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Revisit on representation theory of quantum group $U_q(\mathfrak{sl}_2)$

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Abstract

In this present paper, we recover the well-known finite dimensional representation theory of the classical Drinfeld-Jimbo quantum group $U_q(\mathfrak{sl}_2)$ in a new and elementary way.

Key Words: Krull-Schmidt theorem, quantum group, indecomposable module, q^2-chain module, block.
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1 Introduction

The origin of quantum groups lies in solving the quantum Yang-Baxter equation (abbr. QYBE) appearing in the quantum inverse scattering method [2, Chapter I.1]. In fact, the representation theory of quantum groups can be used to construct interesting and useful solutions for QYBE. In the early 1980s, Kulish and Reshetikhin [7] introduced the first such quantum group $U_q(\mathfrak{sl}_2)$ with its Hopf algebra structure discovered in [8, 9]. Nowdays, $U_q(\mathfrak{sl}_2)$ has become the simplest and most important model in the theory of quantum groups (cf. [2, 4, 6]).

The main results in the finite dimensional representation theory of quantum group $U_q(\mathfrak{sl}_2)$ can be summarized in the theorem below (see Chapter 2 in [4]).

Theorem 1. (1) *Each simple $U_q(\mathfrak{sl}_2)$-module of dimension $n+1$ is isomorphic to a $U_q(\mathfrak{sl}_2)$-module $L(n,\omega)$ with basis $v_0, v_1, \cdots, v_n$ and $\omega^2 = 1$ such that for all $0 \leq i \leq n$*

$$\begin{cases} Kv_i = \omega q^{2i-n} v_i, \\ Ev_i = \begin{cases} \omega[n-i][i+1]v_{i+1}, & \text{if } i < n, \\ 0, & \text{if } i = n, \end{cases} \\ Fv_i = \begin{cases} v_{i-1}, & \text{if } i > 0, \\ 0, & \text{if } i = 0. \end{cases} \end{cases} \tag{1.1}$$

(2) *Each finite dimensional $U_q(sl_2)$-module is semisimple.*

Denote by $U_q(\mathfrak{sl}_2)$-**mod** the category of finite dimensional $U_q(sl_2)$-modules. In this present paper, we reprove Theorem 1 in the following four steps.

(1) In virtue of the notion of q^2-chain module and the classical Krull-Schmidt theorem, we prove that each indecomposable object in $U_q(\mathfrak{sl}_2)$-**mod** is a q^2-chain module, and $U_q(\mathfrak{sl}_2)$-**mod** is the direct sum of its four full subcategories, i.e.,

$$U_q(\mathfrak{sl}_2)\text{-}\mathbf{mod} = \mathcal{O}_1 \oplus \mathcal{O}_{-1} \oplus \mathcal{O}_q \oplus \mathcal{O}_{-q},$$

where $\mathcal{O}_1$ (resp. $\mathcal{O}_q$) is isomorphic to $\mathcal{O}_{-1}$ (resp. $\mathcal{O}_{-q}$) under an additive functor Υ_1 (resp. Υ_q). See Theorem 2.

(2) We deduce the most fundamental observation of our work which says that if the dimensions of all the weight spaces of an indecomposable object M in $\mathcal{O}_1 \oplus \mathcal{O}_q$ are equal, then they must be 1 and the weight set $\Lambda_M = \left\{q^{-n}, q^{-n+2}, \cdots, q^{n-2}, q^n\right\}$, where $n = \dim(M)-1$. See Theorem 3.

(3) Applying the fundamental observation in (2), we construct all the simple objects in $\mathcal{O}_1 \oplus \mathcal{O}_q$ (Theorem 4) and show that the categories $\mathcal{O}_1$ and $\mathcal{O}_q$ are both semisimple (Theorem 5).

(4) Via the additivity and equivalence of the functors Υ_1 and Υ_q, we can reprove Theorem 1.

Throughout the paper, the notations $\mathbb{C}, \mathbb{C}^\times$, $\mathbb{Z}$ and $\mathbb{Z}^{\geq 0}$ denote the complex field, the set of all nonzero complex numbers, the set of all integers and the set of all nonnegative integers, respectively. We always assume that $q \in \mathbb{C}^\times$ is not a root of unity. For $n \in \mathbb{Z}$, we fix the following notation

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

All linear spaces, algebras, modules and unadorned tensors are over the complex field $\mathbb{C}$.

2 Block decomposition theorem for the category $U_q(\mathfrak{sl}_2)$-mod of finite dimensional $U_q(\mathfrak{sl}_2)$-modules

Recall that the classical Drinfeld-Jimbo quantum group $U_q(\mathfrak{sl}_2)$ is the associative algebra with unit 1 generated by four generators K, K^{-1}, E, F and subject to the following relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2EK, \quad KF = q^{-2}FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

Let M be a finite dimensional $U_q(\mathfrak{sl}_2)$-module. The nontrivial linear space

$$M_\lambda = \{v \in M | Kv = \lambda v\}$$

is called a weight space of M, and λ is called a weight of M. If M is the direct sum of its weight spaces, then we call M a weight module of $U_q(\mathfrak{sl}_2)$. In this case, we call the set Λ_M consisting of all the weights of M the weight set of M.

Proposition 1. *[4] Each finite dimensional $U_q(\mathfrak{sl}_2)$-module M is a weight module, and*

$$\Lambda_M \subseteq \Lambda_q = \{\pm q^c | c \in \mathbb{Z}\}.$$

Definition 1. (1) *Let $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_l\}$ be a subset of Λ_q. If there exists $\lambda \in \Lambda$ such that*

$$\Lambda = \left\{\lambda, q^2\lambda, \cdots, q^{2(l-1)}\lambda\right\},$$

then we call Λ a q^2-chain in Λ_q.
(2) *If the weight set Λ_M of a finite dimensional $U_q(\mathfrak{sl}_2)$-module M is a q^2-chain in Λ_q, then*

we call M a q^2-chain module.
(3) *For any $\lambda, \mu \in \Lambda_q$, if there exists an integer $l \in \mathbb{Z}$ such that $\lambda = q^{2l}\mu$, then we say that λ and μ are q^2-linked and denote by $\lambda \overset{q^2}{\sim} \mu$.*

It is easy to check that the relation $\overset{q^2}{\sim}$ on Λ_q is an equivalence relation. For any $\lambda \in \Lambda_q$, denote by $[\lambda]$ the equivalence class containing λ. Set $[\Lambda_q] = \{1, q, -1, -q\}$. Then obviously Λ_q can be expressed as the disjoint union of $[\lambda]$ with $\lambda \in [\Lambda_q]$, i.e., $\Lambda_q = \dot{\bigcup}_{\lambda \in [\Lambda_q]} [\lambda]$.

Definition 2. *For $\lambda \in [\Lambda_q]$, we define the category $\mathcal{O}_\lambda$ to be the full subcategory of $U_q(\mathfrak{sl}_2)$-**mod** with object M satisfying $\Lambda_M \subseteq [\lambda]$. We call $\mathcal{O}_\lambda$ a block of $U_q(\mathfrak{sl}_2)$-**mod**.*

Now we prove the block decomposition theorem for the category $U_q(\mathfrak{sl}_2)$-**mod** which enables us to focus on the blocks $\mathcal{O}_1$ and $\mathcal{O}_q$.

Theorem 2. (1) *Each finite dimensional indecomposable $U_q(\mathfrak{sl}_2)$-module is a q^2-chain module.*
(2) *The category $U_q(\mathfrak{sl}_2)$-**mod** is the direct sum of the blocks $\mathcal{O}_\lambda$ as λ ranges over the set $[\Lambda_q]$, i.e.,*

$$U_q(\mathfrak{sl}_2)\text{-}\mathbf{mod} = \bigoplus_{\lambda \in [\Lambda_q]} \mathcal{O}_\lambda.$$

(3) *Under an additive functor, the block $\mathcal{O}_1$ (resp. $\mathcal{O}_q$) is isomorphic to $\mathcal{O}_{-1}$ (resp. $\mathcal{O}_{-q}$).*

Proof. (1) Let M be a finite dimensional indecomposable $U_q(\mathfrak{sl}_2)$-module. Define a relation $\sim$ on the weight set $\Lambda_M = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ of M as follows: $\lambda_i \sim \lambda_j \iff$ there exists a sequence

$$\lambda_i = \lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_r} = \lambda_j \quad \text{or} \quad \lambda_j = \lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_r} = \lambda_i$$

in Λ_M such that $\lambda_{i_{l+1}} = q^2\lambda_{i_l}$ for $1 \le l \le r-1$. It is easy to check that the relation $\sim$ is an equivalence relation on Λ_M. Denote by $\Lambda_M/\sim = \{\Lambda_1, \Lambda_2, \cdots, \Lambda_s\}$ the set consisting of all the equivalence classes. Since Λ_i is a q^2-chain for any $1 \le i \le s$, then $M_{\Lambda_i} = \underset{\lambda \in \Lambda_i}{\oplus} M_\lambda$ is a q^2-chain submodule of M. Noting that $M = \overset{s}{\underset{i=1}{\oplus}} M_{\Lambda_i}$ and M is indecomposable, we know that $s = 1$. Therefore, M is a q^2-chain module of $U_q(\mathfrak{sl}_2)$.

(2) By (1) and the classical Krull-Schmidt theorem (cf. [1, Section 12.9]), each object in $U_q(\mathfrak{sl}_2)$-**mod** can be decomposed as the direct sum of finitely many indecomposable q^2-chain modules $M_1, M_2, \cdots, M_t$, where each M_i lies in a unique block. On the other hand, it is easy to check that $\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(M, N) = 0$ for any $M \in \mathcal{O}_\lambda$ and $N \in \mathcal{O}_\mu$ with $\lambda, \mu \in [\Lambda_q]$ and $\lambda \neq \mu$.

(3) There is a unique automorphism σ of $U_q(\mathfrak{sl}_2)$ defined by

$$\sigma(K) = -K, \quad \sigma(E) = -E, \quad \sigma(F) = F.$$

For any $\lambda \in [\Lambda_q]$, we define the transitive functor Υ_λ as follows

$$\begin{aligned} \Upsilon_\lambda : \mathcal{O}_\lambda \quad &\longrightarrow \quad \mathcal{O}_{-\lambda}, \\ M \quad &\longmapsto \quad \Upsilon_\lambda(M) = M^\sigma, \\ M \xrightarrow{f} N \quad &\longmapsto \quad \Upsilon_\lambda(M) \xrightarrow{\Upsilon_\lambda(f) = f} \Upsilon_\lambda(N), \end{aligned} \tag{2.1}$$

where $M^\sigma = M$ with the action of $U_q(\mathfrak{sl}_2)$ on M^σ given by

$$K \circ_\sigma m = \sigma(K)m, \quad E \circ_\sigma m = \sigma(E)m, \quad F \circ_\sigma m = \sigma(F)m.$$

It is easy to check that each functor Υ_λ is a well-defined additive functor, and

$$\Upsilon_{-\lambda}\Upsilon_\lambda = \mathrm{Id}_{\mathcal{O}_\lambda}, \quad \Upsilon_\lambda\Upsilon_{-\lambda} = \mathrm{Id}_{\mathcal{O}_{-\lambda}}.$$

Hence Υ_λ is an isomorphism of categories. □

Remark 1. *Though we make use of the term "block", the blocks here just satisfy some but not all the conditions described in Section 1.13 in [3].*

3 A fundamental observation

In this section, we deduce a fundamental observation about the indecomposable objects in $\mathcal{O}_1 \oplus \mathcal{O}_q$ which will play a key role not only in reconstructing all the simple objects in $U_q(\mathfrak{sl}_2)$-**mod** but also in reproving the semisimplicity of $U_q(\mathfrak{sl}_2)$-**mod** later.

Suppose that M is a $(n+1)$-dimensional indecomposable module in the category $\mathcal{O}_1 \oplus \mathcal{O}_q$. It follows from Theorem 2 (1) that M is a q^2-chain module. Assume that $\Lambda_M = \{q^{s+2i} \mid 0 \leq i \leq l\}$ for some $s \in \mathbb{Z}$ and $l \in \mathbb{Z}^{\geq 0}$, then $M = \overset{l}{\underset{i=0}{\oplus}} M_{q^{s+2i}}$. For $0 \leq i \leq l$, set $\dim M_{q^{s+2i}} = n_i$ and choose a basis $\{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$ of $M_{q^{s+2i}}$. Then

$$B_M = \{v_{01}, v_{02}, \cdots, v_{0n_0}, \cdots, v_{i1}, v_{i2}, \cdots, v_{in_i}, \cdots, v_{l1}, v_{l2}, \cdots, v_{ln_l}\}$$

is an ordered basis of M. Since $KE = q^2EK$ and $KF = q^{-2}FK$, then $EM_{q^{s+2i}} \subseteq M_{q^{s+2(i+1)}}$ and $FM_{q^{s+2i}} \subseteq M_{q^{s+2(i-1)}}$. Therefore, the matrices of K, E, F acting on M relative to the ordered basis B_M respectively have the following forms

$$\begin{aligned}
\mathcal{K} &= \begin{pmatrix} q^sI_{n_0} & & & \\ & q^{s+2}I_{n_1} & & \\ & & \ddots & \\ & & & q^{s+2l}I_{n_l} \end{pmatrix}, \\
\mathcal{E} &= \begin{pmatrix} 0 & & & & \\ \mathcal{E}_0 & 0 & & & \\ & \mathcal{E}_1 & \ddots & & \\ & & \ddots & 0 & \\ & & & \mathcal{E}_{l-1} & 0 \end{pmatrix}, \\
\mathcal{F} &= \begin{pmatrix} 0 & \mathcal{F}_0 & & & \\ & 0 & \mathcal{F}_1 & & \\ & & \ddots & \ddots & \\ & & & 0 & \mathcal{F}_{l-1} \\ & & & & 0 \end{pmatrix},
\end{aligned} \tag{3.1}$$

where I_{n_i} is the $n_i \times n_i$ identity matrix, $\mathcal{E}_i$ is a $n_{i+1} \times n_i$ matrix and $\mathcal{F}_i$ is a $n_i \times n_{i+1}$ matrix. Unless otherwise specified, we always assume that $\mathcal{E}_i = 0$ and $\mathcal{F}_i = 0$ when $i \le -1$ or $i \ge l$. Since $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$, then $\mathcal{K}, \mathcal{E}, \mathcal{F}$ must satisfy

$$\mathcal{E} = \mathcal{F} = 0, \quad s = 0, \quad \text{if } l = 0, \tag{3.2}$$

$$\mathcal{E}_{i-1}\mathcal{F}_{i-1} - \mathcal{F}_i\mathcal{E}_i = [s+2i]I_{n_i} \ (0 \le i \le l), \quad \text{if } l \ge 1. \tag{3.3}$$

It is easy to check that

$$\mathrm{End}_{U_q(\mathfrak{sl}_2)}(M) \cong \left\{ \begin{pmatrix} A_0 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_l \end{pmatrix} \middle| \begin{array}{l} \mathcal{E}_i A_i = A_{i+1}\mathcal{E}_i (0 \le i \le l-1), \\ A_i \mathcal{F}_i = \mathcal{F}_i A_{i+1} (0 \le i \le l-1) \end{array} \right\}, \tag{3.4}$$

where A_i is a $n_i \times n_i$ matrix for $0 \le i \le l$.

The following result is the most important observation of this present paper which lays a fundamental foundation for us to recover the representation theory of $U_q(\mathfrak{sl}_2)$.

Theorem 3. *Let M be a $(n+1)$-dimensional indecomposable module in the category $\mathcal{O}_1 \oplus \mathcal{O}_q$. If the dimensions of all the weight spaces of M are equal, then they are all equal to* 1*, and*

$$\Lambda_M = \{q^{-n}, q^{-n+2}, \cdots, q^{n-2}, q^n\}. \tag{3.5}$$

Proof. In the following proof, we will retain all the notations above. Since the dimensions of all the weight spaces of M are equal, then $\dim M_{q^{s+2i}} = n_0$ for all $0 \le i \le l$. When $l = 0$, noting that M is indecomposable, we can see from (3.2) that $n_0 = n+1 = 1$ and $\Lambda_M = \{1\}$.

From now on, we assume that $l \ge 1$. For any $0 \le i \le l-1$, set

$$\begin{cases} a_i = \sum\limits_{k=0}^{i} [-s-2k] = [-s-i][i+1], \\ b_i = \sum\limits_{k=i+1}^{l} [s+2k] = [l-i][s+l+i+1]. \end{cases} \tag{3.6}$$

We claim that $a_i = b_i \ne 0$ for any $0 \le i \le l-1$. It follows from (3.6) that there exists at most one a_i (resp. b_i) with $0 \le i \le l-1$ such that $a_i = 0$ (resp. $b_i = 0$). If there exists some $0 \le i_0 \le l-1$ such that $a_{i_0} = 0$, then by (3.6) one has $s = -i_0$ and $b_i = [l-i][l+1+i-i_0] \ne 0$ for all $0 \le i \le l-1$. By respectively adding the top i_0+1 formulas with $i = 0, 1, \ldots, i_0$ and the bottom $l - i_0$ ones with $i = i_0+1, i_0+2, \ldots, l$ in (3.3), one has

$$\mathcal{F}_{i_0}\mathcal{E}_{i_0} = a_{i_0} I_{n_0} = 0 \quad \text{and} \quad \mathcal{E}_{i_0}\mathcal{F}_{i_0} = b_{i_0} I_{n_0} \ne 0,$$

which is a contradiction. Hence $a_i \ne 0$ for all $0 \le i \le l-1$. Similarly, $b_i \ne 0$ for all $0 \le i \le l-1$. When $a_i \ne 0$ and $b_i \ne 0$ for all $0 \le i \le l-1$, by respectively adding the top $i+1$ formulas and the bottom $l-i$ ones in (3.3), one has

$$\mathcal{F}_i\mathcal{E}_i = a_i I_{n_0} \quad \text{and} \quad \mathcal{E}_i\mathcal{F}_i = b_i I_{n_0}, \tag{3.7}$$

which imply that $a_i = b_i \neq 0$. Now for any $0 \leq i \leq l-1$ one has

$$\mathcal{E}_i\mathcal{F}_i = \mathcal{F}_i\mathcal{E}_i = a_i I_{n_0}. \tag{3.8}$$

Combining (3.4) and (3.8), one has

$$\mathrm{End}_{U_q(\mathfrak{sl}_2)}(M) \cong \mathbf{Mat}_{n_0}(\mathbb{C}), \tag{3.9}$$

where $\mathbf{Mat}_{n_0}(\mathbb{C})$ is the matrix algebra consisting of all $n_0 \times n_0$ complex matrices. Since M is indecomposable, then $\mathrm{End}_{U_q(\mathfrak{sl}_2)}(M)$ is local, which implies $n_0 = 1$.

Next we show that $\Lambda_M = \{q^{-n}, q^{-n+2}, \cdots, q^{n-2}, q^n\}$. Since M is $(n+1)$-dimensional, then $n+1 = \sum_{i=0}^{l} n_i = l+1$. So we can obtain $\Lambda_M = \{q^s, q^{s+2}, \cdots, q^{s+2(n-1)}, q^{s+2n}\}$. By (3.6), one gets

$$a_i - b_i = \sum_{k=0}^{n}[-s-2k] = [-s-n][n+1]. \tag{3.10}$$

Because $a_i = b_i$ and $q \in \mathbb{C}^\times$ is not a root of unity, one must have $s = -n$. The proof is finished. □

4 Reformulation of finite dimensional representation theory of $U_q(\mathfrak{sl}_2)$

In this section, we will apply the fundamental observation in Section 3 to recover the finite dimensional representation theory of $U_q(\mathfrak{sl}_2)$.

Lemma 1. *The blocks $\mathcal{O}_1$ and $\mathcal{O}_q$ of $U_q(\mathfrak{sl}_2)$-***mod** *are both closed under taking submodules and quotient modules.*

Proof. For any object N in $\mathcal{O}_1$ or $\mathcal{O}_q$, denote by $g_N(x)$ the characteristic polynomial of K acting on N. Noting that $g_N(x) = g_L(x)g_{N/L}(x)$ for any submodule L of N, we can finish the proof. □

For all $n \in \mathbb{Z}$, set

$$[K; n] = \frac{q^n K - q^{-n}K^{-1}}{q - q^{-1}}.$$

Recall a formula in Section 1.3 in [4] below:

$$EF^r - F^r E = [r]F^{r-1}[K; 1-r]. \tag{4.1}$$

Let

$$C_q := EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$$

be the Casimir element in $U_q(\mathfrak{sl}_2)$.

Now we can clearly describe the simple $U_q(\mathfrak{sl}_2)$-modules in the category $\mathcal{O}_1 \oplus \mathcal{O}_q$.

Theorem 4. *Let M be a $(n+1)$-dimensional simple $U_q(\mathfrak{sl}_2)$-module in the category $\mathcal{O}_1 \oplus \mathcal{O}_q$.*
(1) *The dimensions of all the weight spaces of M are equal to 1.*
(2) *M is isomorphic to the simple module $L(n,1)$ with basis $w_0, w_1, \cdots, w_n$ and the actions of K, E, F on M given below*

$$\begin{cases} Kw_i = q^{2i-n} w_i, \\ Ew_i = \begin{cases} [n-i][i+1]w_{i+1}, & \text{if } i < n, \\ 0, & \text{if } i = n, \end{cases} \\ Fw_i = \begin{cases} w_{i-1}, & \text{if } i > 0, \\ 0, & \text{if } i = 0. \end{cases} \end{cases} \tag{4.2}$$

(3) *The Casimir element C_q acts on M by the same scalar $c_q(n)$ as on $L(n,1)$, where*

$$c_q(n) = \frac{q^{n+1} + q^{-(n+1)}}{(q - q^{-1})^2}.$$

Proof. In this proof, we also retain the notations in the second paragraph of Section 3.

(1) Choose any nonzero vector $v_l \in M_{q^{s+2l}}$. It follows from (4.1) that $\bigoplus_{i=0}^{l} \mathbb{C}F^{l-i} v_l$ is a submodule of M. The simplicity of M implies that $M = \bigoplus_{i=0}^{l} \mathbb{C}F^{l-i} v_l$. The proof is finished.

(2) It follows from Theorem 3, (3.1), (3.2) and (3.3) that M can be presented by

$$\begin{cases} Kv_i = q^{2i-n} v_i, \\ Ev_i = \begin{cases} \mathcal{E}_i v_{i+1}, & \text{if } i < n, \\ 0, & \text{if } i = n, \end{cases} \\ Fv_i = \begin{cases} \mathcal{F}_{i-1} v_{i-1}, & \text{if } i > 0, \\ 0, & \text{if } i = 0, \end{cases} \end{cases}$$

where $v_0, v_1, \cdots, v_n$ is a basis of M and $\mathcal{E}_i, \mathcal{F}_i \in \mathbb{C}$ $(0 \leq i \leq n-1)$ satisfy $\mathcal{E}_i \mathcal{F}_i = [n-i][i+1]$. Since the following set of linear equations

$$\begin{cases} \mathcal{E}_i \lambda_{i+1} = [n-i][i+1]\lambda_i \ (0 \leq i \leq n-1), \\ \mathcal{F}_i \lambda_i = \lambda_{i+1} \ (0 \leq i \leq n-1) \end{cases}$$

has a nonzero solution $(\lambda_0, \lambda_1, \cdots, \lambda_n)$ with all $\lambda_i \neq 0$, then it is easy to check that the map $M \xrightarrow{\phi} L(n,1)$ defined by $\phi(v_i) = \lambda_i w_i$ is an isomorphism of $U_q(\mathfrak{sl}_2)$-modules.

Next we will prove that $L(n,1)$ is simple. Otherwise, the length t of $L(n,1)$ is at least 2, i.e., there exists a composition series of $L(n,1)$ as follows

$$0 = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_t = L(n,1).$$

Since L_1 is a nontrivial simple submodule of $L(n,1)$, then by (1), Theorem 3 and Lemma 1 one obtains $\Lambda_{L_1} = \{q^{-l_1}, q^{-l_1+2}, \cdots, q^{l_1-2}, q^{l_1}\} \subseteq \Lambda_{L(n,1)}$, where $l_1 = \dim L_1 - 1 < n$. Therefore, $L_1 = \bigoplus_{i=\frac{n-l_1}{2}}^{\frac{n+l_1}{2}} \mathbb{C}w_i$. However, the formulas in (4.2) show that L_1 is not a submodule of $L(n,1)$, which is a contradiction.

(3) Noting that (3.7) and (3.8) both hold for M and $L(n,1)$, and $\Lambda_M = \Lambda_{L(n,1)}$, we can deduce that C_q acts on M by the same scalar $c_q(n)$ as on $L(n,1)$ by direct calculations. □

Corollary 1. *For a given $\lambda \in \{1, q\}$, let L and L' be finite dimensional simple $U_q(\mathfrak{sl}_2)$-modules in the block $\mathcal{O}_\lambda$. If C_q acts on L by the same scalar as on L', then L is isomorphic to L'.*

Proof. Suppose that $\dim L = n+1$ and $\dim L' = n'+1$, then by Theorem 4 (2) one has $L \cong L(n,1)$ and $L' \cong L(n',1)$. By Theorem 4 (3), C_q acts on L (resp. L') by the same scalar $c_q(n)$ (resp. $c_q(n')$) as on $L(n,1)$ (resp. $L(n',1)$). If C_q acts on L by the same scalar as on L', then $c_q(n) = c_q(n')$. By direct calculations, $c_q(n) = c_q(n')$ if and only if

$$q^{-(n+1)}(q^{n+n'+2}-1)(q^{n-n'}-1) = 0,$$

which is equivalent to say $n = n'$. Therefore, $L \cong L(n,1) \cong L'$. □

Now we can prove the semisimplicity of the blocks $\mathcal{O}_1$ and $\mathcal{O}_q$. Although our proof has some similar ideas as that of Theorem 2.9 in [4], the applications of some new strategies contained in Theorem 2, Theorem 3 and Lemma 1 make it different.

Theorem 5. *The blocks $\mathcal{O}_1$ and $\mathcal{O}_q$ of $U_q(\mathfrak{sl}_2)$-**mod** are both semisimple.*

Proof. By Krull-Schmidt theorem and Lemma 1, we only need to show that each indecomposable $U_q(\mathfrak{sl}_2)$-module M in the block $\mathcal{O}_\lambda$ with $\lambda \in \{1, q\}$ is simple, i.e., the length $l(M)$ of M is 1.

Assume that $g(x) = (x-\mu_1)^{r_1}(x-\mu_2)^{r_2}\cdots(x-\mu_s)^{r_s}$ is the characteristic polynomial of C_q acting on M. Then M is the direct sum of the generalized eigenspaces for C_q, i.e., $M = \bigoplus_{k=1}^{s} M^{\mu_i}$, where $M^{\mu_i} = \{v \in M | (C_q - \mu_i)^{r_i} v = 0\}$. Since C_q is central in $U_q(\mathfrak{sl}_2)$, each M^{μ_i} is a submodule of M. Hence $M = M^\mu = \{v \in M | (C_q - \mu)^r v = 0\}$ for some μ because M is indecomposable.

Suppose that $l(M) = l$. By Lemma 1, we can pick a composition series

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_l = M \tag{4.3}$$

of M in the block $\mathcal{O}_\lambda$ containing M. Since $M = M^\mu$, then $C_q - \mu$ acts nilpotently on each $M_i/M_{i-1} (1 \le i \le l)$. On the other hand, by Schur lemma C_q acts by a scalar ν_i on M_i/M_{i-1}. Hence for all $1 \le i \le l$ one has $\nu_i = \mu$. Moreover, by Corollary 1 there exists an integer $n_0 \ge 0$ such that each $M_i/M_{i-1} (1 \le i \le l)$ is isomorphic to $L(n_0, 1)$.

Let N be a submodule of M. Since $\dim M_\nu = \dim N_\nu + \dim(M/N)_\nu$ for any $\nu \in \Lambda_M$, then we apply this repeatedly to the composition series (4.3) and obtain

$$\dim M_\nu = \sum_{i=1}^{l} \dim(M_i/M_{i-1})_\nu = l \dim L(n_0,1)_\nu = l$$

for any $\nu \in \Lambda_M$. It follows from Theorem 3 that the dimensions of all the weight spaces of M are equal to 1, i.e., $l = 1$. Therefore, M is simple. □

Proof of Theorem 1 Note that the transitive functor Υ_λ defined in (2.1) is an additive functor. On one hand, we can obtain all the finite dimensional simple $U_q(\mathfrak{sl}_2)$-modules listed in Theorem 1 (1) by applying $\Upsilon_\lambda(\lambda = 1, q)$ to the simple modules presented in Theorem 4 (2). On the other hand, we can see from Theorem 2 (3) and Theorem 5 that the blocks $\mathcal{O}_{-1}$ and $\mathcal{O}_{-q}$ of $U_q(\mathfrak{sl}_2)$-**mod** are also semisimple.

Remark 2. *The method in this paper can be generalized to deal with the finite dimensional representation theory of the quantum groups $U_q(f(K))$ introduced in [5].*

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