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Some degree sequences are determined by Laplacian spectra of the corresponding graphs

by
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Abstract

Let $\mathcal{G}(\Delta^{n_\Delta},\ldots,2^{n_2},1^{n_1},0^{n_0})$ be the set of simple graphs with the degree sequence $\deg(\Delta^{n_\Delta},\ldots,2^{n_2},1^{n_1},0^{n_0})$, and $\mathcal{G}^*(\Delta^{n_\Delta},\ldots,2^{n_2},1^{n_1})$ the set of connected graphs with the degree sequence $\deg(\Delta^{n_\Delta},\ \ldots,2^{n_2},1^{n_1})$. In this paper, we first give the numbers of spanning tree of a bicyclic graph as well as a tricyclic graph, and then show that some degree sequences of graphs in $\mathcal{G}(2^{n_2},1^{n_1},0^{n_0})$ (resp. $\mathcal{G}^*(3^{n_3},2^{n_2},1^{n_1})$, $\mathcal{G}^*(4^{n_4},2^{n_2},1^{n_1})$ and $\mathcal{G}^*(4^{n_4},3^{n_3},2^{n_2},1^{n_1})$) are determined by Laplacian spectra (write as *DLS* for short) of the corresponding graphs. Moreover, for the non-*DLS* degree sequences we present some L-cospectral mates to indicate that their L-cospectral degree sequences do exist. By the way, all of these extend the previous results about Laplacian spectral determinations of some degree sequences in [23]. Besides, we revise the references of Theorems 6 and 7 in [23].

Key Words: Spanning tree, Laplacian spectrum, determined by Laplacian spectrum, degree sequence, cospectral graph.
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1 Introduction

Throughout this paper, all graphs considered are undirected, finite and simple. Let G be a graph with vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ and edge set $E(G)=\{e_1,e_2,\ldots,e_m\}$, where $|V(G)|=n$ and $|E(G)|=m$. If $\emptyset\neq E'(G)\subseteq E(G)$, then we say that $G[E'(G)]$ is an *edge-induced subgraph* of G. Often, we denote by $d_G(v_i)$ and $N_G(v_i)$ the degree and the neighbor set of a vertex v_i in G, respectively. Let $A(G)$ be the *adjacency matrix*, and $D(G)$ the *diagonal degree matrix* of G. The *Laplacian matrix* and the *signless Laplacian matrix* of G are defined as $L(G)=D(G)-A(G)$ and $Q(G)=D(G)+A(G)$, respectively. The *L-spectrum* of G, denoted by $Spec_L(G)$, is a multiset consisting of the L-eigenvalues together with their multiplicities. Conventionally, the *Laplacian eigenvalues* and the *signless Laplacian eigenvalues* of graph G are ordered respectively in non-increased sequence as follows: $\mu_1(G)\geq\mu_2(G)\geq\cdots\geq\mu_{n-1}(G)\geq\mu_n(G)=0$ and $q_1(G)\geq q_2(G)\geq\cdots\geq q_{n-1}(G)\geq q_n(G)=0$.

A sequence $\mathbf{d}=(\Delta^{n_\Delta},\ldots,k^{n_k},\ldots,1^{n_1},0^{n_0})$ is *graphic* [1] if there is a simple graph with the degree sequence $\mathbf{d}$. We denote by $deg(G)=(\Delta^{n_\Delta},\ldots,k^{n_k},\ldots,1^{n_1},0^{n_0})$ the degree sequence of G, where $n_0+n_1+\ldots+n_\Delta=n$ and k^{n_k} means that G has n_k vertices with degree k. Besides, we also express the degree sequence of G in non-increasing order of nonnegative integers, i.e., $d_1(G)\geq d_2(G)\geq\cdots\geq d_n(G)$, where $d_G(v_i)=d_i(G)$ for $i\in\{1,2,\ldots,n\}$. For a given graphic sequence $(\Delta^{n_\Delta},\ldots,k^{n_k},\ldots,1^{n_1},0^{n_0})$, let $\mathcal{G}(\Delta^{n_\Delta},\ldots,2^{n_2},1^{n_1},0^{n_0})$ be the set of simple graphs with the degree sequence $\deg(\Delta^{n_\Delta}, \ldots, 2^{n_2}, 1^{n_1}, 0^{n_0})$, and $\mathcal{G}^*(\Delta^{n_\Delta}, \ldots, 2^{n_2}, 1^{n_1})$ the set of connected graphs with the degree sequence $\deg(\Delta^{n_\Delta}, \ldots, 2^{n_2}, 1^{n_1})$. In addition, if there is no risk of confusion, we simplify $d_G(v_i)$, $d_i(G)$, $N_G(v_i)$, $\mu_i(G)$ and $q_i(G)$ as $d(v_i)$, d_i, $N(v_i)$, μ_i and q_i, respectively.

As usual, $K_{1,n-1}$, P_n, C_n and K_n always denote the *star*, *path*, *cycle* and *complete graph* with n vertices, respectively. A connected graph G with n vertices and $n+c-1$ edges is called a *c-cyclic graph*. Such a c is called *cyclomatic number* of G and denoted by $c(G)$. A *p-rose graph* is a graph with p (≥ 2) cycles that all cycles meet in one vertex. Let $R(s,t)$ stand for the 2-rose graph (called ∞-graph also) with C_s and C_t where $n = s+t-1$.

Two graphs are called to be *L-cospectral* if they have the same Laplacian spectrum. Especially, if $H \ncong G$, in this case H is said to be a *L-cospectral mate* of G. A graph G is called to be determined by its (signless) Laplacian spectrum, simplified as (resp. *DQS*) *DLS* if there does not exist other non-isomorphic graph H such that H and G are (resp. Q-) L-cospectral. Similarly, a degree sequence of a graph G is *determined by the (signless) Laplacian spectrum* if any graph H is (resp. Q-) L-cospectral with G such that they have the same degree sequence.

Which graphs are determined by their spectra seems to be a difficult problem in the theory of graph spectra. In particular, the Laplacian spectral determination of graphs is an important branch and it had drawn more and more attention. In 2003, van Dam et al.[5] reconsidered this question and showed that some graphs such as C_n and P_n are *DLS*. Up to now, many graphs have been proved to be determined by their L-spectra. Wen et al.[22] proved that the *wind-wheel graphs* are *DLS* as well as *DQS*, and then, Liu et al.[14] gave a Laplacian spectral characterization of the *butterfly-like graphs*. In addition, we refer the readers to [10, 13, 15] and the references therein.

As we all know, in order to prove that a given graph G is *DLS*, we first need to characterize the degree sequence of any graph H that is L-cospectral G, and then prove that G and H are isomorphic. Thus, *considering the Laplacian spectral determinations of the graphic degree sequences is an interesting problem for the Laplacian spectral characterization of a graph.* In 2007, Zhang et al.[24] considered the degree sequence determined by Q-spectrum, and obtained the spectral certainty of degree sequence with some restrictions. After then, Liu et al. in [17] proved that except for two exceptions, the degree sequence of a connected graph G with $d_2(G) = 2$ is determined by Laplacian spectra of G. Wen et al. in [23] investigated the Laplacian spectral determinations of the degree sequences of $\mathcal{G}^*(3^{n_3}, 2^{n_2}, 1^{n_1})$ and $\mathcal{G}^*(4^{n_4}, 2^{n_2}, 1^{n_1})$, and independently of Liu et al. [17] proved that any graph $G \in \mathcal{G}^*(\Delta^1, 2^{n_2}, 1^{n_1})$ $(\Delta \geq 3)$, whose degree sequence is determined by Laplacian spectrum except that G is a bicyclic graph with $\Delta = 4$.

Motivated above, in this paper, we focus to *consider which degree sequences are determined by the Laplacian spectra of the corresponding graphs.* Firstly, we give the numbers of spanning tree of a bicyclic graph and a tricyclic graph, respectively. Next, we show that some degree sequences of graphs in $\mathcal{G}(2^{n_2}, 1^{n_1}, 0^{n_0})$ (resp. $\mathcal{G}^*(3^{n_3}, 2^{n_2}, 1^{n_1})$, $\mathcal{G}^*(4^{n_4}, 2^{n_2}, 1^{n_1})$ and $\mathcal{G}^*(4^{n_4}, 3^{n_3}, 2^{n_2}, 1^{n_1})$) are determined by Laplacian spectra of the corresponding graphs. Moreover, for the non-*DLS* degree sequences we present some L-cospectral mates to indicate that their L-cospectral degree sequences do exist. By the way, all of the above extend the previous results about Laplacian spectral determinations of some degree sequences in [23]. Besides, we revise the references of Theorems 6 and 7, which were recalled as preparative knowledge in [23].

2 Preliminaries

Lemma 1 ([5], Lemma 4; [18], Theorem 3.1). *From the Laplacian matrix of a graph, one can obtain the following cospectral invariants by its Laplacian spectrum.*

(i) the number of vertices;

(ii) the number of edges;

(iii) the number of spanning trees;

(iv) the number of components;

(v) the sum of the squares of degrees of vertices.

Lemma 2 ([3], Corollary 4.3; [13], Lemma 5.1). *Let G be a graph with n vertices, m edges, and t triangles. Then*

$$\sum_{i=1}^{n}\mu_i^3=\sum_{i=1}^{n}d_i^3+3\sum_{i=1}^{n}d_i^2-6t,\qquad \sum_{i=1}^{n}q_i^3=\sum_{i=1}^{n}d_i^3+3\sum_{i=1}^{n}d_i^2+6t.$$

Based on Lemma 1 and the first equation of Lemma 2, Liu and Huang in [12] gave a Laplacian cospectral invariant

$$\varepsilon(G)=6t-\sum_{i=1}^{n}(d_i-2)^3.$$

Lemma 3 ([12], Lemma 3.3). *Let G and H be two graphs with $deg(G)=(d_1,d_2,\ldots,d_n)$, $deg(H)=(d_1',d_2',\ldots,d_n')$, and triangles t and t', respectively. If G and H are L-cospectral, then $\varepsilon(G)=\varepsilon(H)$.*

Lemma 4 ([3], Proposition 2.1, Theorem 4.7; [8], Corollary 2). *If G is a connected graph with n vertices and at least one edge, then*

$$d_1+1\le\mu_1(G)\le q_1(G)\le d_1+d_2$$

where the first equality holds if and only if $d_1=n-1$, the second equality holds if and only if G is bipartite, and the third equality holds if and only if G is regular or $G\cong K_{1,n-1}$.

Lemma 5 ([19], Lemma 5.5). *Let G be a c-cyclic graph. If $c\in\{3,4\}$, then $t(G)\le c+1$, where $t(G)$ is the number of triangles in G.*

Lemma 6 ([8], Corollary 2). *Let G be a graph containing at least one edge. Then $\mu_1(G)\ge d_1(G)+1$, the equality holds if and only if $d_1(G)=n-1$.*

For convenience, we denote by $\mathcal{H}_1=\{P_l,C_{2l+1}|l\ge 1\}$ and $\mathcal{H}_2=\{K_{1,3},K_{1,3}+e,K_4-e,K_4,C_{2k}(k\ge 2)\}$.

Lemma 7 ([21], Corollary 2.3). *Let G be a graph of order n. Then*

(a) $\mu_1(G)<4$ *if and only if G is disjoint union of graphs in $\mathcal{H}_1$; and*

(b) $\mu_1(G) = 4$ *if and only if* G *is disjoint union of graphs in* $\mathcal{H}_1 \cup \mathcal{H}_2$ *with at least one component in* $\mathcal{H}_2$.

Lemma 8 (See [5], Propositions 1 and 5; [15], Lemma 2.5)**.** *For any* $n \geq 3$, C_n, P_n *and* $K_{1,n-1}$ *are* DLS.

Lemma 9. *Let* G *be a connected graph with* $deg(G) = (4^2, 2^{n_2}, 1^{n_1})$. *Then* $t(G) \leq 3$, *and the equality holds if only and if* $G \in \{G_1, G_2\}$, *where* $t(G)$ *is the number of triangles in* G.

Proof. Let G be a graph of $\mathcal{G}^*(4^2, 2^{n_2}, 1^{n_1})$. To attain the maximum number of triangles of G, we first assume that there are three vertices, v_1, v_2 and v_3 (say) forming a triangle $v_1v_2v_3$ in G. Since G has no degree-3 vertex and $\Delta(G) = 4$, there must be two vertices v_4 and v_5 (say), incident with one vertex of the triangle $v_1v_2v_3$. Without loss of generality, we suppose that v_3 is the vertex associated with v_4 and v_5.

Case 1. $v_4 \sim v_5$.

One can see that both v_4 and v_5 are non-adjacent to any vertex of v_1 and v_2 due to $n_3 = 0$. Noticing that $n_4 = 2$, so we claim that one of the vertices v_1, v_2, v_4 and v_5 must be the another degree-4 vertex since if not, there exists a degree-4 vertex on another branch of G, which is linked by a path as G is a connected graph, and so, it will always produce a degree-3 vertex in G, a contradiction. Therefore, we may suppose that v_4 is just the degree-4 vertex, and v_6 and v_7 are the other two adjacent vertices. To maximize the number of triangles in G, v_6 and v_7 should be adjacent and they have no other neighbors in G. Hence $G \cong G_1$ (see Fig. 1), and here $t(G) = 3$.

Case 2. $v_4 \not\sim v_5$.

If $N(v_4) \cap \{v_1, v_2\} = \emptyset$ and $N(v_5) \cap \{v_1, v_2\} = \emptyset$, there is just one vertices of v_1, v_2, v_4 and v_5 appends at most one triangle since G has no degree-3 vertices, and so, $t(G) \leq 2$; if $N(v_4) \cap \{v_1, v_2\} \neq \emptyset$ and $N(v_5) \cap \{v_1, v_2\} = \emptyset$ (say), then we claim that v_4 is possibly adjacent to one of the vertices v_1 and v_2, since if not, G must be contained a vertex with degree 3, which leads to a contradiction, and thus $t(G) = 2$; otherwise, v_4 and v_5 should be adjacent to the vertex in $\{v_1, v_2\}$. We notice that $deg(G) = (4^2, 2^{n_2}, 1^{n_1})$, hence, v_4 and v_5 are only adjacent to one of v_1 and v_2, simultaneously. Thus, $G \cong G_2$ (see Fig. 1), and here $t(G) = 3$.

Therefore, we have $t(G) \leq 3$ for any $G \in \mathcal{G}^*(4^2, 2^{n_2}, 1^{n_1})$, and $t(G) = 3$ if only and if $G \in \{G_1, G_2\}$. □

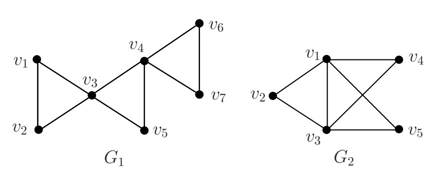


Figure 1: Graphs G_1 and G_2

Corollary 1. *Let* $G \in \mathcal{G}^*(4^2, 2^{n_2}, 1^{n_1})$ *and* $n_2 > 0$. *If* $t(G) = 3$, *then* G *is isomorphic to either* G_1 *or* G_2; *if* $t(G) = 2$, *then* G *is one of* $\mathcal{B}(4^2, 2^{n_2}, 1^{n_1})$ *shown in Fig. 2.*

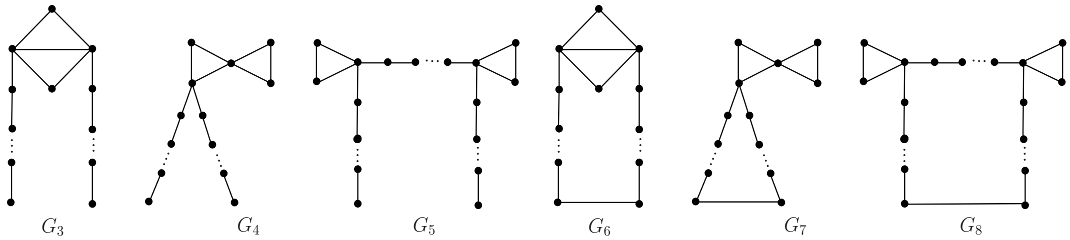


Figure 2: Graphs in $\mathcal{B}(4^2, 2^{n_2}, 1^{n_1})$

Lemma 10. *Let G be a bicyclic graph, and $\tau(G)$ the number of spanning trees of G. Let l_1 and l_2 denote the lengths of the first two shortest cycles of G (not necessary strictly ordered from first shortest length to second shortest length), if the two cycles share s common vertices in G, where $0 \leq s \leq \lfloor \frac{\min\{l_1,l_2\}}{2} \rfloor + 1$. Then*

$$\tau(G) = \begin{cases} l_1 l_2, & \text{if } s = 0; \\ l_1 l_2 - s^2 + 2s - 1, & \text{if } 1 \leq s \leq \lfloor \frac{\min\{l_1,l_2\}}{2} \rfloor + 1. \end{cases}$$

Proof. Let C_1 and C_2 be the first two shortest cycles in G, and let l_1 and l_2 denote the lengths of C_1 and C_2, respectively. Then one can obtain $\tau(G)$ by the well-known **Breaking circle**. Since G is a bicyclic graph, we only need to delete one edge on each cycle of G. So we discuss each case as follows.

If $s = 0$, then barbell graph is the base graph of G. So we have $l_1 l_2$ ways to yield G to be a tree, and thus $\tau(G) = l_1 l_2$.

Otherwise, C_1 and C_2 have s common vertices, and yield $s - 1$ common edges in G. Clearly, there are $l_1 l_2$ ways to break C_1 and C_2. Note that if two edges delete in those $s - 1$ common edges, then the remainder always contains one cycle, and so, there are $(s-1)^2$ ways in this case. Therefore,

$$\tau(G) = l_1 l_2 - (s-1)^2 = l_1 l_2 - s^2 + 2s - 1.$$

□

A tricyclic graph is said to be a *tricyclic base graph* if it contains no pendant vertices. From reference [9] we know that there are exactly 15 types of tricyclic base graphs. Based on these we re-divide into 17 types of such base graphs, and present the numbers of spanning trees of tricyclic graphs below.

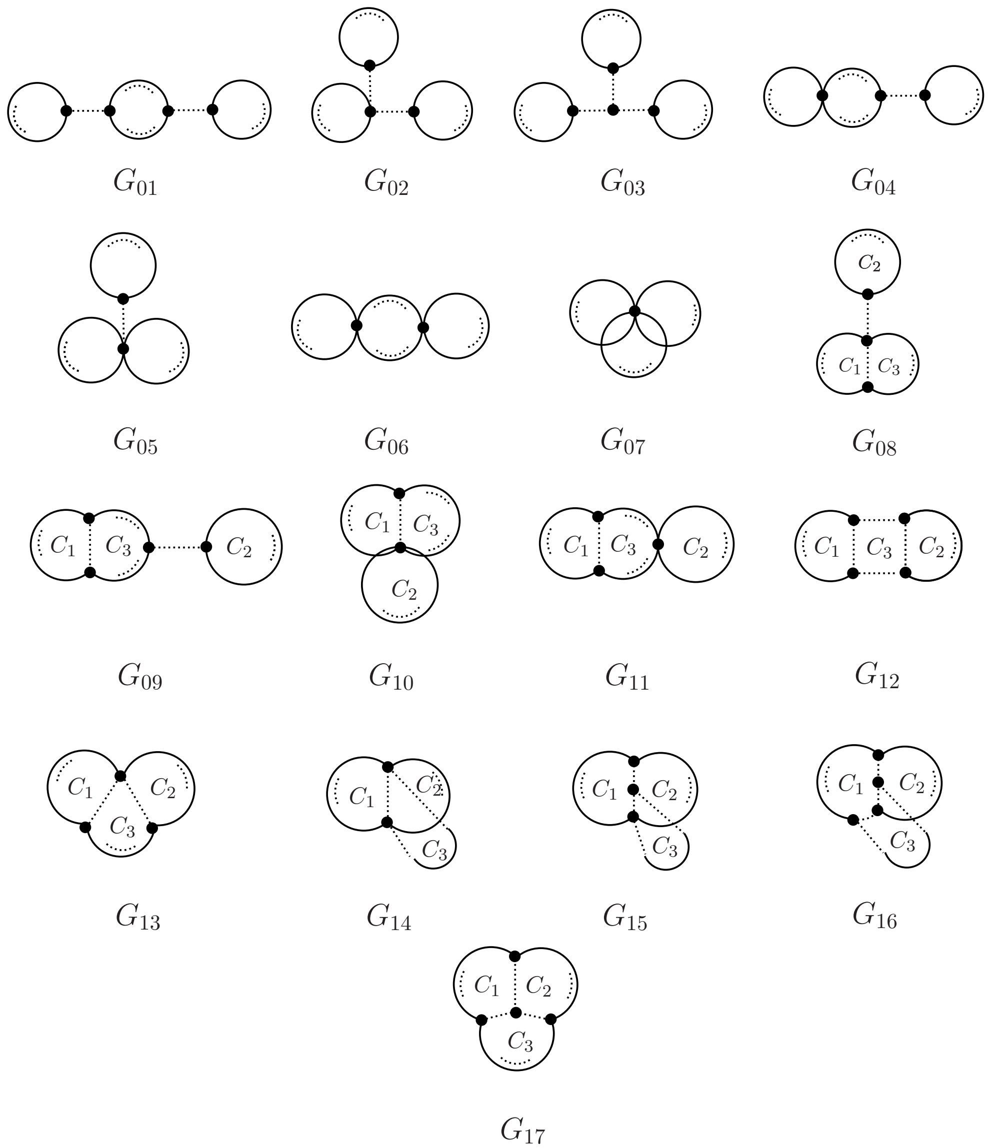


Figure 3: Graphs $G_{01} - G_{17}$

Lemma 11. *Let G be a tricyclic graph. Suppose that $C_i (i = 1, 2, 3)$ are the first three shortest cycles of G, and let l_i denote the length of each cycle C_i in G (not necessary strictly ordered from first shortest length to third shortest length). If C_1 and C_3 share s_1 common vertices, C_2 and C_3 share s_2 common vertices, and C_1 and C_2 share s_3 common vertices in G. Then*

(1) *If $s_1 \leq 1$ and $s_2 \leq 1$ and $s_3 \leq 1$, then $\tau(G) = l_1 l_2 l_3$;*

(2) *If $2 \leq s_1 \leq \lfloor \frac{\min\{l_1, l_3\}}{2} \rfloor + 1$ and $s_2 \leq 1$ and $s_3 \leq 1$, then $\tau(G) = l_1 l_2 l_3 - l_2 (s_1 - 1)^2$;*

(3) *If* $2 \leq s_1 \leq \lfloor \frac{\min\{l_1,l_3\}}{2} \rfloor + 1$ *and* $2 \leq s_2 \leq \lfloor \frac{\min\{l_2,l_3\}}{2} \rfloor + 1$ *and* $s_3 \leq 1$, *then* $\tau(G) = l_1l_2l_3 - l_2s_1^2 + 2l_2s_1 - l_1s_2^2 + 2l_1s_2 - l_1 - l_2$;

(4) *Otherwise, let* E_{13}, E_{23} *and* E_{12} *denote the shared edge sets of* C_1 *and* C_3, C_2 *and* C_3, *and* C_1 *and* C_2, *respectively. Then*

$$\tau(G) = \begin{cases} \alpha, & \text{if } E_{13} = E_{23} = E_{12}; \\ \beta, & \text{if } E_{13} = E_{23} \subset E_{12}; \\ \gamma, & \text{if } E_{13} \neq E_{23} \neq E_{12} \text{ and } E_{13} \cap E_{12} = E_{23}; \\ \eta, & \text{if } E_{13} \neq E_{23} \neq E_{12} \text{ and } E_{13} \cap E_{23} \cap E_{12} = \emptyset; \end{cases}$$

where

$$\begin{aligned} \alpha &= l_1l_2l_3 + 6s_1 + 2s_1l_1 + 2s_1l_2 + 2s_1l_3 + 2s_1^3 - s_1^2l_1 - s_1^2l_2 - s_1^2l_3 - 6s_1^2 - l_1 - l_2 \\ &\quad -l_3 - 2; \\ \beta &= l_1l_2l_3 + 2s_1l_1 + 2s_1l_2 + 2s_3l_3 + 4s_1 + 2s_3 + 2s_1^2s_3 - s_1^2l_1 - s_1^2l_2 - s_3^2l_3 - 2s_1^2 \\ &\quad -4s_1s_3 - l_1 - l_2 - l_3 - 2; \\ \gamma &= l_1l_2l_3 + 2s_1s_2s_3 + 2s_1 + 2s_2 + 2s_3 - 2s_1s_2 - 2s_1s_3 - 2s_2s_3 + 2s_1l_2 + 2s_2l_1 \\ &\quad +2s_3l_3 - s_1^2l_2 - s_2^2l_1 - s_3^2l_3 - l_1 - l_2 - l_3 - 2; \\ \eta &= l_1l_2l_3 - 2s_1s_2s_3 - 2s_1 - 2s_2 - 2s_3 + 2s_1s_2 + 2s_1s_3 + 2s_2s_3 + 2s_1l_2 + 2s_2l_1 \\ &\quad +2s_3l_3 - s_1^2l_2 - s_2^2l_1 - s_3^2l_3 - l_1 - l_2 - l_3 + 2. \end{aligned}$$

Proof. Let $\tau(G)$ be the number of spanning trees of G, and let E_{12}, E_{13} and E_{23} be the shared edge sets of C_1 and C_2, C_1 and C_3, and C_2 and C_3, respectively. Note that a tricyclic graph and its base graph have the same number of spanning tree. Thus, if there is no confusion, we may assume that G is the base graph of a tricyclic graph. For the graph G, one can obtain $\tau(G)$ by the well-known **Breaking circle**. Now, we discuss it in the following.

Case 1. If $s_1 \leq 1$ and $s_2 \leq 1$ and $s_3 \leq 1$, then each two of three cycles C_1, C_2 and C_3 have no common edges in G, that is, $G \in \{G_{01}, G_{02}, \ldots, G_{07}\}$ (see Fig. 3 for instance). Note that l_i is the length of each cycle C_i for $i = 1, 2, 3$. So, there are $l_1l_2l_3$ ways to obtain a spanning tree, and thus $\tau(G) = l_1l_2l_3$.

Case 2. If $2 \leq s_1 \leq \lfloor \frac{\min\{l_1,l_3\}}{2} \rfloor + 1$ and $s_2 \leq 1$ and $s_3 \leq 1$, then G is obtained from θ-graph by attaching a cycle or appending a cycle, see $G_{08} - G_{11}$ in Fig. 3. Since $2 \leq s_1 \leq \lfloor \frac{\min\{l_1,l_3\}}{2} \rfloor + 1$ the θ-graph consists of C_1 and C_3. Together with Lemma 10 we know that, θ-graph has $l_1l_3 - (s_1 - 1)^2$ ways to break the circle. Note that C_2 has l_2 ways to break the cycle. Therefore,

$$\tau(G) = [l_1l_3 - (s_1 - 1)^2]l_2 = l_1l_2l_3 - l_2(s_1 - 1)^2.$$

Case 3. If $2 \leq s_1 \leq \lfloor \frac{\min\{l_1,l_3\}}{2} \rfloor + 1$ and $2 \leq s_2 \leq \lfloor \frac{\min\{l_2,l_3\}}{2} \rfloor + 1$ and $s_3 \leq 1$, G has either the form G_{12} or G_{13}, shown in Fig. 3. Since $s_3 \leq 1$, C_1 and C_2 share no common edges in G, that is, $|E_{12}| = 0$. For the sets of E_{13} and E_{23}, if one of the edges in E_{13} and E_{23} are deleted simultaneously. Then the remainder still contains a cycle. Hence, it is necessary to break the cycle again, and so, we can get $(s_1 - 1)(s_2 - 1)(l_1 + l_2 + l_3 - 2s_1 - 2s_2 + 4)$ ways to make G become a tree; if just one edge is deleted from E_{13} or E_{23}, then the remainder becomes a bicyclic graph with a shared edge set E_{23} or E_{13}. So, we can continue to delete an edge on each of the two cycles except for the common edge set in the bicyclic graph. Thus, one can get $(s_1-1)(l_1+l_3-2s_1-s_2+3)(l_2-s_2+1)+(s_2-1)(l_2+l_3-2s_2-s_1+3)(l_1-s_1+1)$ ways to break the two cycles; otherwise, there are $(l_1-s_1+1)(l_2-s_2+1)(l_3-s_1-s_2+2)$ ways to break all cycles of G. Thus, summing up the above, $\tau(G)=l_1l_2l_3-l_2s_1^2+2l_2s_1-l_1s_2^2+2l_1s_2-l_1-l_2$.

Case 4. Otherwise, any two cycles in G share the common vertices no less than 2. For convenience, we may assume that $|E_{12}|\geq|E_{13}|\geq|E_{23}|$, and here consider the relationships among E_{12}, E_{13} and E_{23} in the following.

Case 4.1. When $E_{12}=E_{13}=E_{23}=E'$ (say), we see that C_1, C_2 and C_3 have the same shared edge set, and thus, G has the form G_{14} (see Fig. 3). If one deletes an edge in E', the remainder becomes a bicyclic graph. Since it is hard to determine which two cycles in the bicyclic graph are the first two shortest cycles, we can proceed to delete one edge from each of any two sets in $E(C_1)\backslash E'$, $E(C_2)\backslash E'$ and $E(C_3)\backslash E'$, so we can obtain $(s_1-1)((l_3-s_1+1)(l_2-s_1+1)+(l_3-s_1+1)(l_1-s_1+1)+(l_1-s_1+1)(l_2-s_1+1))$ ways to break circles. Otherwise, we only delete one edge on each $C_i(i=1,2,3)$ other than E', there are $(l_3-s_1+1)(l_2-s_1+1)(l_1-s_1+1)$ ways to break the cycles. From the above, we can get $\tau(G)=\alpha$.

Case 4.2. When two of E_{13}, E_{23} and E_{12} are the same, without loss of generality, we denote by $E_{12}\neq E_{13}=E_{23}=E''$ (say). Clearly, $E_{13}=E_{23}$ implies that C_1 and C_2 share the same edges as C_3, so we have $E''\subseteq E_{12}$ because E_{12} is the common edge set of C_1 and C_2. Note that $E_{12}\neq E''$. Hence $E''\subset E_{12}$, which implies that G has the form G_{15} as shown in Fig. 3. Now, according to the deleted edges we classify the following cases.

- If one edge is deleted in E'' but not in $E_{12}\backslash E''$, then we can respectively delete one edge from two sets in $E(C_1)\backslash E_{12}$, $E(C_2)\backslash E_{12}$ and $E(C_3)\backslash E_{12}$. So, one can get$(s_1-1)((l_1-s_3+1)(l_2-s_3+1)+(l_1-s_3+1)(l_3-s_1+1)+(l_2-s_3+1)(l_3-s_1+1))$ ways;
- If one edge is deleted in $E_{12}\backslash E''$ but not in E'', then the remainder becomes a bicyclic graph in which the two cycles share no common edges, and thus we have $(s_3-s_1)(l_3-s_1+1)(l_1+l_2-2s_3+2)$ ways;
- If one edge is respectively deleted in E'' and $E_{12}\backslash E''$, then the remaining is a unicyclic graph, we have that $(s_1-1)(s_3-s_1)(l_1+l_2-2s_3+2)$ ways to break its cycle;
- Otherwise, one can delete one edge in each $C_i(i=1,2,3)$ other than in E_{12}, there are $(l_3-s_1+1)(l_1-s_3+1)(l_2-s_3+1)$ ways to break cycles.

Summing up above four cases, we can obtain $\tau(G)=\beta$.

Case 4.3. When $E_{12}\neq E_{13}\neq E_{23}$, we consider the relationships between the those sets.

Case 4.3.1. If $E_{13}\subset E_{12}$, then $E_{13}\subset E(C_2)$ since $E_{12}\subset E(C_2)$. Note that $E_{13}\subset E(C_3)$. Thus $E_{13}\subseteq E_{23}$. Together with $|E_{13}|\geq|E_{23}|$ we have $E_{13}=E_{23}$, a contradiction.

Case 4.3.2. If $E_{13}\nsubseteq E_{12}$ and $E_{13}\cap E_{12}\neq\emptyset$, then E_{13} is compose of partial E_{12} (denoted by E^*) together with some edges in $E(C_1)\backslash E_{12}$. Meanwhile, $G[E_{13}]$ should be a path since G is a tricyclic graph. Moreover, we claim that $E_{23}\cap(E(C_2)\backslash E_{12})=\emptyset$ since if not, it follows from $s_3\leq\lfloor\frac{\min\{l_1,l_2\}}{2}\rfloor+1$ that $|E(C_1)\backslash E_{12}|\geq|E_{12}|$. We notice that G is a tricyclic graph, which implies that $G[E_{13}\cup E_{23}]$ must be a path, thus $|E_{13}|>|E(C_1)\backslash E_{12}|\geq|E_{12}|$, which contradicts $|E_{12}|\geq|E_{13}|$. Therefore, $E_{23}\subset E_{12}$, and we further have $E_{23}\subset E_{13}$ since $E_{23}\subset E_{12}\subset E(C_1)$, $E_{23}\subset E(C_3)$ and $E_{23}\neq E_{13}$. Thus, $E_{23}\subseteq E_{12}\cap E_{13}$. In fact,

$E_{12} \cap E_{13} = E^*$ since E_{13} is compose of $E^*(\subset E_{12})$ and some edges in $E(C_1)\backslash E_{12}$. Hence, $E_{23} \subseteq E^*$. On the other hand, since $E^* \subset E_{13} \subset E(C_3)$ and $E^* \subset E_{12} \subset E(C_2)$, we have $E^* \subseteq E_{23}$. Thus, $E^* = E_{23}$, that is, $E_{23} = E_{12} \cap E_{13}$. Therefore, G has the form G_{16} (see Fig. 3). Now, according to the deleted edges we consider the following subcases.

- If just one edge in E_{23} is deleted, we can proceed to delete one edge from each of any two sets in $E(C_1)\backslash(E_{12} \cup E_{13})$, $E(C_2)\backslash E_{12}$ and $E(C_3)\backslash E_{13}$. Thus, we have $(s_2-1)((l_1-s_1+s_2-s_3+1)(l_2-s_3+1)+(l_1-s_1+s_2-s_3+1)(l_3-s_1+1)+(l_2-s_3+1)(l_3-s_1+1))$ ways to break cycles.

- If just one edge in $E_{12}\backslash E_{23}$ (or $E_{13}\backslash E_{23}$) is deleted, then the remainder becomes a bicyclic graph with a shared edge set of $E_{13}\backslash E_{23}$ (or $E_{12}\backslash E_{23}$). Then, we can continue to delete an edge on each of the two cycles except for the common edge set in the bicyclic graph. So, there are $(s_3-s_2)(l_1+l_2-s_1+s_2-2s_3+2)(l_3-s_1+1)+(s_1-s_2)(l_1+l_3-2s_1+s_2-s_3+2)(l_2-s_3+1)$ ways to break cycles.

- If two of the edge sets E_{23}, $E_{12}\backslash E_{23}$ and $E_{13}\backslash E_{23}$ are selected, and one edge is deleted in each selected edge set, then the remainder still contains a cycle. Hence, it is necessary to break the cycle again, and so, one can get $(s_2-1)(s_3-s_2)(l_1+l_2-s_1+s_2-2s_3+2)+(s_2-1)(s_1-s_2)(l_1+l_3-2s_1+s_2-s_3+2)+(s_3-s_2)(s_1-s_2)(l_1+l_2+l_3-2s_1+s_2-2s_3+3)$ ways to make G become a tree.

- If one edge in E_{23}, $E_{12}\backslash E_{23}$ and $E_{13}\backslash E_{23}$ is respectively deleted, then the remaining becomes a spanning tree of G. Therefore, there are $(s_3-s_2)(s_2-1)(s_1-s_2)$ ways.

- Otherwise, one can delete one edge in each $C_i (i = 1, 2, 3)$ other than in E_{23}, $E_{12}\backslash E_{23}$ and $E_{13}\backslash E_{23}$, there are $(l_1-s_1+s_2-s_3+1)(l_2-s_3+1)(l_3-s_1+1)$ ways to break all cycles of G.

From the five subcases above, $\tau(G) = \gamma$.

Case 4.3.3. If $E_{13} \nsubseteq E_{12}$ and $E_{13} \cap E_{12} = \emptyset$, then $E_{23} \cap E_{12} = \emptyset$ since if not, we suppose that $E_{23} \cap E_{12} = \tilde{E} \neq \emptyset$, then $\tilde{E} \subseteq E_{13}$ because $\tilde{E} \subseteq E_{23} \subset E(C_3)$ and $\tilde{E} \subseteq E_{12} \subset E(C_1)$. Together with $\tilde{E} \subseteq E_{12}$, we therefore have $\tilde{E} \subseteq E_{13} \cap E_{12} \neq \emptyset$, a contradiction. By similar reasoning as above, we can obtain that $E_{23} \cap E_{13} = \emptyset$. Hence, $E_{13} \cap E_{23} \cap E_{12} = \emptyset$. We notice that G is a tricyclic graph and $s_1, s_2, s_3 \geq 2$, so G has the form G_{17} (see Fig. 3). We also consider whether the deleted edges are in the common edge set. Clearly, we can only select at most two sets in E_{13}, E_{23} and E_{12} to delete edges since if not, the remaining will be disconnected. Now, according to the deleted edges we classify the following subcases.

- If just one edge is deleted in the set E_{12} (or E_{13}, or E_{23}), then the remainder becomes a bicyclic graph with the shared edge set of $E_{13} \cup E_{23}$ (or $E_{12} \cup E_{23}$, or $E_{12} \cup E_{13}$). We can continue to delete an edge on each of the two cycles except for the common edge set in the bicyclic graph. Thus, there are $(s_1-1)(l_1+l_3-2s_1-s_2-s_3+4)(l_2-s_2-s_3+2)+(s_2-1)(l_2+l_3-2s_2-s_1-s_3+4)(l_1-s_1-s_3+2)+(s_3-1)(l_1+l_2-2s_3-s_1-s_2+4)(l_3-s_1-s_2+2)$ ways to obtain the spanning tree of G.

- If one selects two edge sets in E_{12}, E_{13} and E_{23}, and then deletes one edge in each selected edge set. Now, the remainder graph still contains a cycle. Hence, it also need to break the cycle again, and so, we can get $((s_1-1)(s_2-1)+(s_1-1)(s_3-1)+(s_2-1)(s_3-1))(l_1+l_2+l_3-2s_1-2s_2-2s_3+6)$ ways to make G become a tree.

- Otherwise, one can delete one edge in each $C_i (i = 1, 2, 3)$ other than in E_{12}, E_{13} and E_{23}, then G has $(l_1 - s_1 - s_3 + 2)(l_2 - s_2 - s_3 + 2)(l_3 - s_1 - s_2 + 2)$ ways to obtain a spanning tree.

To sum up above, $\tau(G) = \eta$. □

3 Degree sequence $deg(G) = (2^{n_2}, 1^{n_1}, 0^{n_0})$

Theorem 1. *Let G be a graph of $\mathcal{G}(2^{n_2}, 1^{n_1}, 0^{n_0})$. If either $n_0 = 0$ or $n_0 \geq 1$ and G contains no even cycle as its component, then $deg(G) = (2^{n_2}, 1^{n_1}, 0^{n_0})$ is determined by Laplacian spectrum of G; otherwise, G has a L-cospectral mate H with $deg(H) = (3^{n_0-k}, 2^{n_2-3n_0+3k}, 1^{n_1+3n_0-3k}, 0^k)$, where $0 \leq k \leq n_0$.*

Proof. From $deg(G)$ we know that G is disjoint union of paths and cycles. Let H be a graph L-cospectral with G. Recall that the L-spectrum of a disjoint union of graphs is obtained by stringing together the spectra of the components (see [4]). So we may suppose that G^* (resp. H^*) is the maximal component with $\mu_1(G) = \mu_1(G^*)$ (resp. $\mu_1(H) = \mu_1(H^*)$). Then by Lemma 4 and Lemma 6 we have

$$d_1(H) + 1 \leq \mu_1(H) = \mu_1(H^*) = \mu_1(G^*) \leq d_1(G^*) + d_2(G^*) \leq d_1(G) + d_2(G) = 4,$$

and thus, $d_1(H) \leq 3$. Suppose that H has degree sequence $deg(H) = (3^{x_3}, 2^{x_2}, 1^{x_1}, 0^{x_0})$. Then by Lemma 1 (i), (ii) and (v), one can obtain that

$$\begin{cases} x_0 + x_1 + x_2 + x_3 = n \\ x_1 + 2x_2 + 3x_3 = n + n_2 - n_0 \\ x_1 + 4x_2 + 9x_3 = n + 3n_2 - n_0 \end{cases}$$

Thus, it follows that

$$x_1 = n_1 + 3n_0 - 3x_0, \quad x_2 = n_2 - 3n_0 + 3x_0, \quad x_3 = n_0 - x_0. \tag{3.1}$$

If $n_0 = 0$, then $x_3 = 0$ since x_3 is non-negative integer, and so $x_1 = n_1$ and $x_2 = n_2$. Hence, $deg(G)$ is determined by Laplacian spectrum of G.

If $n_0 \geq 1$ and G has no even cycle as its components, then from Lemma 7 $\mu_1(G) < 4$, and further we get $d_1(H) + 1 \leq 3$ by Lemma 6. So $d_1(H) \leq 2$, which implies that $x_3 = 0$, that is, $x_0 = n_0$, $x_1 = n_1$ and $x_2 = n_2$. Thus, $deg(G)$ is determined by Laplacian spectrum of G.

Otherwise, G maybe have a L-cospectral mate H with $deg(H) = (3^{n_0-k}, 2^{n_2-3n_0+3k}, 1^{n_1+3n_0-3k}, 0^k)$, where $0 \leq k \leq n_0$ is an integer. □

According to $\mathcal{G}^*(2^{n_2}, 1^{n_1})$ we know that $n_1 = 2$ if G is a tree, since otherwise, if $n_1 \geq 4$, then $4 + 2n_2 \leq 2m = 2(n_1 + n_2 - 1)$, which leads to $n_1 = 3$, a contradiction. Thus, $G \in \mathcal{G}^*(2^{n_2}, 1^{n_1})$ is a path if G is a tree, and furthermore, it is easy to obtain the following corollary.

Corollary 2 ([5], Proposition 1). *For any positive integer n, P_n is determined by its Laplacian spectrum.*

In Theorem 1, if $G \in \mathcal{G}^*(2^{n_2}, 1^{n_1})$ and $n_1 = 0$. Then $G \cong C_n$.

Corollary 3 ([5], Proposition 5). *For any positive integer $n(\geq 3)$, C_n is determined by its Laplacian spectrum.*

Corollary 4 ([21], Theorem 3.1). *The degree sequence of the disjoint union of paths and odd-cycles is determined by the L-spectrum.*

Next, we give a counterexample to illustrate that for some graph G in $\mathcal{G}(2^{n_2}, 1^{n_1}, 0^{n_0})$ is not determined by its Lapalcian spectrum if G contains even cycles. Moreover, G has a L-cospectral mate H with $deg(H) = (3^{n_0-k}, 2^{n_2-3n_0+3k}, 1^{n_1+3n_0-3k}, 0^k)$, where $0 \leq k \leq n_0$.

Example 1. *Let G and H be two graphs shown in Fig. 4, which can be found in [21]. One can see that $deg(G) = (2^5, 1^2, 0^1)$ and $deg(H) = (3^1, 2^2, 1^5, 0^0)$ but they are L-cospectral.*

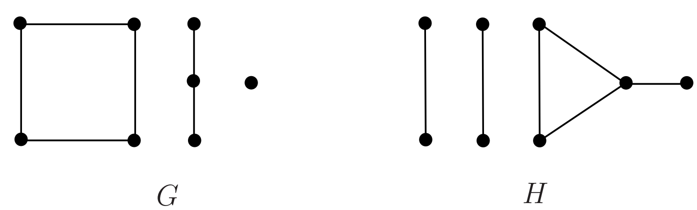


Figure 4: Graphs G and H

4 Degree sequence $deg(G) = (3^{n_3}, 2^{n_2}, 1^{n_1})$

Theorem 2. *Let G be a graph of $\mathcal{G}^*(3^{n_3}, 2^{n_2}, 1^{n_1})$ with $n_1 + n_2 + n_3 = n$. If*

(1) $0 \leq n_3 \leq 1$*;*

(2) $n_3 \geq 2$ *and* $n_1 = 0$*;*

(3) $n_3 \geq 2$ *and* $n_1 \geq 1$*, and G satisfies the following conditions;*

- *G is a tree or unicyclic graph;*
- *G is a c-cyclic graph with maximum number of triangles, where $c \geq 2$;*

Then $deg(G) = (3^{n_3}, 2^{n_2}, 1^{n_1})$ is determined by the Laplacian spectrum of G. Otherwise, if G has L-cospectral mate H, then $deg(H) = (1^{n_1-k}, 2^{n_2+3k}, 3^{n_3-3k}, 4^k)$ where $1 \leq k \leq$ $\min\{n_1, [\frac{n_3}{3}]\}$.

Proof. When $n_3 = 0$, by Theorem 1 the conclusion holds; when $n_3 = 1$, it is obvious from literature [23] (see Theorem 8).

When $n_3 \geq 2$, let H be a graph L-cospectral with G. Then $\mu_{n-1}(H) = \mu_{n-1}(G)$. Since G is a connected graph, $\mu_1(G) > 0$. Thus H is also a connected graph (see [6]). Therefore, it further follows from Lemma 4 that

$$d_1(H) + 1 \leq \mu_1(H) = \mu_1(G) \leq d_1(G) + d_2(G) = 6,$$

and hence, $d_1(H) \leq 5$. We now suppose that H has degree sequence $\deg(H) = (5^{x_5}, 4^{x_4}, 3^{x_3}, 2^{x_2}, 1^{x_1})$. Then by Lemma 1 (i), (ii) and (v),

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = n, \\ x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = n + n_2 + 2n_3, \\ x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 = n + 3n_2 + 8n_3. \end{cases} \tag{4.1}$$

From (4.1) one can get

$$x_3 + 3x_4 + 6x_5 = n_3. \tag{4.2}$$

For $d_1(H)$ we distinct three cases in the following.

Case 1. $d_1(H) \leq 2$.

If $d_1(H) \leq 2$, then $x_3 = x_4 = x_5 = 0$. By Eq. (4.2) we have $n_3 = 0$, it contradicts $n_3 \geq 2$.

Case 2. $d_1(H) = 5$.

From Lemma 4, $6 = d_1(H) + 1 \leq \mu_1(H) = \mu_1(G) \leq q_1(G) \leq d_1(G) + d_2(G) = 6$, it implies that G is either isomorphic to $K_{1,n-1}$ or a bipartite regular graph and $d_1(H) = n-1$.

If $G \cong K_{1,n-1}$, it follows from $d_1(G) = n - 1 = 3$ that $n = 4$ and $d_2(G) = d_3(G) = d_4(G) = 1$. So, it is contrary to $d_2(G) = 3$ due to $n_3 \geq 2$.

If G is a bipartite regular graph with $d_1(G) = 3$, then $n_2 = n_1 = 0$. And $n = 6$ since $d_1(H) = n - 1 = 5$. Since H is L-cospectral G and G is a regular graph, H is also a regular graph of order 6 (see Proposition 2 in [5]). So H is K_6. It contradicts Lemma 1 (ii).

Case 3. $3 \leq d_1(H) \leq 4$.

Clearly, $x_5 = 0$. From Eq. (4.1) we have

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = n, \\ x_1 + 2x_2 + 3x_3 + 4x_4 = n + n_2 + 2n_3, \\ x_1 + 4x_2 + 9x_3 + 16x_4 = n + 3n_2 + 8n_3. \end{cases}$$

Therefore, it follows that

$$x_1 = n_1 - k, \quad x_2 = n_2 + 3k, \quad x_3 = n_3 - 3k, \quad x_4 = k \tag{4.3}$$

where k is non-negative integer.

Case 3.1. When $n_1 = 0$, $k = 0$ by Eq. (4.3). So we have $x_1 = 0$, $x_2 = n_2$, $x_3 = n_3$ and $x_4 = 0$, that is, $deg(G) = (3^{n_3}, 2^{n_2})$ is determined by the Laplacian spectrum of G, where $n_2 + n_3 = n$.

Case 3.2. When $n_1 \geq 1$, we suppose that H and G have t' and t triangles, respectively. Then by Lemma 3 we get

$$\begin{aligned} &6t - (n_1(1-2)^3 + n_2(2-2)^3 + n_3(3-2)^3) \\ &= 6t' - ((n_1 - k)(1-2)^3 + (n_2 + 3k)(2-2)^3 + (n_3 - 3k)(3-2)^3 + k(4-2)^3). \end{aligned}$$

Thus, it follows that $t' - t = k$.

(a) If G is a connected graph with $0 \leq c(G) \leq 1$, then by Lemma 1 (i) and (ii) we know that H has the same cyclomatic number, and further by (iii) we get $k = 0$ since H L-cospectral with G implies that the two graphs have the same number of spanning tree. Hence, the degree sequence of G is determined by its Laplacian spectrum;

(b) Let $t_{\max}$ be maximum number of triangles of a c-cyclic graph, where $c \geq 2$. If G is a c-cyclic graph with $t_{\max}$, then by Lemma 1 (i) and (ii), H is also a c-cyclic graph. Combine with $t' - t = k$ and $t = t_{\max}$ we have $k + t_{\max} = t'$, hence, it further follows $k + t_{\max} = t' \leq t_{\max}$, so we have $k = 0$. Therefore, $deg(G) = (3^{n_3}, 2^{n_2}, 1^{n_1})$ is determined by Laplacian spectrum of G;

(c) Otherwise, G maybe have a L-cospectral mate H with $deg(H) = (1^{n_1-k}, 2^{n_2+3k}, 3^{n_3-3k}, 4^k)$. Note that $n_1 - k \geq 0$ and $n_3 - 3k \geq 0$. Hence $1 \leq k \leq \min\left\{n_1, [\frac{n_3}{3}]\right\}$, where $[x]$ is taken integer of x.

The proof is completed. □

Remark 1. *Under the assumption of Theorem 2, for a graph $G \in \mathcal{G}^*(3^{n_3}, 2^{n_2}, 1^{n_1})$, if $deg(G) = (3^{n_3}, 2^{n_2}, 1^{n_1})$ is not determined by Laplacian spectrum of G, then G has a L-cospectral mate H with $deg(H) = (1^{n_1-k}, 2^{n_2+3k}, 3^{n_3-3k}, 4^k)$ where $1 \leq k \leq \min\left\{n_1, [\frac{n_3}{3}]\right\}$. Here we give a pair of L-cospectral mate G_1 and H_1 (see Fig. 5), which can be found in [23] (see Example 1), to illustrate that the bound of k is best possible.*

From Fig. 5 we see that $deg(G) = (3^4, 2^3, 1^2)$, by Theorem 2 one can get $1 \leq k \leq \min\left\{2, [\frac{4}{3}]\right\} = 1$, that is, $k = 1$. Thus, $deg(H) = (4^1, 3^1, 2^6, 1^1)$.

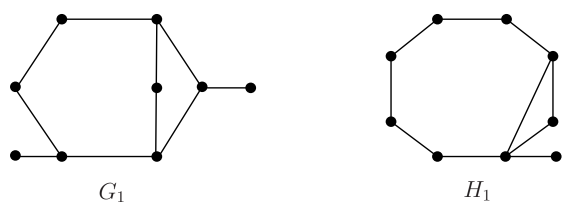


Figure 5: Graphs G_1 and H_1

Corollary 5. *Let T be a tree with $\Delta \leq 3$. Then the degree sequence of T is determined by its Laplacian spectrum.*

A graph G is a *minimal* 3*-degree* [23] graph if $G \in \mathcal{G}^*(3^{n_3}, 2^{n_2}, 1^{n_1})$ where $n_1 + n_2 + n_3 = n$ and $n_1, n_3 > 0$ and $n_2 \geq 0$.

Corollary 6 ([23], Theorem 1)**.** *Let G be a minimal 3-degree graph of order n. If the cyclomatic number $c(G) \leq 1$, Then the degree sequence of G is determined by it Laplacian spectrum.*

A graph G if whose vertex set can be partitioned into two subsets V_1 and V_2 such that $d(v_i) = r$ for $v_i \in V_1$ and $d(v_j) = r + 1$ for $v_j \in V_2$ is called $(r, r+1)$-*almost regular graph* (see [20]). From Theorem 2 one can obtain the following corollary.

Corollary 7. *Let G be a connected $(2,3)$-almost regular graph. Then the degree sequence of G is determined by its Laplacian spectrum.*

Note that bicyclic graph has 2 triangles at most. For c-cyclic graph G with $c \in \{3,4\}$, by Lemma 5 we know that the number of triangles of G is no more than $c+1$. Thus, by Theorem 2 one can obtain the following corollary.

Corollary 8. *Let $G \in \mathcal{G}^*(3^{n_3}, 2^{n_2}, 1^{n_1})$. If G is a bicyclic graph with 2 triangles, or a tricyclic graph with 4 triangles, or 4-cyclic graph with 5 triangles, then $deg(G) = (3^{n_3}, 2^{n_2}, 1^{n_1})$ is determined by Laplacian spectrum of G.*

5 Degree sequence $deg(G) = (4^{n_4}, 2^{n_2}, 1^{n_1})$

Theorem 3. *Let G be a graph of $\mathcal{G}^*(4^{n_4}, 2^{n_2}, 1^{n_1})$ with $n_1 + n_2 + n_4 = n$ and $n_4 \le 2$. If*

(1) *$0 \le n_4 \le 1$ and G is not isomorphic to either $R(3,4)$ or $R(3,5)$ (see Fig. 6);*

(2) *$n_4 = 2$ and G satisfies the following conditions;*

- *G is a tree or unicyclic graph;*
- *G is a c-cyclic graph either with c $(2 \le c \le 3)$ or without triangles;*

Then $deg(G) = (4^{n_4}, 2^{n_2}, 1^{n_1})$ is determined by the Laplacian spectrum of G. Otherwise, if G is a bicyclic graph with $t(G) = 1$, then G has a L-cospectral mate H with $deg(H) = (4^1, 3^3, 2^{n_2-3}, 1^3)$; if G is a tricyclic graph with $t(G) = 1$, then G has a L-cospectral mate H with $deg(H) = (4^1, 3^3, 2^{n_2-3}, 1^1)$, and if G is a tricyclic graph with $t(G) = 2$, then G has a L-cospectral mate H either with $deg(H) = (3^6, 2^{n_2-6}, 1^2)$ or with $deg(H) = (4^1, 3^3, 2^{n_2-3}, 1^1)$.

Proof. When $n_4 = 0$, it is trivial; when $n_4 \ge 1$, let H be a graph L-cospectral with G. Then H is a connected graph since $\mu_{n-1}(H) = \mu_{n-1}(G) > 0$ (see [6]). Furthermore, by Lemma 4 we have

$$d_1(H) + 1 \le \mu_1(H) = \mu_1(G) \le q_1(G) \le d_1(G) + d_2(G) = 8$$

hence, it follows that $d_1(H) \le 7$.

Case 1. $d_1(H) = 7$.

From Lemma 4, $8 = d_1(H) + 1 \le \mu_1(H) = \mu_1(G) \le q_1(G) \le d_1(G) + d_2(G) = 8$, it implies that G is either isomorphic to $K_{1,n-1}$ or a bipartite regular graph and $d_1(H) = n-1$.

If $G \cong K_{1,n-1}$, then $G \cong K_{1,7}$ since $d_1(H) = n-1 = 7$, contrary to $d_1(G) = 4$.

If G is a bipartite regular graph with $d_1(G) = 4$, then $n_2 = n_1 = 0$. Combine with $d_1(H) = n-1 = 7$ we have $n = 8$. Moreover, since H is L-cospectral the regular graph G, H is also a regular graph of order 8 (see Proposition 2 in [5]). Hence $H = K_8$, it contradicts Lemma 1 (ii).

Case 2. $d_1(H) \le 6$.

Suppose that H has the degree sequence $deg(H) = (1^{x_1}, 2^{x_2}, 3^{x_3}, 4^{x_4}, 5^{x_5}, 6^{x_6})$. Then by Lemma 1 (i), (ii) and (iii) we have

$$\begin{cases} \sum_{i=1}^{6} x_i = n, \\ \sum_{i=1}^{6} i x_i = n + n_2 + 3n_4, \\ \sum_{i=1}^{6} i^2 x_i = n + 3n_2 + 15n_4. \end{cases} \tag{5.1}$$

Thus, it follows from Eq. (5.1) that

$$x_3 + 3x_4 + 6x_5 + 10x_6 = 3n_4. \tag{5.2}$$

Let t and t' be the number of triangles in G and H, respectively. Since G and H are L-cospectral, by Lemma 3 it follows that

$$3(t - t') = x_3 - 6x_5 - 20x_6. \tag{5.3}$$

Case 2.1. $n_4 = 1$.

From Eq. (5.2), it follows that either $x_4 = 1$ and $x_3 = x_5 = x_6 = 0$ or $x_3 = 3$ and $x_4 = x_5 = x_6 = 0$.

If $x_4 = 1$ and $x_3 = x_5 = x_6 = 0$, then by Eq. (5.1) $x_1 = n_1$ and $x_2 = n_2$;

If $x_3 = 3$ and $x_4 = x_5 = x_6 = 0$, then by Eq. (5.1) we have $x_1 = n_1 + 1$ and $x_2 = n_2 - 3$. Since $deg(G) = (4^{n_4}, 2^{n_2}, 1^{n_1})$ is graphic degree sequences, n_1 is even. Let $G \in \mathcal{G}^*(4^{n_4}, 2^{n_2}, 1^{n_1})$ be a c-cyclic graph. Then

$$c = \frac{4 + 2n_2 + n_1}{2} - (n_1 + n_2 + 1) + 1 = \frac{4 - n_1}{2} \geq 0,$$

therefore, it follows that $n_1 \leq 4$. We notice that n_1 is even, so we have $n_1 = 0, 2, 4$, and the corresponding graph G is bicyclic graph, unicyclic graph and tree, respectively.

- When $n_1 = 2$, G is a unicyclic graph with $deg(G) = (4^1, 2^{n-3}, 1^2)$. We declare that $deg(G)$ is also determined by Laplacian spectrum of G since if not, then by Lemma 1 (i) and (ii), H is also a unicyclic graph with $deg(H) = (3^3, 2^{n-6}, 1^3)$. From Eq. (5.3) we have $t - t' = 1$. However, it follows from Lemma 1 (iii) that $t = t'$, so we have $0 = 1$, a contradiction.

- When $n_1 = 4$, similar to the case of $n_1 = 2$, one can prove that $deg(G)$ is also determined by Laplacian spectrum of G if G is a tree with $deg(G) = (4^1, 2^{n-5}, 1^4)$;

- When $n_1 = 0$, G is a bicyclic graph with $deg(G) = (4^1, 2^{n-1})$. From Lemma 1 (i) and (ii), H is also a bicyclic graph and $deg(H) = (3^3, 2^{n-4}, 1^1)$. Clearly, $G \in \mathcal{G}^*(4^1, 2^{n-1})$ is a ∞-graph. In fact, He and van Dam in [10] have proved that ∞-graph is determined by its Laplacian spectrum except for two graphs $R(3,4)$ and $R(3,5)$, and $R(3,4)$ and $R(3,5)$ are L-cospectral with B_1 and B_2, respectively. Thus, $deg(G) = (4^1, 2^{n-1})$ is determined by Laplacian spectrum of G except for $deg(G) \in \{(4^1, 2^5), (4^1, 2^6)\}$.

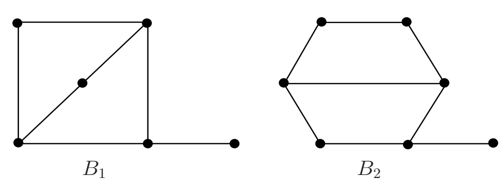


Figure 6: Graphs B_1 and B_2

From the discussion above, $deg(G) = (4^1, 2^{n_2}, 1^{n_1})$ is determined by Laplacian spectrum of G except for that G is a bicyclic graph with $deg(G) \in \big\{(4^1, 2^5), (4^1, 2^6)\big\}$.

Case 2.2. $n_4 = 2$.

It follows from Eq. (5.2) that $x_3 + 3x_4 + 6x_5 + 10x_6 = 3n_4 = 6$, and so $x_6 = 0$ and $x_3 + 3x_4 + 6x_5 = 6$. Thus, one can see that $0 \le x_5 \le 1$.

If $x_5 = 1$, then $x_3 = x_4 = x_6 = 0$. From Eq. (5.1) one can obtain that $deg(H) = (5^1, 2^{n_2+1}, 1^{n_1-1})$. By [16] we know that the degree sequence is determined by L-spectrum, it therefore contradicts $d_1(G) = 4$ since G and H are L-cospectral.

If $x_5 = 0$, it follows from Eq. (5.1) that $x_3 = 6$ and $x_4 = 0$, or $x_3 = 3$ and $x_4 = 1$, or $x_3 = 0$ and $x_4 = 2$. Now we consider three subcases in the following.

Case (1). When $x_3 = 0$ and $x_4 = 2$, we have $x_1 = n_1$, $x_2 = n_2$. Clearly, $deg(H) = deg(G) = (4^2, 2^{n_2}, 1^{n_1})$.

Case (2). When $x_3 = 6$ and $x_4 = 0$, $deg(H) = (3^6, 2^{n_2-6}, 1^{n_1+2})$.

If $0 \le n_2 \le 5$, then $deg(H)$ does not exist; otherwise, $n_2 \ge 6$. Then by Lemma 9, $t \le 3$. Thus, it follows from Eq. (5.3) that $t' \le 1$.

Case (2.1). If $t' = 1$, then $t = 3$. By Corollary 1 we get $n_2 = 5$ or $n_2 = 3$. Note that $x_2 = n_2 - 6$. So, it leads to a contradiction.

Case (2.2). If $t' = 0$, then $t = 2$. Let G be a c-cyclic graph with $deg(G) = (4^2, 2^{n_2}, 1^{n_1})$. From $c = m - n + 1$ we deduce that $n_1 = 2(3 - c)$, that is, $c \le 3$. Note that $t = 2$. Thus, G is just a bicyclic graph or tricyclic graph.

- Suppose that G is a tricyclic graph, then $n_1 = 0$, and so, it follows from Eq.(5.1) that $deg(H) = (3^6, 2^{n_2-6}, 1^2)$. In fact, we find that if $G \cong G_1^*$ and $H \cong H_1^*$ (see Fig. 7), then G and H are L-cospectral. Thus, G has a L-cospectral mate H with $deg(H) = (3^6, 2^{n_2-6}, 1^2)$.

- Suppose that G is a bicyclic graph, then $G \in \{G_3, G_4, G_5\}$ by Corollary 1. By Lemma 10 we have $\tau(G_3) = 8$, $\tau(G_4) = \tau(G_5) = 9$, hence $\tau(G) \le 9$. Combine with Lemma 1 as well as $t' = 0$ we see that, H is also a bicyclic graph without containing triangles. Let C_1 and C_2 be the first two shortest cycles of H, respectively. And let l_i be the length of C_i for $i = 1, 2$. Clearly, $l_1, l_2 \ge 4$. Suppose that C_1 and C_2 share s common vertices, and $l_1 \ge l_2$. Then by Lemma 10, $\tau(H) = l_1 l_2 \ge 16$ if $s = 0$; and $\tau(H) = l_1 l_2 - s^2 + 2s - 1$ if $1 \le s \le \lfloor \frac{l_2}{2} \rfloor + 1$. For the latter one, one can see that $\tau(H)$ is the smallest when $s = \lfloor \frac{l_2}{2} \rfloor + 1$. Note that both $f_1(l_2) = -l_2^2 + (4l_1 + 2)l_2 - 1$ and $f_2(l_2) = -l_2^2 + 4l_1 l_2$ are increasing functions on the interval $4 \le l_2 \le l_1$. Thus,

$$\begin{aligned}\tau(H) &= l_1 l_2 - s^2 + 2s - 1\\ &\ge l_1 l_2 - (\lfloor \tfrac{l_2}{2} \rfloor + 1)^2 + 2(\lfloor \tfrac{l_2}{2} \rfloor + 1) - 1\\ &= \tfrac{-l_2^2 + (4l_1 + 2)l_2 - 1}{4} \ge 21\end{aligned}$$

  if l_2 is odd, and

$$\begin{aligned}\tau(H) &= l_1 l_2 - s^2 + 2s - 1\\ &\ge l_1 l_2 - (\lfloor \tfrac{l_2}{2} \rfloor + 1)^2 + 2(\lfloor \tfrac{l_2}{2} \rfloor + 1) - 1\\ &= \tfrac{-l_2^2 + 4l_1 l_2}{4} \ge 12\end{aligned}$$

  otherwise. Hence, $\tau(H) \ge \min\{16, 21, 12\} = 12$, it therefore contradicts $\tau(G) \le 9$.

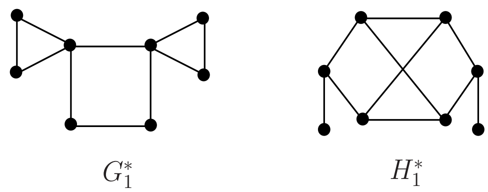


Figure 7: Graphs G_1^* and H_1^*

Case (3). When $x_3 = 3$ and $x_4 = 1$, $deg(H) = (4^1, 3^3, 2^{n_2-3}, 1^{n_1+1})$. Thus, it follows from Lemma 9 and Eq. (5.3), that $t^{'} \leq 2$.

Case (3.1). If $t' = 2$, then $t = 3$. By Corollary 1 we have $G \in \{G_1, G_2\}$. Combine (1) with (4) in Lemma 11, we have $\tau(G_1) = 27$ and $\tau(G_2) = 20$, hence $\tau(G) \leq 27$. Also by Lemma 1 and $t' = 2$ we see that H is also a tricyclic graph with containing 2 triangles. Let $C_i (i = 1, 2, 3)$ be the first three shortest length cycles of H, and l be the length of the third shortest cycle in H, where $l \geq 4$. And we suppose that C_1 and C_3 share s_1 common vertices, C_2 and C_3 share s_2 common vertices, and C_1 and C_2 share s_3 common vertices. And let E_{13}, E_{23} and E_{12} denote the shared edge sets of C_1 and C_3, C_2 and C_3, and C_1 and C_2, respectively. Noting that H contains 2 triangles, we have $0 \leq s_i \leq 2$ for $1 \leq i \leq 3$. Now, according to the value of s_i we classify the following subcases.

(a) When $s_i \leq 1$ for $1 \leq i \leq 3$, by Lemma 11 (1) we have $\tau(H) = l_1 l_2 l_3 = 9l \geq 36$.

(b) When one of s_i $(1 \leq i \leq 3)$ is equal to 2, without loss of generality, we assume that $s_1 = 2$ and $s_2, s_3 \leq 1$. Then by Lemma 11 (2) we have $\tau(H) = l_1 l_2 l_3 - l_2(s_1 - 1)^2 = 9l - l_2$. Since H contains 2 triangles and C_i $(i = 1, 2, 3)$ are the first three shortest cycles of H, l_2 is equal to 3 or l possibly. If $l_2 = 3$, then $\tau(H) = 9l - 3 \geq 36 - 3 = 33$; otherwise, $l_2 = l \geq 4$, then $\tau(H) = 8l \geq 32$. Therefore, $\tau(H) \geq \min\{33, 32\} = 32$.

(c) When two of s_i $(1 \leq i \leq 3)$ are equal to 2, we may suppose that $s_1 = s_2 = 2$ and $s_3 \leq 1$. Then it follows from Lemma 11 (3) that $\tau(H) = 9l - l_1 - l_2$. Similarly, l_1 and l_2 are equal to 3 or l possibly. If one of l_1 and l_2 is equal to l, then $\tau(H) = 9l - 3 - l = 8l - 3 \geq 32 - 3 = 29$. Otherwise, both l_1 and l_2 are equal to 3, and so $\tau(H) = 9l - 6 \geq 36 - 6 = 30$. Thus, $\tau(H) \geq \min\{29, 30\} = 29$.

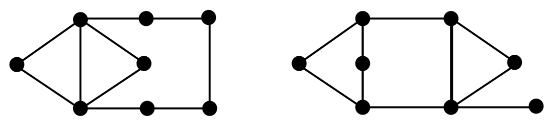
(d) When $s_1 = s_2 = s_3 = 2$, by Lemma 11 (4), if $E_{13} = E_{23} = E_{12}$, we have $\tau(H) = \alpha = 9l - l_1 - l_2 - l_3 + 2 = 8l - 4 \geq 32 - 4 = 28$; and if $E_{13} \neq E_{23} \neq E_{12}$ and $E_{13} \cap E_{23} \cap E_{12} = \emptyset$, then $\tau(H) = \eta = 9l - l_1 - l_2 - l_3 - 2 = 8l - 8$.

Note that $\tau(G_1) = 27$ and $\tau(G_2) = 20$. Since H and $G \in \{G_1, G_2\}$ are L-cospectral, by Lemma 1 (iii) $\tau(H) = \tau(G) \in \{20, 27\}$. If one of above (a), (b) and (c) occurs, then $\tau(H) \geq \min\{29, 32, 36\} = 29 > \tau(G)$, a contradiction; otherwise, if $E_{13} = E_{23} = E_{12}$, similarly, it is also a contradiction; if $E_{13} \cap E_{23} \cap E_{12} = \emptyset$, then $\tau(H) = 8l - 8$. Now we consider the graph G, when $G \cong G_2$, we have $\tau(G) = 20$, which contradicts $\tau(H) \geq 32 - 8 = 24$ due to $l \geq 4$; when $G \cong G_1$, we have $\tau(H) = \tau(G) = 27$, it leads to $\tau(H) = 8l - 8 = 27$. Thus $8l = 35$, it contradicts l being an integer.

Case (3.2). If $t' = 1$, then $t = 2$. By Corollary 1 we see that G is a bicyclic graph or a tricyclic graph.

- Suppose that G is a bicyclic graph, by similar method as Case (2.2), we have that if $s = 0$, $\tau(H) \geq 12$; and if $1 \leq s \leq 2$, $\tau(H) = -s^2 + 2s + 3l - 1 \geq 11$. Therefore, $\tau(H) \geq \min\{12, 11\} = 11$. Therefore, it contradicts $\tau(G) \leq 9$.

- Suppose that G is a tricyclic graph, then $n_1 = 0$. So, it follows from Eq.(5.1) that $deg(H) = (4^1, 3^3, 2^{n_2-3}, 1^1)$. Thus, G has a L-cospectral mate H with $deg(H) = (4^1, 3^3, 2^{n_2-3}, 1^1)$. In fact, we find that the degree sequence exists since if $G \cong G_2^*$ and $H \cong H_2^*$ (see Fig. 8), then G and H are L-cospectral.



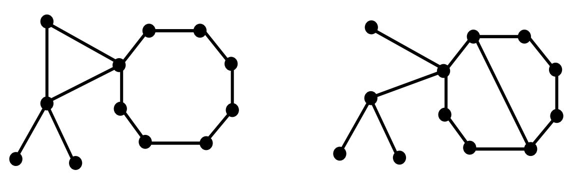
G_2^* H_2^*

Figure 8: Graphs G_2^* and H_2^*

Case (3.3). If $t' = 0$, then $t = 1$. One can see that G is a k-cyclic graph for $1 \leq k \leq 3$.

- Suppose that G is a unicyclic graph, then by Lemma 1 H is also a unicyclic graph. Moreover, G and H have the same number of triangles since H L-cospectral with G implies that the two graphs have the same number of spanning tree. Thus, it is impossible since $t' = 0$ and $t = 1$.

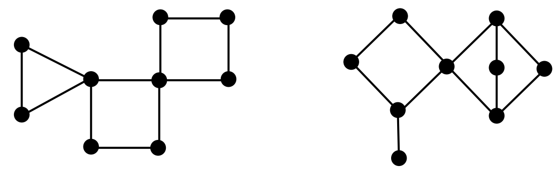
- Suppose that G is a bicyclic graph, then $n_1 = 2$, by Eq. (5.1) we have $deg(H) = (4^1, 3^3, 2^{n_2-3}, 1^3)$. In fact, we find that the degree sequence exists since if $G \cong G_3^*$ and $H \cong H_3^*$ (see Fig. 9), then G and H are L-cospectral. Thus, G has a L-cospectral mate H with $deg(H) = (4^1, 3^3, 2^{n_2-3}, 1^3)$.



G_3^* H_3^*

Figure 9: Graphs G_3^* and H_3^*

- Suppose that G is a tricyclic graph, then $n_1 = 0$, it follows that $deg(H) = (4^1, 3^3, 2^{n_2-3}, 1^1)$. Therefore, G has a L-cospectral mate H with $deg(H) = (4^1, 3^3, 2^{n_2-3}, 1^1)$. In face, we have that G and H are L-cospectral if $G \cong G_4^*$ and $H \cong H_4^*$ (see Fig. 10).



G_4^* H_4^*

Figure 10: Graphs G_4^* and H_4^*

□

Remark 2. *In [23], Wen et al. showed that* $deg(4^1, 2^{n_2}, 1^{n_1})$ *is not determined by Lapalcian spectrum of the corresponding bicyclic graph* G*, where* $G \in \mathcal{G}^*(4^1, 2^{n_2}, 1^{n_1})$*. As a complement, in Theorem 3 we present a precise characterization for the degree sequence with the help of [10].*

6 Degree sequence $deg(G) = (4^{n_4}, 3^{n_3}, 2^{n_2}, 1^{n_1})$

Theorem 4. *Let* G *be a graph of* $\mathcal{G}^*(4^{n_4}, 3^{n_3}, 2^{n_2}, 1^{n_1})$ *with* $c(G) \leq 1$*. If* $n_4 \leq 3$*, then the degree sequence of* G *is determined by Laplacian spectrum.*

Proof. Let H be a graph L-cospectral with G. Then H is a connected graph since $\mu_{n-1}(H) = \mu_{n-1}(G) > 0$ (see [6]). Furthermore, by Lemma 4 we have

$$d_1(H) + 1 \leq \mu_1(H) = \mu_1(G) \leq q_1(G) \leq d_1(G) + d_2(G) = 8$$

hence, it follows that $d_1(H) \leq 7$.

Case 1. $d_1(H) = 7$.

From Lemma 4, $8 = d_1(H) + 1 \leq \mu_1(H) = \mu_1(G) \leq q_1(G) \leq d_1(G) + d_2(G) = 8$, it implies that G is either isomorphic to $K_{1,n-1}$ or a bipartite regular graph and $d_1(H) = n-1$.

If $G \cong K_{1,n-1}$, then $G \cong K_{1,7}$ since $d_1(H) = n-1 = 7$, contrary to $d_1(G) = 4$.

If G is a bipartite regular graph with $d_1(G) = 4$, then $n_2 = n_1 = 0$. Combine with $d_1(H) = n-1 = 7$ we have $n = 8$. And since H is L-cospectral G and G is a regular graph, H is also a regular graph of order 8. So $H = K_8$, it contradicts Lemma 1 (ii).

Case 2. $d_1(H) \leq 6$.

Suppose that H has the degree sequence $deg(H) = (1^{x_1}, 2^{x_2}, 3^{x_3}, 4^{x_4}, 5^{x_5}, 6^{x_6})$. Then by Lemma 1 (i), (ii) and (iii) we have

$$\begin{cases} \sum_{i=1}^{6} x_i = n, \\ \sum_{i=1}^{6} i x_i = n + n_2 + 2n_3 + 3n_4, \\ \sum_{i=1}^{6} i^2 x_i = n + 3n_2 + 8n_3 + 15n_4. \end{cases} \tag{6.1}$$

Thus, it follows from Eq. (6.1) that

$$x_3 + 3x_4 + 6x_5 + 10x_6 = n_3 + 3n_4. \tag{6.2}$$

Let t and t' be the number of triangles in G and H, respectively. Since G and H are L-cospectral, by Lemma 3 it follows that

$$t' - t = x_4 + 4x_5 + 10x_6 - n_4. \tag{6.3}$$

Since $0 \leq c(G) \leq 1$, then by Lemma 1 (i) and (ii) we know that H has the same cyclomatic number, and further by (iii) we get $t' = t$ since H L-cospectral with G implies that the two graphs have the same number of spanning tree. Then $n_4 = x_4 + 4x_5 + 10x_6$. Hence, if $n_4 \leq 3$, then $x_5 = x_6 = 0$ and $n_4 = x_4$. Together with Eq.(6.1) we have $x_1 = n_1$, $x_2 = n_2$, $x_3 = n_3$. Thus, the degree sequence of G is determined by its Laplacian spectrum. □

7 Erratum Remark

Wen et al. in [23] recalled two basic theorems in Section 5 (i.e., Theorems 6 and 7), but didn't label their references for a misprint, the two theorems can be found in [2, 7, 11], we here revise the references of Theorems 6 and 7 of [23] below:

Theorem 5 ([7], Section 3, Lemma 2; [2], Theorem 3.2.2)**.** *Let M be irreducible and η be an eigenvalue of M. Then $|\eta| \leq \rho(|M|)$, with equality if and only if $M = e^{i\phi}N|M|N^{-1}$, where $\rho(|M|)$ is the spectral radius of $|M|$ and $|N| = I$.*

Theorem 6 ([11], Theorem 8.1.22; [7], Section 3, Remark 2)**.** *For any irreducible non-negative matrix M, let $R_i(M)$ be the sum of the i-th row of M. Then*

$$\min\{R_i(M)|1 \leq i \leq n\} \leq \rho(M) \leq \max\{R_i(M)|1 \leq i \leq n\}$$

with equality iff all row sums are equal.

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