For convenience, we set

$$\Omega(x, R) = \Omega \cap B(x, R), \quad f_{x,R} = \frac{1}{|\Omega(x, R)|} \int_{\Omega(x,R)} f(y) dy.$$

Definition 1. *Let $1 < p < \infty$ and $\lambda \ge 0$. We say that $f \in L_{\mathrm{loc}}^p(\Omega)$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ if*

$$\|f\|_{L^{p,\lambda}(\Omega)} = \sup_{x \in \Omega, 0 < \rho < \mathrm{diam}\Omega} \left(\rho^{-\lambda} \int_{\Omega(x,\rho)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty;$$

we say that $f \in L_{\mathrm{loc}}^p(\Omega)$ belongs to the Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$ if

$$\|f\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \sup_{x \in \Omega, 0 < \rho < \mathrm{diam}\Omega} \left(\rho^{-\lambda} \int_{\Omega(x,\rho)} |f(y) - f_{x,\rho}|^p dy \right)^{\frac{1}{p}} < \infty.$$

Definition 2. *Let $\alpha \in (0,1)$. The Hölder space $C_X^{0,\alpha}(\bar{\Omega})$ is a Banach space with respect to the norm*

$$\|f\|_{C_X^{0,\alpha}(\bar{\Omega})} = \sup_{\Omega} |f| + \sup_{\bar{\Omega}} \frac{|f(x)-f(y)|}{[d_X(x,y)]^\alpha} < \infty.$$

We say $f \in C_X^{0,\alpha}(\Omega)$, if $f \in C_X^{0,\alpha}(K)$ for every compact set $K \subset \Omega$.

Definition 3. *We say that $f \in L^1_{\mathrm{loc}}(\Omega)$ belongs to* BMO(Ω) *if*

$$\|f\|_* = \sup_{x\in\Omega, 0<\rho<d_0} \frac{1}{|\Omega(x,\rho)|} \int_{\Omega(x,\rho)} |f(y) - f_{x,\rho}| dy < \infty;$$

f belongs to VMO(Ω) *if $f \in$* BMO(Ω) *and*

$$\eta_r(f) = \sup_{x\in\Omega, 0<\rho<r} \frac{1}{|\Omega(x,\rho)|} \int_{\Omega(x,\rho)} |f(y) - f_{x,\rho}| dy \to 0, \ r \to 0.$$

As in the Euclidian space, the Hölder space $C_X^{0,\alpha}$ can equivalently be characterized by the integral representation, see [30], [33]. So we have the following lemma.

Lemma 2. *If $u \in \mathcal{L}^{p,Q+p\alpha}(\Omega)$, $1<p<\infty$, $0<\alpha<1$, then $u \in C_X^{0,\alpha}(\Omega)$.*

Lemma 3 (Iteration Lemma [22])**.** *Let $\Phi(\rho)$ be a nonnegative and nondecreasing function on $[0,R_0]$ satisfying*

$$\Phi(\rho) \le A\left(\left(\frac{\rho}{R}\right)^a + \varepsilon\right)\Phi(R) + BR^b, \quad 0<\rho\le R\le R_0,$$

where A, a, b and B are nonnegative constants, $b<a$. Then there exists a constant $\varepsilon_0 = \varepsilon_0(A,a,b)$ such that if $\varepsilon < \varepsilon_0$ we have

$$\Phi(\rho) \le C\left(\left(\frac{\rho}{R}\right)^b \Phi(R) + B\rho^b\right), \quad 0<\rho\le R\le R_0,$$

where C is a constant depending on A, a and b.

3 Proof of the main result

For the fixed $x \in \Omega$ and a small $R>0$, let $B_R = B_R(x) \subset\subset \Omega$. We first recall a result about Morrey type estimate for weak solutions to the following subelliptic equation with constant coefficients

$$X^*(\langle A_{R/2}Xw, Xw\rangle^{\frac{p-2}{2}} A_{R/2}Xw) = X^*(\langle A_{R/2}X\psi_1, X\psi_1\rangle^{\frac{p-2}{2}} A_{R/2}X\psi_1), \tag{3.1}$$

where $A_{R/2} = \fint_{B_{R/2}} A(x)dx$ is the integral average of $A(x)$.

Lemma 4 (see [16])**.** *Let $w \in HW^{1,p}(B_{R/2})$ be a weak solution to (3.1) with p close to 2. Then for any $0<\rho<R/2$ and $\varepsilon>0$, we have*

$$\int_{B_\rho} |Xw|^p dx \le c\left(\left(\frac{\rho}{R}\right)^Q + \varepsilon\right)\int_{B_{R/2}} |Xw|^p dx + cR^Q. \tag{3.2}$$

We also need the following higher integrability result for solutions to double obstacle problem for (1.1).

Lemma 5 ([14]). *Let $u \in \mathcal{K}(\Omega)$ be a solution to the double obstacle problem for* (1.1) *and (H1)-(H3) be satisfied. Then $u \in HW^{1,t}_{\mathrm{loc}}(\Omega)$ for some $t > p$ and there exists a constant $c > 0$ such that for any $B_R \subset\subset \Omega$,*

$$\left(\fint_{B_{R/2}} |Xu|^t dx\right)^{\frac{1}{t}} \le c\left[\left(\fint_{B_R} |Xu|^p dx\right)^{\frac{1}{p}} + \left(\fint_{B_R} (|X\psi_1|^t + |X\psi_2|^t) dx\right)^{\frac{1}{t}}\right], \tag{3.3}$$

where c does not depend on R.

Lemma 6 (see [27]). *Suppose that $A(x)$ satisfies (H1). Then for any $\xi, \eta \in \mathbb{R}^m$, we have*

$$\langle \langle A\xi, \xi\rangle^{\frac{p-2}{2}} A\xi - \langle A\eta, \eta\rangle^{\frac{p-2}{2}} A\eta, \xi - \eta\rangle \ge C(p)(|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2; \tag{3.4}$$

particularly, for $p \ge 2$,

$$\langle \langle A\xi, \xi\rangle^{\frac{p-2}{2}} A\xi - \langle A\eta, \eta\rangle^{\frac{p-2}{2}} A\eta, \xi - \eta\rangle \ge C(p)|\xi - \eta|^p. \tag{3.5}$$

On the basis of the above lemmas, we can prove the following Morrey type estimates of solutions to the double obstacle problem for (1.1).

Lemma 7. *If $u \in \mathcal{K}(\Omega)$ is a solution to the double obstacle problem for* (1.1) *and p is close to 2, then for any $0 < \rho \le R$, $B_R \subset\subset \Omega$, and $\eta, \sigma, \varepsilon > 0$ it holds*

$$\int_{B_\rho} |Xu|^p dx \le c\left(\left(\frac{\rho}{R}\right)^Q + \vartheta\right) \int_{B_R} |Xu|^p dx + cR^Q, \tag{3.6}$$

where $\vartheta = \|A\|^{(t-p)/t}_{*,R/2} + \eta + \sigma + \varepsilon$.

Proof. Let $w \in HW^{1,p}(B_{R/2})$ be a weak solution to the Dirichlet problem for (3.1) (the existence of weak solutions can be found in [26, Theorem 2.13]), i.e.,

$$\begin{aligned} &\int_{B_{R/2}} \langle \langle A_{R/2} Xw, Xw\rangle^{\frac{p-2}{2}} A_{R/2} Xw, X\phi\rangle dx \\ &= \int_{B_{R/2}} \langle \langle A_{R/2} X\psi_1, X\psi_1\rangle^{\frac{p-2}{2}} A_{R/2} X\psi_1, X\phi\rangle dx \end{aligned} \tag{3.7}$$

for all $\phi \in C_0^\infty(B_{R/2})$ and $u - w \in HW_0^{1,p}(B_{R/2})$. Since $u - w \in HW_0^{1,p}(B_{R/2})$ and $u \ge \psi_1$ on $\partial B_{R/2}$, we know that $w \ge \psi_1$ in $B_{R/2}$ by the maximum principle. On the other hand, we note that $\psi_1 \le w \wedge \psi_2 \le \psi_2$ in $B_{R/2}$ and $w \wedge \psi_2 - u \in HW_0^{1,p}(B_{R/2})$, where $w \wedge \psi_2 = \min\{w, \psi_2\}$. Hence we can choose $w \wedge \psi_2$ as a test function in $B_{R/2}$. Applying $w \wedge \psi_2$ to (1.2), it follows

$$\int_{B_{R/2}} \langle \langle AXu, Xu\rangle^{\frac{p-2}{2}} AXu, X(u - w \wedge \psi_2)\rangle dx \le \int_{B_{R/2}} B(x, u, Xu)(w \wedge \psi_2 - u) dx. \tag{3.8}$$

On the other hand, from (H1) and (3.7)

$$\begin{aligned}\int_{B_{R/2}} |Xw|^p dx &\le c\int_{B_{R/2}} \langle A_{R/2}Xw, Xw\rangle^{\frac{p}{2}} dx \\ &= c\int_{B_{R/2}} \langle\langle A_{R/2}Xw, Xw\rangle^{\frac{p-2}{2}} A_{R/2}Xw, Xu\rangle dx \\ &\quad + c\int_{B_{R/2}} \langle\langle A_{R/2}X\psi_1, X\psi_1\rangle^{\frac{p-2}{2}} A_{R/2}X\psi_1, Xw - Xu\rangle dx \\ &\le c\int_{B_{R/2}} |Xw|^{p-1}|Xu| dx + \int_{B_{R/2}} |X\psi_1|^{p-1}|Xw - Xu| dx \\ &\le \varepsilon\int_{B_{R/2}} |Xw|^p dx + c_\varepsilon\int_{B_{R/2}} |Xu|^p dx + c_\varepsilon\int_{B_{R/2}} |X\psi_1|^p dx.\end{aligned}$$

Taking ε small enough, we have

$$\int_{B_{R/2}} |Xw|^p dx \le c\int_{B_{R/2}} |Xu|^p dx + c(\|X\psi_1\|_{L^\infty})R^Q. \tag{3.9}$$

Then it follows by (3.2) and (3.9) that for any $0 < \rho < R/2$ and $\varepsilon > 0$,

$$\begin{aligned}\int_{B_\rho} |Xu|^p dx &\le 2^p\int_{B_\rho} |Xw|^p dx + 2^p\int_{B_\rho} |Xu - Xw|^p dx \\ &\le c\left(\left(\frac{\rho}{R}\right)^Q + \varepsilon\right)\int_{B_{R/2}} |Xw|^p dx + cR^Q + 2^p\int_{B_\rho} |Xu - Xw|^p dx \\ &\le c\left(\left(\frac{\rho}{R}\right)^Q + \varepsilon\right)\int_{B_{R/2}} |Xu|^p dx + cR^Q + 2^p\int_{B_{R/2}} |Xu - Xw|^p dx. \qquad (3.10)\end{aligned}$$

To estimate the last term in the right hand side of (3.10), we consider two cases: $p \ge 2$ and $p < 2$.

Assume $p \ge 2$. From (3.5) and (3.7), we have

$$\begin{aligned}&\int_{B_{R/2}} |Xu - Xw|^p dx \\ &\le c\int_{B_{R/2}} \langle\langle A_{R/2}Xu, Xu\rangle^{\frac{p-2}{2}} A_{R/2}Xu - \langle A_{R/2}Xw, Xw\rangle^{\frac{p-2}{2}} A_{R/2}Xw, X(u - w)\rangle dx \\ &= c\int_{B_{R/2}} \langle\langle A_{R/2}Xu, Xu\rangle^{\frac{p-2}{2}} A_{R/2}Xu, X(u - w \wedge \psi_2)\rangle dx \\ &\quad + c\int_{B_{R/2}} \langle\langle A_{R/2}Xu, Xu\rangle^{\frac{p-2}{2}} A_{R/2}Xu, X(w \wedge \psi_2 - w)\rangle dx \\ &\quad - c\int_{B_{R/2}} \langle\langle A_{R/2}X\psi_1, X\psi_1\rangle^{\frac{p-2}{2}} A_{R/2}X\psi_1, Xu - Xw\rangle dx \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \qquad (3.11)\end{aligned}$$

By Young's inequality and Hölder's inequality we conclude that

$$\begin{aligned}\mathrm{III} &\leq \varepsilon \int_{B_{R/2}} |Xu - Xw|^p dx + c\int_{B_{R/2}} |X\psi_1|^p dx\\ &\leq \varepsilon \int_{B_{R/2}} |Xu - Xw|^p dx + cR^Q\end{aligned}$$

and

$$\mathrm{II} \leq \eta \int_{B_{R/2}} |Xu|^p dx + c\int_{w\geq\psi_2} (|Xw|^p + |X\psi_2|^p)dx.$$

Choosing $\phi = w - w\wedge\psi_2$ as the test function in (3.7), it follows

$$\begin{aligned}\int_{w\geq\psi_2} |Xw|^p dx &\leq c\int_{w\geq\psi_2} \langle A_{R/2}Xw, Xw\rangle^{\frac{p}{2}} dx\\ &= c\int_{w\geq\psi_2} \langle\langle A_{R/2}Xw, Xw\rangle^{\frac{p-2}{2}} A_{R/2}Xw, X(w\wedge\psi_2)\rangle dx\\ &\quad + c\int_{w\geq\psi_2} \langle\langle A_{R/2}X\psi_1, X\psi_1\rangle^{\frac{p-2}{2}} A_{R/2}X\psi_1, X(w - w\wedge\psi_2)\rangle dx\\ &\leq c\int_{w\geq\psi_2} |Xw|^{p-1}|X\psi_2| dx + \int_{w\geq\psi_2} |X\psi_1|^{p-1}|Xw - X\psi_2| dx\\ &\leq \epsilon\int_{B_{R/2}} |Xw|^p dx + c_\epsilon\int_{B_{R/2}} (|X\psi_1|^p + |X\psi_2|^p)dx.\end{aligned}$$

Hence

$$\int_{w\geq\psi_2} |Xw|^p dx \leq c\int_{B_{R/2}} (|X\psi_1|^p + |X\psi_2|^p)dx. \tag{3.12}$$

Therefore, the integral II can be estimated by

$$\mathrm{II} \leq \eta\int_{B_{R/2}} |Xu|^p dx + c\int_{B_{R/2}} (|X\psi_1|^p + |X\psi_2|^p)dx.$$

By (3.8), one gets

$$\begin{aligned}\mathrm{I} &= c\int_{B_{R/2}} \langle\langle A_{R/2}Xu, Xu\rangle^{\frac{p-2}{2}} A_{R/2}Xu, X(u - w\wedge\psi_2)\rangle dx\\ &\leq c\int_{B_{R/2}} \langle\langle A_{R/2}Xu, Xu\rangle^{\frac{p-2}{2}} A_{R/2}Xu - \langle AXu, Xu\rangle^{\frac{p-2}{2}} AXu, X(u - w\wedge\psi_2)\rangle dx\\ &\quad + c\int_{B_{R/2}} B(x, u, Xu)(w\wedge\psi_2 - u)dx\\ &:= \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

For every $\xi \in \mathbb{R}^m$, we have (see [27])

$$|\langle A\xi, \xi\rangle^{\frac{p-2}{2}} A\xi - \langle A_{R/2}\xi, \xi\rangle^{\frac{p-2}{2}} A_{R/2}\xi| \leq c(p, \Lambda)|A - A_{R/2}||\xi|^{p-1}. \tag{3.13}$$

Using (3.13), (3.12) and noting that $w \wedge \psi_2 = \psi_2$ for $x \in \operatorname{supp}(w - w \wedge \psi_2)$, we infer

$$
\begin{aligned}
\mathrm{I}_1 &\le c \int_{B_{R/2}} |A - A_{R/2}||Xu|^{p-1}(|Xu - Xw| + |X(w - w \wedge \psi_2)|)dx \\
&\le \varepsilon \int_{B_{R/2}} |Xu - Xw|^p dx + c \int_{w \ge \psi_2} |X(w - \psi_2)|^p dx + c_\varepsilon \int_{B_{R/2}} |A - A_{R/2}|^{\frac{p}{p-1}} |Xu|^p dx \\
&\le \varepsilon \int_{B_{R/2}} |Xu - Xw|^p dx \\
&\quad + c \int_{B_{R/2}} (|X\psi_1|^p + |X\psi_2|^p) dx + c_\varepsilon \int_{B_{R/2}} |A - A_{R/2}|^{\frac{p}{p-1}} |Xu|^p dx.
\end{aligned} \tag{3.14}
$$

For I_2, we have by the subcritical growth (1.4) and Hölder's inequality that

$$
\begin{aligned}
\mathrm{I}_2 &\le c \int_{B_{R/2}} |B(x, u, Xu)||w \wedge \psi_2 - u| dx \\
&\le c \int_{B_{R/2}} |Xu|^q |w \wedge \psi_2 - u| dx \\
&\le c \left(\int_{B_{R/2}} |Xu|^p dx \right)^{\frac{q}{p}} \left(\int_{B_{R/2}} |u - w \wedge \psi_2|^{\frac{p}{\delta}} dx \right)^{\frac{\delta}{p}},
\end{aligned}
$$

where $\delta = p - q > 0$. Since $\psi_1 \le u \le \psi_2$ and $\psi_1, \psi_2 \in C_X^{1,\gamma}$, we deduce that $|u| \le M$ for some $M > 0$. In view of $w = u$ on $\partial B_{R/2}$, we have by the maximum principle that $|w| \le M$ in $B_{R/2}$. Consequently, we conclude by Young's inequality that

$$
\begin{aligned}
\mathrm{I}_2 &\le c R^{\frac{Q\delta}{p}} \sup_{x \in B_{R/2}} (|u| + |w| + |\psi_2|) \left(\int_{B_{R/2}} |Xu|^p dx \right)^{\frac{q}{p}} \\
&\le c(p, q, M, L) R^{\frac{Q\delta}{p}} \left(\int_{B_{R/2}} |Xu|^p dx \right)^{\frac{q}{p}} \\
&\le \eta \int_{B_{R/2}} |Xu|^p dx + c(p, q, M, L, \eta) R^Q.
\end{aligned} \tag{3.15}
$$

Combining (3.14) and (3.15) gives

$$
\begin{aligned}
\mathrm{I} &\le \varepsilon \int_{B_{R/2}} |Xu - Xw|^p dx + \eta \int_{B_{R/2}} |Xu|^p dx \\
&\quad + c_\varepsilon \int_{B_{R/2}} |A - A_{R/2}|^{\frac{p}{p-1}} |Xu|^p dx + c(p, q, M, L, \eta, \|X\psi_1\|_{L^\infty}, \|X\psi_2\|_{L^\infty}) R^Q.
\end{aligned}
$$

Putting the above estimates of I, II and III into (3.11) and then taking $\varepsilon = 1/4$, it follows

$$\int_{B_{R/2}} |Xu - Xw|^p dx \le c \int_{B_{R/2}} |A_{R/2} - A|^{\frac{p}{p-1}} |Xu|^p dx + 2\eta \int_{B_{R/2}} |Xu|^p dx + cR^Q. \tag{3.16}$$

We continue to estimate the first integral in the right hand side of (3.16). From Hölder's inequality and Lemma 5 we know that there exists $t > p$ such that

$$\begin{aligned} & c \int_{B_{R/2}} |A_{R/2} - A|^{\frac{p}{p-1}} |Xu|^p dx \\ & \le c|B_{R/2}| \left(\fint_{B_{R/2}} |A_{R/2} - A|^{\frac{pt}{(p-1)(t-p)}} dx \right)^{\frac{t-p}{t}} \left(\fint_{B_{R/2}} |Xu|^t dx \right)^{\frac{p}{t}} \\ & \le c\|A\|_{*,R/2}^{(t-p)/t} |B_{R/2}| \left(\fint_{B_{R/2}} |Xu|^t dx \right)^{\frac{p}{t}} \\ & \le c\|A\|_{*,R/2}^{(t-p)/t} \int_{B_R} |Xu|^p dx + c\|A\|_{*,R/2}^{(t-p)/t} |B_R| \left(\fint_{B_R} (|X\psi_1|^t + |X\psi_2|^t) dx \right)^{\frac{p}{t}}. \end{aligned} \tag{3.17}$$

Inserting (3.17) into (3.16) yields

$$\int_{B_{R/2}} |Xu - Xw|^p dx \le c(\|A\|_{*,R/2}^{(t-p)/t} + \eta) \int_{B_R} |Xu|^p dx + cR^Q. \tag{3.18}$$

Thus for any $0 < \rho < R/2$, the estimate (3.6) follows by taking (3.18) into (3.10) and letting $\vartheta(R, \eta, \varepsilon) = \|A\|_{*,R/2}^{(t-p)/t} + \eta + \varepsilon$.

When $1 < p < 2$, using Hölder's inequality, Young's inequality and (3.9) we have

$$\begin{aligned} & \int_{B_{R/2}} |Xu - Xw|^p dx \\ & = \int_{B_{R/2}} \left(|Xu - Xw|^2 \left(|Xu| + |Xw| \right)^{p-2} \right)^{\frac{p}{2}} \left(|Xu| + |Xw| \right)^{\frac{p(2-p)}{2}} dx \\ & \le c \left(\int_{B_{R/2}} |Xu - Xw|^2 \left(|Xu| + |Xw| \right)^{p-2} dx \right)^{\frac{p}{2}} \left(\int_{B_{R/2}} \left(|Xu| + |Xw| \right)^p dx \right)^{\frac{2-p}{2}} \\ & \le c \left(\int_{B_{R/2}} |Xu - Xw|^2 \left(|Xu|^2 + |Xw|^2 \right)^{\frac{p-2}{2}} dx \right)^{\frac{p}{2}} \left(\int_{B_{R/2}} |Xu|^p dx + R^Q \right)^{\frac{2-p}{2}} \\ & \le \sigma \left(\int_{B_{R/2}} |Xu|^p dx + R^Q \right) + c_\sigma \int_{B_{R/2}} |Xu - Xw|^2 \left(|Xu|^2 + |Xw|^2 \right)^{\frac{p-2}{2}} dx. \end{aligned} \tag{3.19}$$

To estimate the last term in (3.19), we apply the inequality (3.4) and the estimates of I-III and (3.17) in the case $p \geq 2$. It follows that

$$\begin{aligned}
&\int_{B_{R/2}} |Xu - Xw|^2 \left(|Xu|^2 + |Xw|^2\right)^{\frac{p-2}{2}} dx \\
&\leq c\int_{B_{R/2}} \langle \langle A_{R/2}Xu, Xu\rangle^{\frac{p-2}{2}} A_{R/2}Xu - \langle A_{R/2}Xw, Xw\rangle^{\frac{p-2}{2}} A_{R/2}Xw, Xu - Xw\rangle dx \\
&\leq 2\varepsilon \int_{B_{R/2}} |Xu - Xw|^p dx + c(\|A\|_{*,R/2}^{(t-p)/t} + \eta) \int_{B_R} |Xu|^p dx + cR^Q. \qquad (3.20)
\end{aligned}$$

Combining (3.20) with (3.19) and choosing ε small enough, we find that

$$\int_{B_{R/2}} |Xu - Xw|^p dx \leq c\left(\|A\|_{*,R/2}^{(t-p)/t} + \eta + \sigma\right) \int_{B_R} |Xu|^p dx + cR^Q. \qquad (3.21)$$

Now for $0 < \rho < R/2$, the desired estimate (3.6) follows from (3.21) and (3.10).

On the other hand, it is easy to see that for $R/2 \leq \rho \leq R$,

$$\int_{B_\rho} |Xu|^p dx \leq \int_{B_R} |Xu|^p dx \leq 2^Q \left(\frac{\rho}{R}\right)^Q \int_{B_R} |Xu|^p dx$$

and this finishes the proof of Lemma 7. □

Proof of Theorem 1 Let $u \in \mathcal{K}(\Omega)$ be a solution to the double obstacle problem for (1.1) with p close to 2 and let $B_R(x) \subset\subset \Omega$. It follows from Lemma 7 that for any $0 < \rho \leq R$ and $\eta, \sigma, \varepsilon > 0$,

$$\int_{B_\rho} |Xu|^p dx \leq c\left(\left(\frac{\rho}{R}\right)^Q + \vartheta\right) \int_{B_R} |Xu|^p dx + cR^Q,$$

where $\vartheta = \|A\|_{*,R/2}^{(t-p)/t} + \eta + \sigma + \varepsilon$. Since $A(x) \in \mathrm{VMO}(\Omega)$, we can choose R, η, σ, and ε so small that ϑ is small enough. By virtue of Lemma 3 we see that for any $0 \leq \alpha < 1$,

$$\int_{B_\rho} |Xu|^p dx \leq c\left(\frac{\rho}{R}\right)^{Q-p+p\alpha} \int_{B_R} |Xu|^p dx + c\rho^{Q-p+p\alpha},$$

which implies $Xu \in L^{p,\lambda}_{\mathrm{loc}}(\Omega)$, $\lambda = Q - p + p\alpha$. By Poincaré's inequality,

$$\int_{B_\rho} |u - u_\rho|^p dx \leq c\rho^p \int_{B_\rho} |Xu|^p dx \leq C\rho^{Q+p\alpha},$$

where C is is independent of x, ρ. By Lemma 2, we have

$$u \in C^{0,\alpha}_X(\Omega),\ \forall\ 0 \leq \alpha < 1.$$

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