Bull. Math. Soc. Sci. Math. Roumanie
Tome 66 (114), No. 4, 2023, 433–444

Relative Matlis duality with respect to a semidualizing module

by

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Abstract

Let R be a commutative Noetherian ring and let C be a semidualizing R-module. The aim of this paper is to introduce and study the relative version of Matlis duality with respect to C and some other related topics. In particular, it is shown that over local ring R, the relative Matlis dual of a Noetherian R-module is Artinian, and in the case that R is complete the relative Matlis dual of an Artinian R-module is Noetherian.

Key Words: Semidualizing, C-injective, Matlis duality.
2020 Mathematics Subject Classification: Primary 13D05; Secondary 13D45, 18G20.

1 Introduction

Throughout this paper R is a commutative Noetherian ring and we use the notation $\mathrm{E}_R(M)$ for the injective envelope of an R-module M. The notion of a "semidualizing module" is a central notion in relative homological algebra. The study of semidualizing modules was independently initiated by Foxby [6], Vasconcelos [14] and Golod [7], which are common generalizations of dualizing modules and finitely generated projective modules of rank one. This notion has been investigated by many authors from different points of view; see for example [3], [8], [12], and [13]. In [9], Holm and White defined the so-called C-injective, C-projective and C-flat modules to characterize the Auslander class $\mathcal{A}_C(R)$ and the Bass class $\mathcal{B}_C(R)$, where C is a semidualizing R-module. The notion of C-injective (C-projective, C-flat) modules is important for the study of the relative homological algebra with respect to semidualizing modules. For example in [8], Holm and Jørgensen used these modules to define C-Gorenstein injective (projective, flat) modules, introduced the notions of C-Gorenstein projective, C-Gorenstein injective, and C-Gorenstein flat dimensions, and investigated the properties of these dimensions. Many other properties of C-injective modules, especially Tor-modules, are investigated in [5] and [10].

The first part of this paper is focused on the class of C-injective modules. We do some preliminary work in Section 2. In particular, we review some of the results, demonstrating the extent to which C-injective modules act like injective modules. In Section 3, we give a generalization of the notion of Matlis duality with respect to a semidualizing module and some related topics. For an R-module M over a local ring $(R,\mathfrak{m})$, we denote by $M^{\vee_C}$ the relative Matlis dual $\mathrm{Hom}_R(M,\mathrm{Hom}_R(C,\mathrm{E}_R(R/\mathfrak{m})))$ with respect to C. There is a natural R-homomorphism $\psi : M \to (M^{\vee_C})^{\vee_C}$ defined by $\psi(x)(f) = f(x)$ for all $x \in M$ and $f \in M^{\vee_C}$. We say that an R-module M is *C-Matlis reflexive* if $M \cong (M^{\vee_C})^{\vee_C}$ under the homomorphism ψ. It is known that if R is a complete local ring, then $\mathrm{E}_R(R/\mathfrak{m})$ is a Matlis reflexive R-module. Along these lines, we shown that $\mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{m}))$ is C-Matlis reflexive, in the case that R is complete local. Also, we investigate some properties of the relative Matlis duality functor with respect to C which are similar to the properties of the classical Matlis duality functor. For example it is shown that over local ring R, the relative Matlis dual of a Noetherian R-module is Artinian, and in the case that R is complete the relative Matlis dual of an Artinian R-module is Noetherian.

2 Background and preliminary results

We begin with a definition due to Foxby [6], generalizing Grothendieck's notion of a dualizing module, and introduced independently by Golod [7] and Vasconcelos [14].

Definition 2.1. A finitely generated R-module C is called *semidualizing* if the natural homothety morphism $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$.

Many of the primary properties of semidualizing modules are investigated in [11]. In the following, we recall some of them from [11] that will be used in the next section.

Fact 2.2. Let C be a semidualizing R-module. Then the following statements hold.

(i) $\mathrm{Supp}_R(C) = \mathrm{Spec}(R)$.

(ii) If M is a non-zero R-module, then $\mathrm{Hom}_R(C, M) \neq 0$ and $C \otimes_R M \neq 0$.

(iii) If $f : R \to S$ is a flat ring homomorphism, then $C \otimes_R S$ is a semidualizing S-module.

The classes defined next are collectively known as *Foxby classes*. The definitions are due to Foxby; see [1] and [3].

Definition 2.3. The *Auslander class* with respect to C is the class $\mathcal{A}_C(R)$ of R-modules M such that:

(i) $\mathrm{Tor}_i^R(C, M) = 0 = \mathrm{Ext}_R^i(C, C \otimes_R M)$ for all $i \geqslant 1$, and

(ii) the natural map $\gamma_C^M : M \to \mathrm{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

The *Bass class* with respect to C is the class $\mathcal{B}_C(R)$ of R-modules M such that:

(i) $\mathrm{Ext}_R^i(C, M) = 0 = \mathrm{Tor}_i^R(C, \mathrm{Hom}_R(C, M))$ for all $i \geqslant 1$, and

(ii) the natural evaluation map $\xi_M^C : C \otimes_R \mathrm{Hom}_R(C, M) \to M$ is an isomorphism.

In the following, we collect some properties of Foxby classes from [11].

Fact 2.4. Let C be a semidualizing R-module. Then the following statements hold.

(i) The class $\mathcal{A}_C(R)$ contains all the R-modules of finite flat dimension and the class $\mathcal{B}_C(R)$ contains all the R-modules of finite injective dimension.

(ii) If $M \in \mathcal{A}_C(R)$, then $C \otimes_R M \in \mathcal{B}_C(R)$. If $M \in \mathcal{B}_C(R)$, then $\mathrm{Hom}_R(C, M) \in \mathcal{A}_C(R)$.

(*iii*) The classes $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ satisfy the "two-of-three property": If any two R-modules in a short exact sequence are in $\mathcal{A}_C(R)$, respectively $\mathcal{B}_C(R)$, then so is the third.

Definition 2.5. For a semidualizing R-module C, we set

$$\mathcal{I}_C(R) = \{\mathrm{Hom}_R(C, I) |\ I \text{ is an injective } R\text{-module}\}.$$

The R-modules in $\mathcal{I}_C(R)$ are called *C-injective*.

Proposition 2.6. *Let $f : R \to S$ be a flat ring homomorphism, and let C be a semidualizing R-module. If E is an injective R-module, then $\mathrm{Hom}_R(C, \mathrm{Hom}_R(S, E))$ is a $(C \otimes_R S)$-injective S-module.*

Proof. By Fact 2.2(*iii*), $C \otimes_R S$ is a semidualizing S-module. Also, $\mathrm{Hom}_R(S, E)$ is an injective S-module, by [4, Proposition 3.1.6]. Hence

$$\mathrm{Hom}_R(C, \mathrm{Hom}_R(S, E)) \cong \mathrm{Hom}_S(C \otimes_R S, \mathrm{Hom}_R(S, E))$$

is a $(C \otimes_R S)$-injective S-module. □

Proposition 2.7. *Let C be a semidualizing R-module, and let F be a flat R-module. Assume that E and E' are two injective R-modules. Then the following statements hold.*

(*i*) $\mathrm{Hom}_R(F, \mathrm{Hom}_R(C, E))$ *is a C-injective R-module.*

(*ii*) $\mathrm{Hom}_R(C, E) \otimes_R F$ *is a C-injective R-module.*

(*iii*) $\mathrm{Hom}_R(C, E) \otimes_R (C \otimes_R F)$ *is an injective R-module.*

(*iv*) $\mathrm{Hom}_R(\mathrm{Hom}_R(C, E), \mathrm{Hom}_R(C, E'))$ *is a flat R-module.*

Proof. (*i*) By adjointness, we have

$$\begin{aligned}\mathrm{Hom}_R(F, \mathrm{Hom}_R(C, E)) &\cong \mathrm{Hom}_R(C \otimes_R F, E)\\ &\cong \mathrm{Hom}_R(C, \mathrm{Hom}_R(F, E)).\end{aligned}$$

By [4, Theorem 3.2.16], $\mathrm{Hom}_R(F, E)$ is an injective R-module. So, we get the assertion.

(*ii*) By [4, Theorem 3.2.14], $\mathrm{Hom}_R(C, E) \otimes_R F \cong \mathrm{Hom}_R(C, E \otimes_R F)$. So $\mathrm{Hom}_R(C, E) \otimes_R F$ is a C-injective R-module, since $E \otimes_R F$ is injective by [4, Theorem 3.2.16].

(*iii*) In the following sequence, the second isomorphism follows from [4, Theorem 3.2.14], and the third isomorphism follows from [4, Theorem 3.2.16] and Fact 2.4(*i*).

$$\begin{aligned}\mathrm{Hom}_R(C, E) \otimes_R (C \otimes_R F) &\cong (\mathrm{Hom}_R(C, E) \otimes_R F) \otimes_R C\\ &\cong \mathrm{Hom}_R(C, E \otimes_R F) \otimes_R C\\ &\cong E \otimes_R F\end{aligned}$$

(*iv*) By [11, Proposition 3.1.10] and Fact 2.4(*i*), we have

$$\mathrm{Hom}_R(\mathrm{Hom}_R(C, E), \mathrm{Hom}_R(C, E')) \cong \mathrm{Hom}_R(E, E').$$

Also, [4, Proposition 3.2.16] implies that $\mathrm{Hom}_R(E, E')$ is a flat R-module, as desired. □

Parallel to the class of injective modules in $\mathcal{B}_C(R)$, we have the class of C-injective modules in $\mathcal{A}_C(R)$. Thus, C-injective modules are expected to play the role of the injective objects of $\mathcal{A}_C(R)$. In the following two propositions, we review some of the results, demonstrating the extent to which C-injective modules act like injective modules.

Proposition 2.8. *Let C be a semidualizing R-module, and let*

$$(*):\ 0 \to W' \to W \to W'' \to 0$$

be a short exact sequence of R-modules. Then the following statements hold.

(i) [9, Proposition 5.2 (c)] If W' and W'' are C-injective, then W is also C-injective and the sequence splits.

(ii) If W' and W are C-injective, then W'' is also C-injective and the sequence splits.

Proof. (i) Let E' and E'' be injective R-modules such that $W' = \mathrm{Hom}_R(C, E')$, and $W'' = \mathrm{Hom}_R(C, E'')$. Applying functor $- \otimes_R C$ to the exact sequence $(*)$ to get the split exact sequence $(**):\ 0 \to E' \to C \otimes_R W \to E'' \to 0$, since $\mathcal{I}_C(R) \subseteq \mathcal{A}_C(R)$. Hence $C \otimes_R W$ is an injective R-module and [13, Theorem 2.11 (b)] implies that W is C-injective. Applying the functor $\mathrm{Hom}_R(C, -)$ on the sequence $(**)$ to get the split exact sequence $0 \to \mathrm{Hom}_R(C, E') \to \mathrm{Hom}_R(C, C \otimes_R W) \to \mathrm{Hom}_R(C, E'') \to 0$ of C-injective R-modules. Also, we have the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(C, E') & \longrightarrow & \mathrm{Hom}_R(C, C \otimes_R W) & \longrightarrow & \mathrm{Hom}_R(C, E'') & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow f & & \downarrow \cong & & \\
0 & \longrightarrow & \mathrm{Hom}_R(C, E') & \longrightarrow & W & \longrightarrow & \mathrm{Hom}_R(C, E'') & \longrightarrow & 0
\end{array}$$

Now the Five Lemma implies that f is an isomorphism, which implies that the sequence $(*)$ is split exact as desired.

(ii) It is proved the same line as (i). ◻

Proposition 2.9. *Let C be a semidualizing R-module, and*

$$\begin{array}{ccccc}
0 & \longrightarrow & M & \xrightarrow{f} & N \\
 & & \downarrow g & & \\
 & & \mathrm{Hom}_R(C, E) & &
\end{array}$$

be a diagram of R-modules with exact row such that $M, N \in \mathcal{A}_C(R)$ and E is an injective R-module. Then there exists an R-homomorphism $h : N \to \mathrm{Hom}_R(C, E)$ making the following diagram commute.

$$\begin{array}{ccccc}
0 & \longrightarrow & M & \xrightarrow{f} & N \\
 & & \downarrow g & \swarrow h & \\
 & & \mathrm{Hom}_R(C, E) & &
\end{array}$$

Proof. Basically, we need to show that the sequence

$$\mathrm{Hom}_R(N, \mathrm{Hom}_R(C, E)) \to \mathrm{Hom}_R(M, \mathrm{Hom}_R(C, E)) \to 0,$$

is exact. Since $M, N \in \mathcal{A}_C(R)$, we conclude by Fact 2.4(*iii*) that $L = \mathrm{Coker}\, f \in \mathcal{A}_C(R)$. Since L and $\mathrm{Hom}_R(C, E)$ belong to $\mathcal{A}_C(R)$, we have

$$\begin{aligned} \mathrm{Ext}^1_R(L, \mathrm{Hom}_R(C, E)) &\cong \mathrm{Ext}^1_R(C \otimes_R L, C \otimes_R \mathrm{Hom}_R(C, E)) \\ &\cong \mathrm{Ext}^1_R(C \otimes_R L, E) \\ &= 0. \end{aligned}$$

In the above sequence, the first isomorphism follows from [11, Lemma 3.1.13], and the second isomorphism follows from Fact 2.4(*i*). □

Theorem 2.10. *Let C be a semidualizing R-module and consider the following two short exact sequences of R-modules*

$$0 \longrightarrow M \longrightarrow \mathrm{Hom}_R(C, E_1) \longrightarrow K_1 \longrightarrow 0$$

$$0 \longrightarrow M \longrightarrow \mathrm{Hom}_R(C, E_2) \longrightarrow K_2 \longrightarrow 0,$$

where E_1 and E_2 are injective and $M \in \mathcal{A}_C(R)$. Then

$$K_2 \oplus \mathrm{Hom}_R(C, E_1) \cong K_1 \oplus \mathrm{Hom}_R(C, E_2).$$

Proof. Note that $K_1, K_2 \in \mathcal{A}_C(R)$, by Fact 2.4(*iii*). Therefore applying the functor $C \otimes_R -$ on the above two exact sequences, we get the following two exact sequences:

$$0 \longrightarrow C \otimes_R M \longrightarrow C \otimes_R \mathrm{Hom}_R(C, E_1) \longrightarrow C \otimes_R K_1 \longrightarrow 0,$$

$$0 \longrightarrow C \otimes_R M \longrightarrow C \otimes_R \mathrm{Hom}_R(C, E_2) \longrightarrow C \otimes_R K_2 \longrightarrow 0.$$

On the other hand, $C \otimes_R \mathrm{Hom}_R(C, E_i) \cong E_i$ for $i = 1, 2$. Now, the dual of Schanuel Lemma implies that

$$(C \otimes_R K_2) \oplus E_1 \cong (C \otimes_R K_1) \oplus E_2.$$

Applying the functor $\mathrm{Hom}_R(C, -)$ on the above isomorphism, implies the assertion. □

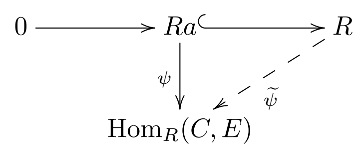
Let M be an R-module, and let $x \in M$ and $a \in R$. By the notation $a \underset{M}{|} x$, we mean that $x = ay$ for some $y \in M$. Recall that M is called *divisible* R-module if for every non zero-divisor element $r \in R$, and every element $m \in M$ we have $r \underset{M}{|} m$.

Proposition 2.11. *Let C be a semidualizing R-module and let E be an injective R-module. Then the following statements hold.*

(*i*) *Let a be a non zero-divisor element of R. Then for every $f \in \mathrm{Hom}_R(C, E)$ we have $a \underset{\mathrm{Hom}_R(C,E)}{|} f$.*

(*ii*) *Suppose that* $Ra \in \mathcal{A}_C(R)$ *for every non zero-divisor element* a *of* R. *Then the* R*-module* $\mathrm{Hom}_R(C,E)$ *is a divisible.*

Proof. (*i*) Let $f \in \mathrm{Hom}_R(C,E)$. Since a is a non zero-divisor element of R, the map $\psi : Ra \to \mathrm{Hom}_R(C,E)$ is well defined R-module homomorphism given by $\psi(ra) = rf$, for each $r \in R$. Since $Ra \cong R$ belongs to $\mathcal{A}_C(R)$, Proposition 2.9 implies that there exists an R-homomorphism $\widetilde{\psi} : R \to \mathrm{Hom}_R(C,E)$ such that the following diagram commutes.



Note that $f = \psi(a) = \widetilde{\psi}(a) = a\widetilde{\psi}(1)$, and so $a|_{\mathrm{Hom}_R(C,E)} f$.
(*ii*) It follows from item (*i*). □

Remark 2.12. Let R be a PID, and let C be a semidualizing R-module. Assume that M is a divisible R-module. Then $\mathrm{Hom}_R(C,M)$ is a C-injective R-module.

Proposition 2.13. *Let* C *be a semidualizing* R*-module, and let* $\mathfrak{p},\mathfrak{q} \in \mathrm{Spec}(R)$. *Then the following statements hold.*

(*i*) *The multiplication by* $r \in R - \mathfrak{p}$ *is an automorphism on* $\mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{p}))$.

(*ii*) $\mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{p})) \cong \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{q}))$ *if and only if* $\mathfrak{p} = \mathfrak{q}$.

(*iii*) $\mathrm{Ass}_R(\mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{p}))) = \{\mathfrak{p}\}$.

(*iv*) *If* $\varphi \in \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{p}))$, *then there exists a positive integer* t *such that* $\mathfrak{p}^t\varphi = 0$.

(*v*) $\mathrm{Hom}_R(\mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{p})), \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{q}))) \neq 0$ *if and only if* $\mathfrak{p} \subseteq \mathfrak{q}$.

Proof. (*i*) Let $r \in R - \mathfrak{p}$. Then $\mathrm{E}_R(R/\mathfrak{p}) \xrightarrow{r} \mathrm{E}_R(R/\mathfrak{p})$ is an isomorphism, by [4, Theorem 3.3.8 (1)] and so $\mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{p})) \xrightarrow{r} \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{p}))$ is an R-isomorphism.

(*ii*) Assume that $\mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{p})) \cong \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{q}))$. Therefore,

$$C \otimes_R \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{p})) \cong C \otimes_R \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{q})).$$

By Fact 2.4(*i*), we have $\mathrm{E}_R(R/\mathfrak{p}) \cong \mathrm{E}_R(R/\mathfrak{q})$ and then [4, Theorem 3.3.8 (2)] implies that $\mathfrak{p} = \mathfrak{q}$. For the reverse, suppose that $\mathfrak{p} = \mathfrak{q}$, then [4, Theorem 3.3.8 (2)] implies that $\mathrm{E}_R(R/\mathfrak{p}) \cong \mathrm{E}_R(R/\mathfrak{q})$ and therefore, $\mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{p})) \cong \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{q}))$.

(*iii*) In the following sequence, the first equality follows from [2, Exercise 1.2.27], and the second equality follows from Fact 2.2(*i*) and [4, Theorem 3.3.8 (3)].

$$\begin{aligned}\mathrm{Ass}_R(\mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{p}))) &= \mathrm{Supp}_R(C) \cap \mathrm{Ass}_R(\mathrm{E}_R(R/\mathfrak{p})) \\ &= \mathrm{Spec}(R) \cap \{\mathfrak{p}\} \\ &= \{\mathfrak{p}\}.\end{aligned}$$

(iv) Let $0 \neq \varphi \in \operatorname{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{p}))$. Then $\operatorname{Ass}_R(R\varphi) = \{\mathfrak{p}\}$, by ($iii$). So, $\mathfrak{p}$ is the unique minimal element in $\operatorname{Supp}_R(R\varphi)$. On the other hand, $\operatorname{Supp}_R(R\varphi) = \{\mathfrak{q} \in \operatorname{Spec}(R) | \operatorname{Ann}(\varphi) \subset \mathfrak{q}\}$. Hence $\mathfrak{p}$ is the radical of $\operatorname{Ann}(\varphi)$, and so $\operatorname{Ann}(\varphi)$ is $\mathfrak{p}$-primary.

(v) By the proof of Proposition 2.7(iv), we have

$$\operatorname{Hom}_R(\operatorname{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{p})), \operatorname{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{q}))) \cong \operatorname{Hom}_R(\mathrm{E}_R(R/\mathfrak{p}), \mathrm{E}_R(R/\mathfrak{q})).$$

Now the assertion follows from [4, Theorem 3.3.8 (5)]. □

3 Relative Matlis duality

Throughout this section $(R, \mathfrak{m})$ is a local ring. Let M be an R-module. We denote by $M^\vee$ the Matlis dual $\operatorname{Hom}_R(M, \mathrm{E}_R(R/\mathfrak{m}))$ of M. There is a natural homomorphism $\varphi : M \to M^{\vee\vee}$ defined by $\varphi(x)(f) = f(x)$ for $x \in M$ and $f \in M^\vee$. Recall that M is *Matlis reflexive* if $M \cong M^{\vee\vee}$ under the homomorphism φ. In this section, we introduce the notion of relative Matlis duality with respect to a semidualizing R-module which gives a generalization of the notion Matlis duality.

Definition 3.1. Let C be a semidualizing R-module. For an R-module M, we denote by $M^{\vee_C}$ the *relative Matlis dual* of M with respect to C, and define

$$M^{\vee_C} = \operatorname{Hom}_R(M, C^\vee).$$

There is a natural R-homomorphism $\psi : M \to (M^{\vee_C})^{\vee_C}$ defined by $\psi(x)(f) = f(x)$ for all $x \in M$ and $f \in M^{\vee_C}$. We say that an R-module M is *C-Matlis reflexive* if $M \cong (M^{\vee_C})^{\vee_C}$ under the homomorphism ψ.

Proposition 3.2. *Let C be a semidualizing R-module, and let M be an R-module. Then the following statements hold.*

(i) $M^{\vee_C} \cong (C \otimes_R M)^\vee$.

(ii) $M^{\vee_C} \cong \operatorname{Hom}_R(C, M^\vee)$.

(iii) $(M^{\vee_C})^{\vee_C} \cong \operatorname{Hom}_R(C, C \otimes_R M^{\vee\vee})$.

(iv) $(M^{\vee_C})^{\vee_C} \cong (\operatorname{Hom}_R(C, C \otimes_R M))^{\vee\vee}$.

Proof. The items (i) and (ii) follow from adjointness.

(iii) In the following sequence, the first and second isomorphisms follow from item (ii), and the third isomorphism follows from [4, Theorem 3.2.11].

$$\begin{aligned}
(M^{\vee_C})^{\vee_C} &\cong (\operatorname{Hom}_R(C, M^\vee))^{\vee_C} \\
&\cong \operatorname{Hom}_R(C, \operatorname{Hom}_R(C, M^\vee)^\vee) \\
&= \operatorname{Hom}_R(C, \operatorname{Hom}_R(\operatorname{Hom}_R(C, M^\vee), \mathrm{E}_R(R/\mathfrak{m}))) \\
&\cong \operatorname{Hom}_R(C, C \otimes_R \operatorname{Hom}_R(M^\vee, \mathrm{E}_R(R/\mathfrak{m}))) \\
&= \operatorname{Hom}_R(C, C \otimes_R M^{\vee\vee}).
\end{aligned}$$

(*iv*) In the following sequence, the first and second isomorphisms follow from items (*i*), and the third isomorphism follows from [4, Theorem 3.2.11].

$$\begin{aligned}
(M^{\vee_C})^{\vee_C} &\cong ((C \otimes_R M)^{\vee})^{\vee_C} \\
&\cong (C \otimes_R (C \otimes_R M)^{\vee})^{\vee} \\
&= (C \otimes_R \operatorname{Hom}_R(C \otimes_R M, \operatorname{E}_R(R/\mathfrak{m})))^{\vee} \\
&\cong (\operatorname{Hom}_R(\operatorname{Hom}_R(C, C \otimes_R M), \operatorname{E}_R(R/\mathfrak{m})))^{\vee} \\
&= (\operatorname{Hom}_R(C, C \otimes_R M))^{\vee\vee}.
\end{aligned}$$

□

Proposition 3.3. *Let C be a semidualizing R-module, and let M be an R-module. Then the following statements hold.*

(*i*) *If M is Matlis reflexive and $M \in \mathcal{A}_C(R)$, then M is C-Matlis reflexive.*

(*ii*) *If $l_R(M) < \infty$ and $M \in \mathcal{A}_C(R)$, then M is C-Matlis reflexive.*

(*iii*) *If $l_R(M) < \infty$, then $l_R(M^{\vee_C}) < \infty$.*

Proof. (*i*) It follows from Proposition 3.2(*iii*).

(*ii*) Let $l_R(M) < \infty$. Then M is Matlis reflexive by [4, Theorem 3.4.1]. Now the assertion follows from item (*i*).

(*iii*) Let $l_R(M) < \infty$. Then $l_R(M^{\vee}) < \infty$, by [4, Theorem 3.4.1]. By Proposition 3.2(*ii*), $M^{\vee_C} \cong \operatorname{Hom}_R(C, M^{\vee})$ and so, $l_R(M^{\vee_C}) = l_R(\operatorname{Hom}_R(C, M^{\vee})) \leqslant t l_R(M^{\vee})$, where t is a number of generators of C. □

Remark 3.4. A Standard fact for finite length module M is that $l_R(M^{\vee}) = l_R(M)$. It is worth noting that this fails in general for C-Matlis duality, where C is semidualizing. For example, if $(R, \mathfrak{m})$ is Artinian local and not Gorenstein, with $M = R/\mathfrak{m}$ and $C = \operatorname{E}_R(R/\mathfrak{m})$, then $M^{\vee_C} \cong \operatorname{Hom}_R(R/\mathfrak{m}, R)$, so $l_R(M)^{\vee_C} = \operatorname{type} R > 1 = l_R(M)$. This example also shows that modules of finite length will rarely be C-Matlis reflexive.

It is known that if R is a complete ring, then $\operatorname{E}_R(R/\mathfrak{m})$ is a Matlis reflexive R-module. In this regard, in the following it is shown that $\operatorname{Hom}_R(C, \operatorname{E}_R(R/\mathfrak{m}))$ is a C-Matlis reflexive R-module, where C is a semidualizing module over complete ring R.

Corollary 3.5. *Let R be a complete ring and let C be a semidualizing R-module. Then $\operatorname{Hom}_R(C, \operatorname{E}_R(R/\mathfrak{m}))$ is C-Matlis reflexive.*

Proof. By [4, Theorem 3.4.1(6)], we have $\operatorname{Hom}_R(\operatorname{E}_R(R/\mathfrak{m}), \operatorname{E}_R(R/\mathfrak{m})) \cong R$, since R is complete. Therefore, $(\operatorname{Hom}_R(C, \operatorname{E}_R(R/\mathfrak{m})))^{\vee\vee} \cong \operatorname{Hom}_R(C, \operatorname{E}_R(R/\mathfrak{m}))$ by [4, Theorem 3.2.11]. Now the assertion follows from Proposition 3.3(*i*) and Fact 2.4. □

Remark 3.6. Note that $\mathrm{E}_R(R/\mathfrak{m})$ is an injective cogenerator for R-modules. That means, $\mathrm{Hom}_R(M, \mathrm{E}_R(R/\mathfrak{m})) \neq 0$ for any R-module $M \neq 0$. Also, for any R-module $M \neq 0$, we have $C \otimes_R M \neq 0$, by Fact 2.2(*ii*), and so

$$\begin{aligned}\mathrm{Hom}_R(M, \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{m}))) &\cong \mathrm{Hom}_R(C \otimes_R M, \mathrm{E}_R(R/\mathfrak{m})) \\ &\neq 0.\end{aligned}$$

Theorem 3.7. *Let C be a semidualizing R-module, and let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. Then the following statements hold.*

(*i*) $\mathrm{Hom}_R(C^\vee, C^\vee) \cong \widehat{R}$.

(*ii*) $\widehat{R} \otimes_R \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{m})) \cong \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{m}))$.

(*iii*) $\mathrm{Hom}_{\widehat{R}}(\widehat{C}, \mathrm{E}_{\widehat{R}}(\widehat{R}/\widehat{\mathfrak{m}})) \cong \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{m}))$*, as* $\widehat{R}$*-modules.*

(*iv*) *If M is a finitely generated R-module, then* $(M^{\vee_C})^{\vee_C} \cong \mathrm{Hom}_R(C, C \otimes_R \widehat{M})$.

(*v*) $\mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{m}))$ *is Artinian as R-module and $\widehat{R}$-module.*

Proof. (*i*) In the following sequence, the first isomorphism follows from adjointness, the second isomorphism follows from [4, Theorem 3.4.1], and the last one follows from [4, Theorem 3.2.14], since $\widehat{R}$ is a flat R-module.

$$\begin{aligned}\mathrm{Hom}_R(C^\vee, C^\vee) &= \mathrm{Hom}_R(C^\vee, \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{m}))) \\ &\cong \mathrm{Hom}_R(C, C^{\vee\vee}) \\ &\cong \mathrm{Hom}_R(C, \widehat{C}) \\ &\cong \mathrm{Hom}_R(C, C \otimes_R \widehat{R}) \\ &\cong \widehat{R}.\end{aligned}$$

(*ii*) In the following sequence, the first isomorphism follows from [4, Theorem 3.2.14], since $\widehat{R}$ is a flat R-module, and the second isomorphism follows from [4, Theorem 3.4.1(4)].

$$\begin{aligned}\widehat{R} \otimes_R \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{m})) &\cong \mathrm{Hom}_R(C, \widehat{R} \otimes_R \mathrm{E}_R(R/\mathfrak{m})) \\ &\cong \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{m})).\end{aligned}$$

(*iii*) In the following sequence, the first isomorphism follows form [4, Theorem 3.4.1(5)], and the second isomorphism follows from adjointness.

$$\begin{aligned}\mathrm{Hom}_{\widehat{R}}(\widehat{C}, \mathrm{E}_{\widehat{R}}(\widehat{R}/\widehat{\mathfrak{m}})) &\cong \mathrm{Hom}_{\widehat{R}}(C \otimes_R \widehat{R}, \mathrm{E}_R(R/\mathfrak{m})) \\ &\cong \mathrm{Hom}_R(C, \mathrm{Hom}_{\widehat{R}}(\widehat{R}, \mathrm{E}_R(R/\mathfrak{m}))) \\ &\cong \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{m})).\end{aligned}$$

(*iv*) By Proposition 3.2(*ii*), we have $(M^{\vee_C})^{\vee_C} \cong \mathrm{Hom}_R(C, C \otimes_R M^{\vee\vee})$. Now the assertion follows from [4, Theorem 3.4.1 (8)].

(*v*) The assertion follows from [4, Corollary 3.4.4], since C is a Noetherian R-module. ∎

In the following theorem, we give a characterization of Artinian modules.

Theorem 3.8. *Let C be a semidualizing R-module and let M be an R-module. If M is Artinian, then $\mathrm{Hom}_R(C,M) \subseteq \mathrm{Hom}_R(C,\mathrm{E}_R(R/\mathfrak{m})^n)$ for some $n \geqslant 1$. In the case that M is finitely generated the converse also holds.*

Proof. Let M be an Artinian R-module. Then there exists $n \geqslant 1$ such that $M \subseteq \mathrm{E}_R(R/\mathfrak{m})^n$, by [4, Theorem 3.4.3]. So, $\mathrm{Hom}_R(C,M) \subseteq \mathrm{Hom}_R(C,\mathrm{E}_R(R/\mathfrak{m})^n)$. For the reverse, let M be a finitely generated R-module such that $\mathrm{Hom}_R(C,M) \subseteq \mathrm{Hom}_R(C,\mathrm{E}_R(R/\mathfrak{m})^n)$ for some $n \geqslant 1$. Note that $\mathrm{Hom}_R(C,\mathrm{E}_R(R/\mathfrak{m})^n) \cong (C^\vee)^n$ is an Artinian R-module, by [4, Corollary 3.4.4]. Hence $\mathrm{Hom}_R(C,M)$ is an Artinian R-module. Assume that $M \neq 0$. Then $\mathrm{Hom}_R(C,M) \neq 0$, by Fact 2.2($ii$). So,

$$\begin{aligned}\{\mathfrak{m}\} &= \mathrm{Ass}_R(\mathrm{Hom}_R(C,M))\\ &= \mathrm{Supp}_R(C) \cap \mathrm{Ass}_R(M)\\ &= \mathrm{Spec}(R) \cap \mathrm{Ass}_R(M).\end{aligned}$$

Therefore, $\mathrm{Ass}_R(M) = \{\mathfrak{m}\}$ and so, M is an Artinian R-module. □

Remark 3.9. Let M and N be R-modules such that $\mathrm{Supp}_R(N) = \mathrm{Spec}(R)$. Then the proof of Theorem 3.8 shows that M is Artinian if and only if $\mathrm{Hom}_R(N,M) \subseteq \mathrm{Hom}_R(N,\mathrm{E}_R(R/\mathfrak{m})^n)$ for some $n \geqslant 1$.

Theorem 3.10. *Let C be a semidualizing R-module, and let M be an R-module. Then the following statements hold.*

(i) *If M is Noetherian, then $M^{\vee_C}$ is Artinian.*

(ii) *If $M^{\vee_C}$ is Artinian, then $C \otimes_R M$ is Noetherian.*

(iii) *If M is Artinian, then $M^{\vee_C}$ is Noetherian provided that R is complete.*

Proof. (i): Assume that M is finitely generated. Then so is $C \otimes_R M$, which implies that $M^{\vee_C} \cong (C \otimes_R M)^\vee$ is Artinian by [4, Corollary 3.4.4].

(ii): Assume that $M^{\vee_C}$ is Artinian. Then so is $(C \otimes_R M)^\vee$. Now the assertion follows from [4, Corollary 3.4.4].

(iii) Assume that R is complete and M is Artinian. Then $M^\vee$ is Noetherian by [4, Theorem 3.4.7]. Also Proposition 3.2(ii) implies that $M^{\vee_C} \cong \mathrm{Hom}_R(C,M^\vee)$, and so $M^{\vee_C}$ is Noetherian. □

Remark 3.11. Note that Theorem 3.10 holds true for any finitely generated R-module C; we do not have to assume that C is semidualizing.

Acknowledgement *The authors are very grateful to the referee for several suggestions and comments that greatly improved the paper.*

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Received: 31.03.2022
Revised: 02.06.2022
Accepted: 02.08.2022

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