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ACM bundles on abelian surfaces of Picard rank 4

by
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Abstract

Let X be an abelian surface. It is a recent result due to Beauville that all polarized abelian surfaces possess an Ulrich (hence ACM) bundle of rank 2. However, the case of ACM line bundles was left open. In this note we solve the problem in the affirmative by discussing their existence on a member of the isogeny class of any X in the case of Picard number 4, and finally extend to any such X.

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1 Introduction

Let X be a complex projective manifold of dimension n and choose a polarization L. An *arithmetically Cohen-Macaulay* (henceforth ACM) bundle on X is a vector bundle M such that for all $0 < j < n$ and $t \in \mathbf{Z}$, $H^j(X, M(tL)) = 0$.

ACM sheaves are defined in such a way that they represent the analogous of maximal Cohen-Macaulay (MCM) modules in commutative algebra, that is, as a generalization of projective varieties with minimal projective dimension, i.e. length of a minimal free resolution. For curves, the condition on cohomology is vacuous, so the first interesting case is that of surfaces. In the case of a projective space $X = \mathbf{P}^n$, the result of Horrocks [7] (which predates the algebraic theory and anyway is a completely geometric one) states that these bundles are precisely direct sums of line bundles $\mathcal{O}(k)$.

There is also an analogy with the representation theory of quivers: if a variety admits finitely many indecomposable ACM bundles, then it is called of *finite representation type*; it is of *tame* representation type if for any rank there are finitely many families of dimension at most one of indecomposable ACM bundles. Otherwise, there are families of arbitrary large dimension, and the variety is said to be of *wild* representation type (which is the case for instance with Segre varieties, see [3]). Other results concern the decomposition as sums of (twists) of spinor bundles and the trivial bundle on quadrics [10] and on determinantal varieties [12]. See also [5], [6], [8], [11].

The exposition follows the following course. In the preliminaries, we will give a remainder of the necessary notions from the theory of abelian varieties and adapt as needed to the case of surfaces. The material is drawn from the two monographs of Birkenhake and Lange [2] and [9]. We then state our main result, and proceed immediately to the proof in the case of a representative of an isogeny class. Finally, we discuss existence for $\rho = 4$. This is complementary to the recent result of Beauville in [1], which states that any polarized abelian surface carries a rank 2 Ulrich bundle and also to a recent work by Yoshioka [13] discussing the $\rho = 1$ case.

2 Preliminaries

Let (X, L) be a polarized abelian surface. Since $h^{1,1}(X) = 4$ while X carries non-zero line bundles, $1 \leq \rho(X) \leq 4$. Now, by [2, 2.7.1], assuming $\rho = 4$ is equivalent to X being isogeneous to a product $E \times E$, where E is an elliptic curve with complex multiplication. This means that for $X = E \times E$ we can write the lattice in the form

$$\Lambda = \begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & 0 & \tau \end{pmatrix}$$

where $E = \mathbf{C}/\langle 1, \tau \rangle$ has CM. This means that $\tau = \frac{p + \mathrm{i}\sqrt{q}}{r}$, where $q, r > 0$ and p are integers. Now, the Appell-Humbert Theorem [9, 2.2.3] states that there is a surjective map $p : (H, \chi) \mapsto H$ from the set of pairs $(H, \chi) \in \mathcal{P}(\Lambda)$ to $\mathrm{NS}(X)$, where H are Hermitian forms such that $\operatorname{Im} H(\Lambda, \Lambda) \subset \mathbf{Z}$ and χ are semicharacters for H.

3 The result

We state here our

Theorem. Let X be an abelian surface of Picard rank 4. Then there are infinitely many classes of polarizations in $NS(X)$ for which there exists at least one corresponding ACM line bundle on X.

Proof. We make use of the theorem [9, 3.5.5]. Let L be a line bundle corresponding to (H, χ) in $\mathcal{P}(\Lambda)$ such that its alternating form $E = \operatorname{Im} H$ can be expressed in some basis for Λ as

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}.$$

Here D is given by the elementary divisor theorem as $D = \mathrm{diag}(d_1, d_2)$ with $d_1 | d_2$, so that the Pfaffian of the given matrix is $d_1 d_2$. We denote by $\mathrm{Pfr}(E)$ the *reduced Pfaffian*, that is, the product of the nonzero values in $\{d_1, d_2\}$ (or 1 if $d_1 = 0$). Then (3.5.5) states that $h^1(L)$ is zero for the signature of the form H either $(2, 0)$ or $(0, 2)$ and $\mathrm{Pfr}(E) \neq 0$ otherwise.

Returning to our problem, if we consider a polarized surface $(E \times E, L)$, it suffices to find a line bundle M such that $H_{M(tL)}$ has eigenvalues of the same sign for all $t \in \mathbf{Z}$.

Write the Hermitian matrices for the polarization L and for the line bundle sought M

$$H_M = \begin{pmatrix} a & b + c\mathrm{i} \\ b - c\mathrm{i} & d \end{pmatrix}, \; H_L = \begin{pmatrix} a' & b' + c'\mathrm{i} \\ b' - c'\mathrm{i} & d' \end{pmatrix}, \qquad a, b, c, d, a', b', c', d' \in \mathbf{R}$$

so that the conditions $\operatorname{Im} H_M(\Lambda,\Lambda) \subset \mathbf{Z}$, $\operatorname{Im} H_L(\Lambda,\Lambda) \subset \mathbf{Z}$ amount to

$$c, c' \in \mathbf{Z} \quad \text{and} \quad a = \frac{a_0 r}{\sqrt{q}}, d = \frac{d_0 r}{\sqrt{q}}, a' = \frac{a_0' r}{\sqrt{q}}, d' = \frac{d_0' r}{\sqrt{q}}, b = \frac{b_0}{\sqrt{q}}, b' = \frac{b_0'}{\sqrt{q}}$$

with $a_0, d_0, a_0', d_0', b_0, b_0' \in \mathbf{Z}$. Since the Hermitian matrix associated with $M(tL)$ is $H_M + tH_L$, we require $\det(H_M + tH_L) > 0$ for all $t \in \mathbf{Z}$, that is,

$$r^2(a_0 + ta_0')(d_0 + td_0') > (b_0 + tb_0')^2 + q(c + tc')^2, \; t \in \mathbf{Z}.$$

Now it is clear that for $s | \gcd(a_0', b_0', c', d_0')$, where s is a prime, we can choose $a_0 = \frac{a_0'}{s}, b_0 = \frac{b_0'}{s}, c_0 = \frac{c'}{s}, d_0 = \frac{d_0'}{s}$ so that

$$\frac{1}{q}\det(H_M + tH_L) = \left(\frac{1}{s} + t\right)^2 \left(\frac{r^2}{q} a_0' d_0' - \frac{1}{q} {b_0'}^2 - {c'}^2\right)$$

which is positive since $t \neq -\frac{1}{s}$ for $t \in \mathbf{Z}$ and the paranthesized expression equals $\det(H_L) > 0$ for L ample, by [9, 4.5.2]. Starting with non-coprime b', c' we can produce enough a', d' so that, upon scaling by prime numbers, yield the desired matrices. Now use the surjectivity of the map defined in the preliminaries.

To finalize the proof, let X be any abelian surface of Picard number 4 and $f : X \to E \times E$ an isogeny of degree k. For any pair (M, L) on $E \times E$ such that L is very ample and M ACM with respect to L, we consider (f^*M, f^*L) on X. By Serre's characterization of ampleness, since f is finite, f^*L will be ample on X.

Now $h^1(X, f^*M(f^*L)^t) = h^1(E \times E, f_*(f^*M(tL))) = h^1(E \times E, f_*\mathcal{O}_X \otimes M(tL))$ by Leray's spectral sequence followed by the projection formula. The canonical map $\mathcal{O}_{E\times E} \to f_*\mathcal{O}_X$ has a section $f_*\mathcal{O}_X \to \mathcal{O}_{E\times E}$ induced by the trace map [4, 9.18(iii)], so that $f_*\mathcal{O}_X = \mathcal{O}_{E\times E} \oplus F$, where F is a bundle of rank $k-1$. For any other line bundle $N \in \operatorname{Pic}^0(E \times E)$ such that $f^*N = \mathcal{O}_X$, $f_*f^*N = f_*O_X \otimes N = N \oplus (N \otimes E)$ and thus N is another direct summand of f_*O_X.[1] Continuing in this fashion

$$h^1(f_*\mathcal{O}_X \otimes M(tL)) = h^1(\mathcal{O}_{E\times E} \otimes M(tL)) + h^1((N_1 \otimes M)(tL)) + \cdots = kh^1(M(tL)) = 0, \forall t \in \mathbf{Z}$$

since all $N \otimes M$ lie in the same numerical class as M, which was to be proved. □

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