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On Mahler's U_m-numbers in fields of formal power series over finite fields

by
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Abstract

Let K be a finite field, $K(x)$ be the quotient field of the ring of polynomials in x with coefficients in K and $\mathbb{K}$ be the field of formal power series over K. In this paper, we treat polynomials whose coefficients belong to a field extension of degree m over $K(x)$. We show that the values of these polynomials at certain U_1-numbers in the field $\mathbb{K}$ are U_m- numbers in $\mathbb{K}$.

Key Words: Mahler's classification of transcendental formal power series over a finite field, U-number, continued fraction, transcendence measure.
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1 Introduction

1.1 The field of formal power series over a finite field

Let K be a finite field with q elements. We denote the ring of all polynomials over K by $K[x]$, the quotient field of $K[x]$ by $K(x)$ and the degree of a non-zero polynomial $a(x)$ in $K[x]$ by $\deg(a)$. A non-Archimedean absolute value $|\cdot|$ is defined on $K(x)$ by

$$|0| = 0 \qquad \text{and} \qquad \left|\frac{a(x)}{b(x)}\right| = q^{\deg(a)-\deg(b)},$$

where $a(x)$ and $b(x)$ are non-zero polynomials in $K[x]$. The completion of $K(x)$ with respect to $|\cdot|$ is called the field of formal power series over K and is denoted by $\mathbb{K}$. The absolute value $|\cdot|$ is uniquely extended from $K(x)$ to $\mathbb{K}$ and denoted by the same notation $|\cdot|$. Every non-zero element ξ of $\mathbb{K}$ can be written uniquely as

$$\xi = \sum_{n=r}^{\infty} a_n x^{-n},$$

where $a_n \in K$ for $n = r, r+1, \ldots$ with $a_r \neq 0$ and r is the rational integer such that $|\xi| = q^{-r}$. The elements of $\mathbb{K}$ are called formal power series.

Let $P(y) = a_0 + a_1 y + \cdots + a_n y^n$ be a non-zero polynomial with coefficients in $K[x]$. The height $H(P)$ of $P(y)$ is defined as $H(P) = \max\{|a_0|, |a_1|, \ldots, |a_n|\}$ and the degree of $P(y)$ with respect to y is denoted by $\deg(P)$. An element ξ of $\mathbb{K}$ is called an algebraic formal power series if it is algebraic over $K(x)$ and ξ is called a transcendental formal power series otherwise. Let α be an algebraic formal power series and $P(y)$ be its minimal polynomial over $K[x]$. Then the height $H(\alpha)$ and the degree $\deg(\alpha)$ of α are defined by $H(P)$ and $\deg(P)$, respectively. Moreover, the roots of $P(y)$ are called the conjugates of α over $K(x)$. Throughout, by algebraic formal power series, we mean algebraic formal power series in $\mathbb{K}$.

1.2 Mahler's classification in $\mathbb{K}$

In 1932, Mahler [10] gave a classification of complex numbers and separated transcendental complex numbers into three disjoint classes called $S-$, $T-$ and $U-$numbers. (See Bugeaud [2] for detailed information about Mahler's classification of complex numbers.) In 1978, Bundschuh [3] introduced a classification similar to Mahler's classification and separated transcendental formal power series into three disjoint classes as follows.

Let ξ be a transcendental formal power series and let n and H be any positive rational integers. Set

$$w_n(H,\xi) = \min\big\{|P(\xi)| \,:\, P(y) \in K[x][y]\backslash\{0\},\, \deg(P) \le n, H(P) \le H\big\},$$

$$w_n(\xi) = \limsup_{H\to\infty} \frac{-\log w_n(H,\xi)}{\log H}, \qquad w(\xi) = \limsup_{n\to\infty} \frac{w_n(\xi)}{n}.$$

Bundschuh [3] proved that

$$w_n(H,\xi) < H^{-n} q^n \max\{1, |\xi|\}^n.$$

This gives us $w_n(\xi) \ge n$ for $n = 1, 2, ...$ and therefore $w(\xi) \ge 1$. If $w_n(\xi)$ is infinite for some integers n, then denote by $\nu(\xi)$ the smallest such integer. If $w_n(\xi)$ is finite for $n = 1, 2, \ldots$, put $\nu(\xi) = \infty$. Then ξ is called

- an $S-$number if $1 \le w(\xi) < \infty$ and $\nu(\xi) = \infty$,
- a $T-$number if $w(\xi) = \infty$ and $\nu(\xi) = \infty$,
- a $U-$number if $w(\xi) = \infty$ and $\nu(\xi) < \infty$.

Moreover, ξ is called a U_m-number if $\nu(\xi) = m$.

In 1980, in the field of formal power series $\mathbb{K}$, the first explicit examples of U_m-numbers were constructed by Oryan [14]. Recently, [4], [6], [7] and [8] contributed to constructing explicit examples of $U-$numbers in $\mathbb{K}$. Observe the field $\mathbb{K}$ is not algebraically closed.

1.3 Continued fractions in $\mathbb{K}$

As in the classical continued fraction theory of real numbers, any formal power series can be represented as a continued fraction. A formal power series ξ is in $K(x)$ if and only if its continued fraction expansion is finite. Let ξ be a formal power series in $\mathbb{K}\backslash K(x)$, then there is a unique representation of ξ as follows.

$$\xi = [b_0, b_1, b_2, \ldots] := b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots}}},$$

where $b_0, b_i \in K[x]$ with $|b_i| > 1$ for $i = 1, 2, \ldots$.

Define $p_{-1} = 1$, $p_0 = b_0$, $q_{-1} = 0$, $q_0 = 1$ and

$$p_n = b_n p_{n-1} + p_{n-2}, \qquad q_n = b_n q_{n-1} + q_{n-2} \qquad (n = 1, 2, \ldots).$$

By induction on n, it is easily seen that

$$\frac{p_n}{q_n} = [b_0, b_1, \ldots, b_n] := b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots + \cfrac{1}{b_n}}}}.$$

Moreover, by induction on n, we have the following properties:

(1) $\dfrac{\beta p_n + p_{n-1}}{\beta q_n + q_{n-1}} = [b_0, b_1, \ldots, b_n, \beta] \quad (n = 0, 1, 2, \ldots)$, where $\beta \in \mathbb{K}\backslash\{0\}$,

(2) $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \quad (n = 0, 1, 2, \ldots)$,

(3) $|q_n| > |q_{n-1}| \quad (n = 0, 1, 2, \ldots)$,

(4) $|q_n| = |b_1 b_2 \cdots b_n| \quad (n = 1, 2, \ldots)$,

(5) $|p_n| = |b_0 b_1 b_2 \cdots b_n| = |b_0||q_n| \quad (n = 0, 1, 2, \ldots)$,

(6) $\left|\xi - \dfrac{p_n}{q_n}\right| = \dfrac{1}{|b_{n+1}||q_n|^2} < \dfrac{1}{|q_n|^2} \quad (n = 1, 2, \ldots)$.

It follows from the properties (3) and (6) that $\lim_{n\to\infty} p_n/q_n = \xi$. Therefore, p_n/q_n ($n = 0, 1, 2, ...$) are called the convergents of the continued fraction expansion of ξ. The reader is directed to [5], [12] and [15] for detailed information, proofs and further results on continued fractions in $\mathbb{K}$.

1.4 Construction of our main results

In 1979, Alnıaçık [1, Chapter I, Theorem I, Theorem III] gave a method to construct explicit examples of complex U_m-numbers. In the present paper, in Theorem 1 and Theorem 2, we establish the following analogues of Theorem I and Theorem III of Alnıaçık [1, Chapter I] over $\mathbb{K}$, respectively.

Theorem 1. *Let $\alpha_0, \alpha_1, \ldots, \alpha_k$ ($k \geq 1, \alpha_k \neq 0$) be algebraic formal power series and m be the degree of the field extension $K(x)(\alpha_0, \ldots, \alpha_k)$ over $K(x)$. Let ξ be a U_1-number with convergents p_n/q_n ($n = 0, 1, 2, ...$). For $n \geq 0$, let ω_n be defined by the condition*

$$\left|\xi - \frac{p_n}{q_n}\right| = |q_n|^{-\omega_n} \ .$$

If $\liminf_{n\to\infty} \omega_n > km(m-1)[(km+1)(m-1)+2] + m + 1$, then $\alpha_0 + \alpha_1\xi + \cdots + \alpha_k\xi^k$ is a U_m-number.

Theorem 2. *Let $\alpha_0, \alpha_1, \ldots, \alpha_k$ ($k \geq 1, \alpha_k \neq 0$) be algebraic formal power series and m be the degree of the field extension $K(x)(\alpha_0, \ldots, \alpha_k)$ over $K(x)$. Let ξ be in $\mathbb{K}\backslash K(x)$ and $\{p_n/q_n\}_{n=0}^{\infty}$ be a sequence in $K(x)$ with $p_n, q_n \in K[x]$ and $|q_n| > 1$ such that the following conditions*

1. $\displaystyle\limsup_{n\to\infty} \frac{\log|q_{n+1}|}{\log|q_n|} = \infty$,

2. $\displaystyle\limsup_{n\to\infty} \frac{\log|q_{n+1}|}{\log\left|\xi - \frac{p_n}{q_n}\right|^{-1}} < \infty$

are satisfied. Then there exist a subsequence $\{p_{n_j}/q_{n_j}\}$ *such that* $\lim_{j\to\infty} p_{n_j}/q_{n_j} = \xi$ *and* ξ *is a* U_1-*number. Further,* $\alpha_0 + \alpha_1\xi + \cdots + \alpha_k\xi^k$ *is a* U_m-*number.*

We recommend the reader to see LeVeque [9] and compare it with Theorem 1. In the next section, we cite some results we need to prove Theorem 1 and Theorem 2. We prove Theorem 1 in Section 3 and Theorem 2 in Section 4.

2 Auxiliary Results

Lemma 1 (Müller [11], page 291)**.** *Let* α *be an algebraic formal power series and* $P(y)$ *be a non-zero polynomial in* y *with coefficients in* $K[x]$*. If* $P(\alpha) \neq 0$*, then*

$$|P(\alpha)| \geq H(P)^{1-\deg(\alpha)} H(\alpha)^{-\deg(P)}.$$

Lemma 2 (Ooto [13], Lemma 3.2)**.** *Let* α, β *be in* $\mathbb{K}$ *and* $P(y) = \alpha_0 + \alpha_1 y + \cdots + \alpha_k y^k \in \mathbb{K}[y]\,(\alpha_k \neq 0)$ *be a non-constant polynomial. Let* $C \geq 0$ *be a real number such that* $|\alpha - \beta| \leq C$*. Then*

$$|P(\alpha) - P(\beta)| \leq \max_{i=1,\ldots,k}\{C, |\alpha|\}^{i-1}|\alpha - \beta| \max_{i=1,\ldots,k}\{|\alpha_i|\}.$$

Theorem 3 (Can and Kekeç [4], Theorem 1.2)**.** *Let* L *be a finite extension of degree* m *over* $K(x)$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *be in* L*. Let* η *be an algebraic formal power series. Assume that* $F(\eta, \alpha_1, \ldots, \alpha_k) = 0$*, where* $F(y, y_1, \ldots, y_k)$ *is a polynomial in* $y, y_1, \ldots, y_k$ *over* $K[x]$ *with degree at least 1 in* y*. Then*

$$\deg(\eta) \leq dm$$

and

$$H(\eta) \leq H^m H(\alpha_1)^{l_1 m} \cdots H(\alpha_k)^{l_k m},$$

where d *is the degree of* $F(y, y_1, \ldots, y_k)$ *in* y*,* l_j *is the degree of* $F(y, y_1, \ldots, y_k)$ *in* y_j $(j = 1, \ldots, k)$ *and* H *is the maximum of the absolute values of the coefficients of* $F(y, y_1, \ldots, y_k)$*.*

Lemma 3 (Kekeç [8], Lemma 2.1)**.** *Let* $\alpha_0, \ldots, \alpha_k\,(k \geq 1, \alpha_k \neq 0)$ *be algebraic formal power series. Then for* $\theta \in K(x)$ *the algebraic formal power series* $\alpha_0 + \alpha_1\theta + \cdots + \alpha_k\theta^k$ *is a primitive element of* $K(x)(\alpha_0, \ldots, \alpha_k)$ *over* $K(x)$ *except for only finitely many* $\theta \in K(x)$*.*

3 Proof of Theorem 1

We prove Theorem 1 by adapting the method of the proof of Alnıaçık [1, Chapter I, Theorem I] to the field $\mathbb{K}$.

By the assumption of the theorem,

$$\left|\xi - \frac{p_n}{q_n}\right| = |q_n|^{-\omega_n} \leq 1 \qquad (n = 0, 1, \ldots). \tag{3.1}$$

We apply Lemma 2 with

$$P(y) = C(y) := \alpha_0 + \alpha_1 y + \cdots + \alpha_k y^k \in \mathbb{K}[y], \;\; \alpha = \xi, \;\; \beta = \frac{p_n}{q_n} \qquad (n = 0, 1, \ldots)$$

and get

$$\left|C(\xi) - C\left(\frac{p_n}{q_n}\right)\right| \leq c_1 \left|\xi - \frac{p_n}{q_n}\right|, \tag{3.2}$$

where $c_1 = \max_{i=1,\ldots,k}\{1, |\xi|\}^{i-1} \max_{i=1,\ldots,k}\{|\alpha_i|\}$. Since $\lim_{n\to\infty} p_n/q_n = \xi$, there exists a positive integer n_0 such that for all $n \geq n_0$

$$\left|\frac{p_n}{q_n}\right| < 2|\xi|. \tag{3.3}$$

Let $P_n(y)$ be the minimal polynomial of $C(p_n/q_n)$ over $K[x]$ for $n > n_0$. By Lemma 3, there exists a positive integer n_1 with $n_1 > n_0$ such that $\deg(C(p_n/q_n)) = \deg(P_n) = m$ holds for all $n \geq n_1$. It follows from (3.1) and (3.2) that

$$\left|C(\xi) - C\left(\frac{p_n}{q_n}\right)\right| \leq c_1.$$

We apply Lemma 2 with

$$P(y) = P_n(y) \in K[x][y], \;\; \alpha = C(\xi), \;\; \beta = C\left(\frac{p_n}{q_n}\right) \qquad (n \geq n_1)$$

and get

$$|P_n(C(\xi)) - P_n(C\left(\frac{p_n}{q_n}\right))| \leq c_2 \left|C(\xi) - C\left(\frac{p_n}{q_n}\right)\right| H(P_n) \qquad (n \geq n_1), \tag{3.4}$$

where $c_2 = max_{i=1,\ldots,m}\{c_1, |C(\xi)|\}^{i-1}$. Note that $P_n(C(p_n/q_n)) = 0$. Using (3.2) in (3.4),

$$|P_n(C(\xi))| \leq c_3 H(P_n)|q_n|^{-\omega_n} \qquad (n \geq n_1), \tag{3.5}$$

where $c_3 = c_2 c_1$. We will give an upper bound for $H(P_n)$. Put

$$\gamma_n := C\left(\frac{p_n}{q_n}\right) \qquad (n \geq n_1).$$

Then the polynomial

$$F(y, y_0, y_1, \ldots, y_k) = q_n^k y - q_n^k y_0 - p_n q_n^{k-1} y_1 - \cdots - p_n^k y_k$$

is zero for $y = \gamma_n$ and $y_i = \alpha_i \, (i = 0, \ldots, k)$. From (3.3),

$$H \leq |q_n|^k \max\{1, (2|\xi|)^k\}, \tag{3.6}$$

where H is the maximum of the absolute values of the coefficients of $F(y, y_0, y_1, \ldots, y_k)$. We apply Theorem 3 with $\eta = \gamma_n, d = 1, l_i = 1 \, (i = 0, \ldots, k)$ and $F(y, y_0, y_1, \ldots, y_k)$ and get

$$H(\gamma_n) \leq H^m H(\alpha_0)^m H(\alpha_1)^m \cdots H(\alpha_k)^m.$$

Using (3.6) and the fact that $H(\gamma_n) = H(P_n)$ in the inequality above, we obtain

$$H(P_n) \leq c_4 |q_n|^{km} \qquad (n \geq n_1), \tag{3.7}$$

where $c_4 = (\max\{1, (2|\xi|)^k\}H(\alpha_0)\cdots H(\alpha_k))^m$. Since $\lim_{n\to\infty}|q_n| = \infty$, there exists a positive integer n_2 with $n_2 > n_1$ such that

$$H(P_n) \le |q_n|^{km+1} \tag{3.8}$$

holds for all $n \ge n_2$. Combining (3.5) and (3.8),

$$0 < |P_n(C(\xi))| \le \frac{c_3}{H(P_n)^{\frac{\omega_n}{km+1}-1}} \qquad (n \ge n_2). \tag{3.9}$$

Note that $P_n(C(\xi)) \ne 0$ as ξ is transcendental over $K(x)$. Since $\limsup_{n\to\infty}\omega_n = \infty$, we have a subsequence $\{\omega_{n_j}\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty}\omega_{n_j} = \infty$. We infer from (3.9) that

$$0 < |P_{n_j}(C(\xi))| \le c_3 H(P_{n_j})^{-\theta_{n_j}} \tag{3.10}$$

for sufficiently large j, where

$$\theta_{n_j} = \frac{\omega_{n_j}}{km+1} - 1, \qquad \lim_{j\to\infty}\theta_{n_j} = \infty.$$

The sequence $\{H(P_{n_j})\}$ is not bounded from above and so it has a subsequence $\{H(P_{n_{j_t}})\}_{t=1}^{\infty}$ such that

$$1 < H(P_{n_{j_1}}) < H(P_{n_{j_2}}) < H(P_{n_{j_3}}) < \cdots, \qquad \lim_{t\to\infty} H(P_{n_{j_t}}) = \infty.$$

Since $\deg(P_{n_j}) = m$, (3.10) implies that $C(\xi)$ is a $U-$number with

$$\nu(C(\xi)) \le m. \tag{3.11}$$

Now we wish to show that $\nu(C(\xi)) \ge m$ to complete the proof. If $m = 1$, then $\nu(C(\xi)) = m$. Let $m > 1$ and $P(y)$ be a polynomial over $K[x]$ with $1 \le \deg(P) \le m-1$ and with sufficiently large height $H(P)$. Similar to (3.4), we apply Lemma 2 with $P(y)$ and get

$$|P(C(\xi)) - P(\gamma_n)| \le c_2|C(\xi) - \gamma_n|H(P) \qquad (n \ge n_2). \tag{3.12}$$

Using (3.1) and (3.2) in (3.12),

$$|P(C(\xi))| \ge |P(\gamma_n)| - c_3|q_n|^{-\omega_n}H(P) \qquad (n \ge n_2). \tag{3.13}$$

Since $\deg(\gamma_n) = m\,(n \ge n_2)$ and $\deg(P) < m$, it follows that $P(\gamma_n) \ne 0\,(n \ge n_2)$. Hence we apply Lemma 1 with $\alpha = \gamma_n$ $(n \ge n_2)$ and get

$$|P(\gamma_n)| \ge H(P)^{1-m}H(\gamma_n)^{1-m} \;\; (n \ge n_2). \tag{3.14}$$

Using (3.7) and the fact that $H(P_n) = H(\gamma_n)$ in (3.14),

$$|P(\gamma_n)| \ge \frac{c_5}{H(P)^{m-1}|q_n|^{km(m-1)}} \qquad (n \ge n_2),$$

where $c_5 = c_4^{1-m}$. Combining this with (3.13),

$$|P(C(\xi))| \ge \frac{c_5}{H(P)^{m-1}|q_n|^{km(m-1)}} - \frac{c_3 H(P)}{|q_n|^{\omega_n}} \qquad (n \ge n_2). \tag{3.15}$$

It follows from $|\xi - p_n/q_n| = |q_n|^{-\omega_n}$ and the properties of continued fractions in $\mathbb{K}$ that

$$|q_n|^{\omega_n} = |q_{n+1}||q_n|. \tag{3.16}$$

By the assumption of the theorem, there exists a positive integer n_3 with $n_3 > n_2$ such that

$$\omega_n > km(m-1)[(km+1)(m-1)+2] + m + 1 \qquad (n \geq n_3). \tag{3.17}$$

Let v be the unique positive integer satisfying $|q_v| \leq H(P) < |q_{v+1}|$. It follows from (3.16) and (3.17) that

$$|q_v| < |q_{v+1}|^{\frac{1}{(km+1)(m-1)+2}} \qquad (v \geq n_3).$$

If $|q_v| \leq H(P) < |q_{v+1}|^{1/[(km+1)(m-1)+2]}$ holds, then we take $n = v$ in (3.15), (3.16) and get

$$|P(C(\xi))| \geq \frac{c_5}{H(P)^{(km+1)(m-1)}} - \frac{c_3}{H(P)^{(km+1)(m-1)+1}}.$$

Since $H(P)$ is sufficiently large, we have $H(P) > 2c_3/c_5$. This implies together with the inequality above that

$$|P(C(\xi))| \geq \frac{c_5}{2H(P)^{(km+1)(m-1)}}. \tag{3.18}$$

If $|q_{v+1}|^{1/[(km+1)(m-1)+2]} \leq H(P) < |q_{v+1}|$ holds, then we take $n = v+1$ in (3.15) and get

$$|P(C(\xi))| \geq \frac{c_5}{H(P)^{km(m-1)[(km+1)(m-1)+2]+m-1}} - \frac{c_3}{H(P)^{\omega_{v+1}-1}}.$$

Using (3.17) and $H(P) > 2c_3/c_5$ in the inequality above, we obtain

$$|P(C(\xi))| \geq \frac{c_5}{2H(P)^{km(m-1)[(km+1)(m-1)+2]+m-1}}. \tag{3.19}$$

We infer from (3.18) and (3.19) that

$$|P(C(\xi))| \geq \frac{c_5}{2H(P)^{km(m-1)[(km+1)(m-1)+2]+m-1}}$$

holds for all polynomials $P(y) \in K[x][y]$ with $1 \leq \deg(P) \leq m-1$ and with sufficiently large height $H(P)$. This gives us

$$\nu(C(\xi)) \geq m. \tag{3.20}$$

We deduce from (3.11) and (3.20) that $\nu(C(\xi)) = m$. Thus, $C(\xi)$ is a U_m−number in $\mathbb{K}$.

4 Proof of Theorem 2

We prove Theorem 2 by adapting the method of the proof of Alnıaçık [1, Chapter I, Theorem III] to the field $\mathbb{K}$.

From the conditions of the theorem, since there exist a subsequence $\{\log|q_{n_j+1}|/\log|q_{n_j}|\}$ such that $\lim_{j\to\infty} \log|q_{n_j+1}|/\log|q_{n_j}| = \infty$, we have

$$\lim_{j\to\infty} |q_{n_j}| = \infty, \qquad \lim_{j\to\infty} \frac{\log\left|\xi - \frac{p_{n_j}}{q_{n_j}}\right|^{-1}}{\log|q_{n_j}|} = \infty. \tag{4.1}$$

This implies that $\lim_{j\to\infty} p_{n_j}/q_{n_j} = \xi$ and ξ is a U_1-number in $\mathbb{K}$.

To prove the last assertion of the theorem, we put $C(y) = \alpha_0 + \alpha_1 y + \cdots + \alpha_k y^k$ and will first show that $\nu(C(\xi)) \le m$. Set

$$\left|\xi - \frac{p_{n_j}}{q_{n_j}}\right| = |q_{n_j}|^{-\omega_{n_j}}.$$

From (4.1), we have

$$\lim_{j\to\infty} \omega_{n_j} = \infty. \tag{4.2}$$

Put

$$\gamma_{n_j} := C\left(\frac{p_{n_j}}{q_{n_j}}\right) \in \mathbb{K}.$$

Let $P_{n_j}(y)$ be the minimal polynomial of γ_{n_j} over $K[x]$. By Lemma 3, $\deg(\gamma_{n_j}) = \deg(P_{n_j}) = m$ holds for sufficiently large j. We apply Lemma 2 with

$$P(y) = C(y) \in \mathbb{K}[y],\ \ \alpha = \xi,\ \ \beta = \frac{p_{n_j}}{q_{n_j}} \qquad (j = 0, 1, \dots).$$

Using the steps between (3.2) and (3.9) in the proof of Theorem 1, we get

$$0 < |P_{n_j}(C(\xi))| \le \frac{c_6}{H(P_{n_j})^{\frac{\omega_{n_j}}{km+1}-1}} \qquad (j \ge N_1), \tag{4.3}$$

where N_1 is a positive rational integer and c_6 is a real constant. As in the proof of Theorem 1, we infer from (4.2) and (4.3) that $C(\xi)$ is a $U-$number with

$$\nu(C(\xi)) \le m. \tag{4.4}$$

Now we wish to show that $\nu(C(\xi)) \ge m$. If $m = 1$, then $\nu(C(\xi)) = m$. Assume that $m > 1$. As in the steps between (3.2) and (3.7) in the proof of Theorem 1, there exist positive real constants c_7 and c_8 such that

$$\left|C(\xi) - C\left(\frac{p_{n_j}}{q_{n_j}}\right)\right| \le c_7 \left|\xi - \frac{p_{n_j}}{q_{n_j}}\right| \qquad (j \ge N_2) \tag{4.5}$$

and

$$H(P_{n_j}) \le c_8 |q_{n_j}|^{km} \qquad (j \ge N_2), \tag{4.6}$$

where N_2 is a positive integer with $N_2 > N_1$. Let $P(y)$ be a polynomial over $K[x]$ with $1 \le \deg(P) \le m-1$ and with sufficiently large height $H(P)$. As in (3.12), we apply Lemma 2 and get

$$|P(C(\xi)) - P(\gamma_{n_j})| \le c_9 |C(\xi) - \gamma_{n_j}| H(P) \qquad (j \ge N_2).$$

Using (4.5) in the inequality above,

$$|P(C(\xi))| \ge |P(\gamma_{n_j})| - c_{10} \left|\xi - \frac{p_{n_j}}{q_{n_j}}\right| H(P) \qquad (j \ge N_2), \tag{4.7}$$

where $c_{10} = c_7 c_9$. Put

$$\lambda := \limsup_{n\to\infty} \frac{\log|q_{n+1}|}{\log\left|\xi - \frac{p_n}{q_n}\right|^{-1}}.$$

From the second condition of the theorem, λ is a non-negative real number. There exists a positive integer t such that $t > \lambda$. Then

$$\frac{\log|q_{n_j+1}|}{\log\left|\xi - \frac{p_{n_j}}{q_{n_j}}\right|^{-1}} < t \qquad (j \geq N_3),$$

where N_3 is a positive integer with $N_3 > N_2$. So we have

$$\left|\xi - \frac{p_{n_j}}{q_{n_j}}\right| < \frac{1}{|q_{n_j+1}|^{1/t}} \qquad (j \geq N_3). \tag{4.8}$$

Combining (4.7) and (4.8),

$$|P(C(\xi))| \geq |P(\gamma_{n_j})| - \frac{c_{10}H(P)}{|q_{n_j+1}|^{1/t}} \qquad (j \geq N_3). \tag{4.9}$$

Since $\deg(\gamma_{n_j}) = m\,(j \geq N_3)$ and $\deg(P) < m$, it follows that $P(\gamma_{n_j}) \neq 0\,(j \geq N_3)$. Hence we apply Lemma 1 and get

$$|P(\gamma_{n_j})| \geq H(P)^{1-m} H(P_{n_j})^{1-m} \qquad (j \geq N_3). \tag{4.10}$$

Combining (4.6), (4.9) and (4.10),

$$|P(C(\xi))| \geq \frac{c_{11}}{H(P)^{m-1}|q_{n_j}|^{km(m-1)}} - \frac{c_{10}H(P)}{|q_{n_j+1}|^{1/t}} \qquad (j \geq N_3), \tag{4.11}$$

where $c_{11} = c_8^{1-m}$. Since $\lim_{j\to\infty} \frac{\log|q_{n_j+1}|}{\log|q_{n_j}|} = \infty$, there exists a positive integer N_4 with $N_4 > N_3$ such that

$$\frac{\log|q_{n_j+1}|}{\log|q_{n_j}|} > \mu \tag{4.12}$$

holds for $j \geq N_4$, where $\mu = km(m-1)[(km+1)(m-1)+2]t^2 + (m+1)t$. Let v be the unique positive integer satisfying

$$|q_{n_v}| \leq H(P) < |q_{n_v+1}|.$$

If $|q_{n_v}| \leq H(P) < |q_{n_v+1}|^{1/t[(km+1)(m-1)+2]}$ holds, then we take $n_j = n_v$ in (4.11) and get

$$|P(C(\xi))| \geq \frac{c_{11}}{2H(P)^{km(m-1)+m-1}}. \tag{4.13}$$

If $|q_{n_v+1}|^{1/t[(km+1)(m-1)+2]} \leq H(P) < |q_{n_v+1}|$ holds, we take $n_j = n_v + 1$ in (4.11) and get

$$|P(C(\xi))| \geq \frac{c_{11}}{H(P)^{km(m-1)t[(km+1)(m-1)+2]+m-1}} - \frac{c_{10}H(P)}{|q_{n_v+2}|^{1/t}}. \tag{4.14}$$

Now we take $n_j = n_v + 1$ in (4.12) and get

$$|q_{n_v+2}|^{1/t} > |q_{n_v+1}|^{km(m-1)t[(km+1)(m-1)+2]+m+1}.$$

Using the inequality $H(P) < |q_{n_v+1}|$ in the inequality above, we obtain

$$|q_{n_v+2}|^{1/t} > H(P)^{km(m-1)t[(km+1)(m-1)+2]+m+1}. \tag{4.15}$$

Combining (4.14) and (4.15),

$$|P(C(\xi))| > \frac{c_{11}}{2H(P)^{km(m-1)t[(km+1)(m-1)+2]+m-1}}. \tag{4.16}$$

We infer from (4.13) and (4.16) that

$$|P(C(\xi))| > \frac{c_{11}}{2H(P)^{km(m-1)t[(km+1)(m-1)+2]+m-1}}$$

holds for all polynomials $P(y) \in K[x][y]$ with $1 \leq \deg(P) \leq m-1$ and with sufficiently large height $H(P)$. This gives us

$$\nu(C(\xi)) \geq m. \tag{4.17}$$

We deduce from (4.4) and (4.17) that $\nu(C(\xi)) = m$. Thus, $C(\xi)$ is a U_m-number in $\mathbb{K}$.

We give the following example to illustrate Theorem 2.

Example 1. *In Theorem 2, let us take*

$$\xi = \sum_{i=0}^{\infty} x^{-3^{i!}}, \quad p_n = x^{3^{n!}} \sum_{i=0}^{n} x^{-3^{i!}} \quad \textit{and} \quad q_n = x^{3^{n!}}\ (n = 0, 1, \dots).$$

Then, by Theorem 2, ξ is a U_1-number and $1 + \xi + \cdots + \xi^{k-1} + \sqrt[m]{x}\xi^k$ is a U_m-number, where m is any positive rational integer, 1 denotes the identity element of K and $\sqrt[m]{x}$ is defined as a root of the polynomial $y^m - x$.

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