Bull. Math. Soc. Sci. Math. Roumanie
Tome 66 (114), No. 4, 2023, 449–453

On a conjecture related to integer-valued polynomials

by
Victor J. W. Guo

Abstract

Using the following ${}_4F_3$ transformation formula

$$\sum_{k=0}^{n}\binom{-x-1}{k}^2\binom{x}{n-k}^2=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2\binom{x+k}{2k},$$

which can be proved by Zeilberger's algorithm, we confirm some special cases of a recent conjecture of Z.-W. Sun on integer-valued polynomials.

Key Words: Zeilberger's algorithm, Chu-Vandermonde summation, integer-valued polynomials, multi-variable Schmidt polynomials.
2020 Mathematics Subject Classification: Primary 33C20; Secondary 11A07, 11B65, 05A10.

1 Introduction

Recall that a polynomial $P(x)\in\mathbb{Q}[x]$ is called *integer-valued*, if $P(x)\in\mathbb{Z}$ for all $x\in\mathbb{Z}$. During the past few years, integer-valued polynomials have been investigated by several authors (see, for example, [3, 6, 13]). Recently, Z.-W. Sun [14, Conjectures 35(i)] proposed the following conjecture.

Conjecture 1 (Z.-W. Sun)**.** *Let l,m,n be positive integers and $\varepsilon=\pm1$. Then the polynomial*

$$\frac{1}{n}\sum_{k=0}^{n-1}\varepsilon^k(2k+1)^{2l-1}\sum_{j=0}^{k}\binom{-x-1}{j}^m\binom{x}{k-j}^m$$

is integer-valued.

By the Chu-Vandermonde summation formula, we have

$$\sum_{j=0}^{k}\binom{-x-1}{j}\binom{x}{k-j}=\binom{-1}{k}=(-1)^k.$$

Thus, by [9, Lemmas 2.3 and 2.4], we see that Conjecture 1 is true for $m=1$. In this note, we shall confirm Conjecture 1 for $m=2$.

Theorem 1. *Let l and n be positive integers and $\varepsilon = \pm 1$. Then the polynomial*

$$\frac{1}{n}\sum_{k=0}^{n-1}\varepsilon^k(2k+1)^{2l-1}\sum_{j=0}^{k}\binom{-x-1}{j}^2\binom{x}{k-j}^2 \tag{1.1}$$

is integer-valued.

We shall also prove the following result, which confirms the $l = 1$ cases of [14, Conjectures 35(ii)].

Theorem 2. *Let n be a positive integer. Then the polynomial*

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)\sum_{j=0}^{k}\binom{-x-1}{j}^2\binom{x}{k-j}^2 \tag{1.2}$$

is integer-valued.

2 Proof of Theorem 1

We first require the following ${}_4F_3$ transformation formula.

Lemma 1. *Let n be a non-negative integer. Then*

$$\sum_{k=0}^{n}\binom{-x-1}{k}^2\binom{x}{n-k}^2 = \sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2\binom{x+k}{2k}. \tag{2.1}$$

Proof. Denote the left-hand side or the right-hand side of (2.1) by $S_n(x)$. Applying Zeilberger's algorithm (see [1, 10]), we obtain

$$(n+2)^3S_{n+2}(x) - (2n+3)(n^2+2x^2+3n+2x+3)S_{n+1}(x) + (3n^2+3n+1)S_n(x) = 0.$$

That is to say, both sides of (2.1) satisfy the same recurrence relation of order 2. Moreover, the two sides of (2.1) are equal for $n = 0, 1$. This completes the proof. □

Using Zeilberger's algorithm, Z.-W. Sun [11, Eq. (3.1)] found the following identity:

$$16^n\sum_{k=0}^{n}\binom{-1/2}{k}^2\binom{-1/2}{n-k}^2 = \sum_{k=0}^{n}\binom{2k}{k}^3\binom{k}{n-k}(-16)^{n-k}, \tag{2.2}$$

and he [12, Eq. (3.1)] gave the following formula:

$$64^n\sum_{k=0}^{n}\binom{-1/4}{k}^2\binom{-3/4}{n-k}^2 = \sum_{k=0}^{n}\binom{2k}{k}^3\binom{2n-2k}{n-k}16^{n-k}. \tag{2.3}$$

Here we point out that, for $x = -1/2$ and $-3/4$, Eq. (2.1) gives identities different from (2.2) and (2.3).

In [2], Chen and the author introduced the multi-variable Schmidt polynomials

$$S_n(x_0,\dots,x_n)=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}x_k.$$

In order to prove Theorem 1, we also need the following result, which is a special case of the last congruence in [2, Section 4].

Lemma 2. *Let l and n be positive integers and $\varepsilon=\pm 1$. Then all the coefficients in*

$$\sum_{k=0}^{n-1}\varepsilon^k(2k+1)^{2l-1}S_k(x_0,\dots,x_k).$$

are multiples of n.

Proof of Theorem 1. For any non-negative integer k, define

$$x_k=\binom{2k}{k}\binom{x+k}{2k}.$$

Then the identity (2.1) may be rewritten as

$$\sum_{k=0}^{n}\binom{-x-1}{k}^2\binom{x}{n-k}^2=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}x_k. \tag{2.4}$$

It is easy to see that $x_0,\dots,x_n$ are all integers on condition that x is an integer. By Eq. (2.4) and Lemma 2, we see that the polynomial (1.1) is integer-valued. ∎

3 Proof of Theorem 2

We need the following result, which can be easily proved by induction on n. See also [2, Eq. (2.4)].

Lemma 3. *Let n and k be non-negative integers with $k\leqslant n-1$. Then*

$$\sum_{m=k}^{n-1}(2m+1)\binom{m+k}{2k}\binom{2k}{k}=n\binom{n}{k+1}\binom{n+k}{k}. \tag{3.1}$$

Proof of Theorem 2. Using the identities (2.1) and (3.1), we have

$$\begin{aligned}\sum_{m=0}^{n-1}(2m+1)\sum_{k=0}^{m}\binom{-x-1}{k}^2\binom{x}{n-k}^2&=\sum_{m=0}^{n-1}(2m+1)\sum_{k=0}^{m}\binom{n+k}{2k}\binom{2k}{k}^2\binom{x+k}{2k}\\&=\sum_{k=0}^{n-1}n\binom{n}{k+1}\binom{n+k}{k}\binom{2k}{k}\binom{x+k}{2k}.\end{aligned}$$

It follows that the expression (1.2) can be written as

$$\sum_{k=0}^{n-1}\frac{1}{n}\binom{n}{k+1}\binom{n+k}{k}\binom{2k}{k}\binom{x+k}{2k}=\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n-1}{k}\binom{n+k}{k}\binom{2k}{k}\binom{x+k}{2k}. \tag{3.2}$$

Since $\frac{1}{k+1}\binom{2k}{k}=\binom{2k}{k}-\binom{2k}{k-1}$ is clearly an integer (the n-th Catalan number), we conclude that the right-hand side of (3.2) is also an integer whenever x is an integer. This proves the theorem. ∎

4 Concluding remarks

Z.-W. Sun [14, Conjecture 35(ii)] conjectured that, for all positive integers l and n, the polynomial

$$\frac{(2l-1)!!}{n^2}\sum_{k=0}^{n-1}(2k+1)^{2l-1}\sum_{j=0}^{k}\binom{-x-1}{j}^2\binom{x}{k-j}^2$$

is integer-valued. Here $(2l-1)!!=(2l-1)(2l-3)\cdots 3\cdot 1$.

We believe that the following (stronger) result is true.

Conjecture 2. *Let l and n be positive integers and k a non-negative integer with $k\leqslant n-1$. Then*

$$(2l-1)!!\sum_{m=k}^{n-1}(2m+1)^{2l-1}\binom{m+k}{2k}\binom{2k}{k}^2\equiv 0 \pmod{n^2}. \tag{4.1}$$

Our proof of Theorem 2 implies that the above conjecture is true for $l=1$. In view of (2.1), Sun's conjecture follows from (4.1) too.

Recently, q-analogues of congruences have been studied by many authors. See [4, 5, 7, 8, 15] and references therein. For $l=1$, we have a q-analogue of (4.1) as follows:

$$\sum_{m=k}^{n-1}[2m+1]{m+k\brack 2k}{2k\brack k}^2 q^{-(k+1)m}\equiv 0 \pmod{[n]^2}, \tag{4.2}$$

where $[n]=1+q+\cdots+q^{n-1}$ is the q-integer and ${n\brack k}=\prod_{j=1}^{k}(1-q^{n-k+j})/(1-q^j)$ denotes the q-binomial coefficient. The proof of (4.2) is similar to that of Theorem 2. However, we cannot find any q-analogue of (4.1) for $l>1$.

References

[1] M. Apagodu, D. Zeilberger, Multi-variable Zeilberger and Almkvist–Zeilberger algorithms and the sharpening of Wilf–Zeilberger theory, *Adv. Appl. Math.*, **37**, 139–152 (2006).

[2] Q.-F. Chen, V. J. W. Guo, On the divisibility of sums involving powers of multi-variable Schmidt polynomials, *Int. J. Number Theory*, **14**, 365–370 (2018).

[3] V. J. W. Guo, Proof of Sun's conjectures on integer-valued polynomials, *J. Math. Anal. Appl.*, **444**, 182–191 (2016).

[4] V. J. W. Guo, M. J. Schlosser, A family of q-supercongruences modulo the cube of a cyclotomic polynomial, *Bull. Aust. Math. Soc.*, **105**, 296–302 (2022).

[5] V. J. W. Guo, W. Zudilin, A q-microscope for supercongruences, *Adv. Math.*, **346**, 329–358 (2019).

[6] J.-C. Liu, Proof of some divisibility results on sums involving binomial coefficients, *J. Number Theory*, **180**, 566–572 (2017).

[7] J.-C. Liu, X.-T. Jiang, On the divisibility of sums of even powers of q-binomial coefficients, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.*, **116**, Art. 76 (2022).

[8] Y. Liu, X. Wang, Some q-supercongruences from a quadratic transformation by Rahman, *Results Math.*, **77**, Art. 44 (2022).

[9] G.-S. Mao, Proof of some congruence conjectures of Guo and Liu, *Ramanujan J.*, **48**, 233–244 (2019).

[10] M. Petkovšek, H. S. Wilf, D. Zeilberger, $A = B$, A K Peters, Ltd., Wellesley, MA (1996).

[11] Z.-W. Sun, On sums involving products of three binomial coefficients, *Acta Arith.*, **156**, 123–141 (2012).

[12] Z.-W. Sun, Some new series for $1/\pi$ and related congruences, *Nanjing Univ. J. Math. Biquarterly*, **131**, 150–164 (2014).

[13] Z.-W. Sun, Supercongruences involving dual sequences, *Finite Fields Appl.*, **46**, 179–216 (2017).

[14] Z.-W. Sun, Open conjectures on congruences, *Nanjing Univ. J. Math. Biquarterly*, **36**, 1–99 (2019).

[15] X. Wang, C. Xu, q-Supercongruences on triple and quadruple sums, *Results Math.*, **78**, Art. 27 (2023).

Received: 08.04.2022
Accepted: 12.05.2022

School of Mathematics and Statistics, Huaiyin Normal University,
Huai'an, Jiangsu 223300, People's Republic of China
E-mail: `jwguo@math.ecnu.edu.cn`