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Weighted second main theorems and algebraic dependences for holomorphic maps from disks

by

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Abstract

In this paper, we first prove second main theorems with weighted counting functions for holomorphic maps from a complex disk $\Delta(R) \subset \mathbb{C}$ with finite growth index into $\mathbb{P}^n(\mathbb{C})$. Then we apply these theorems to solve the algebraic dependence problems of holomorphic maps sharing moving hyperplanes in general position.

Key Words: Nevanlinna theory, weighted second main theorem, holomorphic map, moving hyperplane, algebraic dependence.

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1 Introduction

In 1990's, W. Stoll, M. Ru [11] as well as M. Shirosaki [15, 16] began to study the second main theorems for meromorphic maps into projective spaces $\mathbb{P}^n(\mathbb{C})$ with moving hyperplanes. M. Ru [12] was also the first one who proved the second main theorem with truncated counting functions for holomorphic maps of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ and moving hyperplanes in 2001. After that, these results have been reproved and extended by many authors, such as [18], [13], [6], [4], [7], ...

In 2020, M. Ru and N. Sibony [14] introduced a new class of holomorphic maps into $\mathbb{P}^n(\mathbb{C})$, which has finite growth index. In their paper, they gave the second main theorem for holomorphic maps from a disk $\Delta(R) \subset \mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ involving hyperplanes in general position, where $\Delta(R) = \{z \in \mathbb{C}; |z| < R\}$, $(0 < R \le +\infty)$. Very recently, S. D. Quang [8] proved the second main theorem for holomorphic maps from disks with moving hyperplanes.

To state Quang's results as well as ours, we recall the following.

Let $\{a_i\}_{i=1}^q$ be q holomorphic maps of $\Delta(R)$ into $\mathbb{P}^n(\mathbb{C})^*$ with reduced representations $a_i = (a_{i0} : \cdots : a_{in})$ $(1 \le i \le q)$. The family $\{a_i\}_{i=1}^q$ is said to be in general position if $\det(a_{i_j l}; 0 \le j \le n, 0 \le l \le n) \not\equiv 0$ for any $1 \le i_0 < \cdots < i_n \le q$. Let $\mathcal{M}_m$ be the field of all meromorphic functions on $\Delta(R)$. Denote by $\mathcal{R}_{\{a_i\}_{i=1}^q} \subset \mathcal{M}_m$ the smallest subfield which contains $\mathbb{C}$ and all $\frac{a_{ik}}{a_{il}}$ with $a_{il} \not\equiv 0$ (for brevity we will write $\mathcal{R}$ if there is no confusion).

Let $f : \Delta(R) \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map with a reduced representation $(f_0 : \cdots : f_n)$, where $f_0, \ldots, f_n$ are holomorphic functions on $\Delta(R)$ without common zeros. Then $\operatorname{rank}_{\mathcal{R}}(f)$ is defined as the rank of the set $\{f_0, \ldots, f_n\}$ over the field $\mathcal{R}$.

According to M. Ru-N. Sibony [14], the growth index of f is defined by

$$c_f = \inf\left\{ c > 0 \,\middle|\, \int_0^R \exp(cT_f(r))dr = +\infty \right\}.$$

For convenient, we will set $c_f = +\infty$ if $\Big\{c > 0 \,\big|\int_0^R \exp(cT_f(r))dr = +\infty\Big\} = \varnothing$.

In [8] and [9], S. D. Quang proved the second main theorems for holomorphic maps from disks with moving hyperplanes as well as moving hypersurfaces. We state here the second main theorems with moving hyperplanes.

Theorem A (see [8, Theorems 1.1]) *Let $f : \Delta(R) \to \mathbb{P}^n(\mathbb{C})$ $(0 < R \le +\infty)$ be a holomorphic map. Let $\{a_i\}_{i=1}^q$ $(q \ge 2n-k+2)$ be holomorphic maps of $\Delta(R)$ into $\mathbb{P}^n(\mathbb{C})^*$ in general position such that $(f, a_i) \not\equiv 0$ $(1 \le i \le q)$, where $k+1 = \operatorname{rank}_{\mathcal{R}}(f)$. Let $\gamma(r)$ be a non-negative measurable function defined on $(0, R)$ with $\int_0^R \gamma(r)dr = \infty$. Then, for every $\varepsilon > 0$, the following assertions hold:*

$$\text{(a)}\ \big\|_E T_f(r) \le \frac{n+2}{q-(n-k)} \sum_{i=1}^q N^{[k]}_{(f,a_i)}(r) + \delta(n,k)\big((1+\varepsilon)\log\gamma(r) + \varepsilon\log r\big) + S(r),$$

$$\text{(b)}\ \big\|_E T_f(r) \le \frac{k(k+2)}{q-2(n-k)} \sum_{i=1}^q N^{[1]}_{(f,a_i)}(r) + \delta(n,k)\big((1+\varepsilon)\log\gamma(r) + \varepsilon\log r\big) + S(r),$$

where $S(r) = O\big(\log T_f(r) + \max_{1\le i\le q} T_{a_i}(r)\big)$ and $\delta(n,k) = \dfrac{k(k+2)(n+1)}{2(n+2)}$.

If we assume further that $q > (n-k)(k+1) + n + 2$ then

$$\text{(c)}\ \big\|_E T_f(r) \le \frac{k+2}{q} \sum_{i=1}^q N^{[k]}_{(f,a_i)}(r) + \frac{k(k+1)}{2}\big((1+\varepsilon)\log\gamma(r) + \varepsilon\log r\big) + S(r).$$

Here the notation $\|_E$ means the inequality holds for all $r \in (0, R)$ outside a subset E with $\int_E \gamma(r)dr < +\infty$.

In 2016, S. D. Quang [6] first gave the second main theorem with weighted counting functions and thanks to this result, he can study the algebraic problem to the more general case. Our first purpose of this paper is to prove some weighted second main theorems and then apply them to prove the algebraic theorems for the holomorphic maps with finite growth index from disks into $\mathbb{P}^n(\mathbb{C})$.

Theorem 1 and Theorem 2 stated below are our generalizations of Theorem A(b) and Theorem A(c) respectively to the case of weighted ones.

Theorem 1. *Let $f : \Delta(R) \to \mathbb{P}^n(\mathbb{C})$ $(0 < R \le +\infty)$ be a holomorphic map. Let $\{a_i\}_{i=1}^q$ $(q \ge 2n-k+2)$ be holomorphic maps of $\Delta(R)$ into $\mathbb{P}^n(\mathbb{C})^*$ in general position such that $(f, a_i) \not\equiv 0$ $(1 \le i \le q)$, where $k+1 = \operatorname{rank}_{\mathcal{R}}(f)$. Let $\gamma(r)$ be a non-negative measurable function defined on $(0, R)$ with $\int_0^R \gamma(r)dr = \infty$. Let $\lambda_1, \ldots, \lambda_q$ be q positive numbers with $2(n-k)\max_{1\le i\le q}\lambda_i \le \sum_{i=1}^q \lambda_i$. Then for every positive number η satisfying $\eta \ge \max_{1\le i\le q}\lambda_i,\ 2(n-k)\eta \le \sum_{i=1}^q \lambda_i$, and for every $\varepsilon > 0$, the following assertion holds:*

$$\begin{aligned}\big\|_E \frac{\sum_{i=1}^q \lambda_i - 2(n-k)\eta}{k(k+2)} T_f(r) &\le \sum_{i=1}^q \lambda_i N^{[1]}_{(f,a_i)}(r)\\ &+ \frac{k(k+2)(n+1)(\sum_{i=1}^q \lambda_i - (n-k)\eta)}{2(n+2)}\big((1+\varepsilon)\log\gamma(r) + \varepsilon\log r\big) + S(r),\end{aligned}$$

where $S(r) = O\big(\log T_f(r) + \max_{1\le i\le q} T_{a_i}(r)\big)$.

Theorem 2. *Let $f : \Delta(R) \to \mathbb{P}^n(\mathbb{C})$ $(0 < R \le +\infty)$ be a holomorphic map. Let $\{a_i\}_{i=1}^q$ $(q > (n-k+1)(k+2))$ be holomorphic maps of $\Delta(R)$ into $\mathbb{P}^n(\mathbb{C})^*$ in general position such that $(f, a_i) \not\equiv 0$ $(1 \le i \le q)$, where $k+1 = \operatorname{rank}_{\mathcal{R}}(f)$. Let $\gamma(r)$ be a non-negative measurable function defined on $(0, R)$ with $\int_0^R \gamma(r)dr = \infty$. Let $\lambda_1, \ldots, \lambda_q$ be q positive numbers with $(n-k+1)(k+2)\max_{1\le i\le q}\lambda_i \le \sum_{i=1}^q \lambda_i$. Then for every $\varepsilon > 0$, we have*

$$\|_E \frac{\sum_{i=1}^q \lambda_i}{k+2} T_f(r) \le \sum_{i=1}^q \lambda_i N_{(f,a_i)}^{[k]}(r) + \frac{k(k+1)\sum_{i=1}^q \lambda_i}{2(k+2)}\big((1+\varepsilon)\log\gamma(r) + \varepsilon\log r\big) + S(r),$$

where $S(r) = O\big(\log T_f(r) + \max_{1\le i\le q} T_{a_i}(r)\big)$.

Remark. In case f has finite growth index (i.e., $c_f < +\infty$), then in Theorem 1 and Theorem 2, we may take $\gamma(r) = \exp\big((c_f+\varepsilon)T_f(r)\big)$.

As applications of Theorem 1 and Theorem 2, in the next section we will prove algebraic dependence theorems for holomorphic maps from disks sharing moving hyperplanes, where all zeros with multiplicities more than a certain number are omitted. To state our results, we need the following.

In this paper, we call each holomorphic map from $\Delta(R)$ into $\mathbb{P}^n(\mathbb{C})^*$ a moving hyperplane in $\mathbb{P}^n(\mathbb{C})$. A moving hyperplane a in $\mathbb{P}^n(\mathbb{C})$ is said to be slow with respect to a set $\{f^1, \ldots, f^M\}$ of finite growth index holomorphic maps from $\Delta(R)$ into $\mathbb{P}^n(\mathbb{C})$ if

$$\|_E\ T_a(r) = o\big(\sum_{i=1}^M T_{f^i}(r)\big),$$

where E is a subset of $(0, R)$ with $\int_E e^{(\min_{1\le j\le M} c_{f^j}+\varepsilon)(\sum_{i=1}^M T_{f^i}(r))}dr < +\infty$ for some $\varepsilon > 0$ and $\int_E dr < +\infty$ if $R = +\infty$.

Let $f_i : \Delta(R) \to \mathbb{P}^n(\mathbb{C})$ $(1 \leqslant i \leqslant \lambda)$ be holomorphic maps with reduced representations $f_i := (f_{i0} : \cdots : f_{in})$. Let $a_i : \Delta(R) \to \mathbb{P}^n(\mathbb{C})^*$ $(1 \leqslant i \leqslant q)$ be slowly moving hyperplanes in general position with reduced representations $a_i := (a_{i0} : \cdots : a_{in})$. Suppose that $(f_t, a_i) := \sum_{j=0}^n f_{tj}a_{ij} \ne 0$ for each $1 \le t \le \lambda,\ 1 \le i \le q$ and $(f_1, a_i)^{-1}\{0\} = \cdots = (f_\lambda, a_i)^{-1}\{0\}$. We put $A_i = (f_1, a_i)^{-1}\{0\}$.

Denote by $T[n+1, q]$ the set of all injective maps from $\{1, \ldots, n+1\}$ to $\{1, \ldots, q\}$. For each $z \in \Delta(R)\backslash \cup_{\beta\in T[n+1,q]}\{z \mid a_{\beta(1)}(z)\wedge\cdots\wedge a_{\beta(n+1)}(z) = 0\}$, we define $\rho(z) = \sharp\{j | z \in A_j\}$. Similarly as [6], we can show that $\rho(z) \le n$.

For any positive number $r > 0$, define $\rho(r) = \sup\{\rho(z) \mid |z| \ge r\}$, where the supremum is taken over all $z \in \Delta(R) \setminus \cup_{\beta\in T[n+1,q]}\{z \mid g_{\beta(1)}(z) \wedge \cdots \wedge g_{\beta(n+1)}(z) = 0\}$. Then $\rho(r)$ is a decreasing function. Let $d := \lim_{r\to+\infty} \rho(r)$ then $d \le n$.

In 2001, M. Ru [12] gave an algebraic dependence theorem for meromorphic maps sharing several moving hyperplanes. After that, the result of Ru has been generalized by many authors. As we stated above, by giving the weighted second main theorem (Theorem 1.2, [6]), S. D. Quang could deal with algebraic dependence problems in case the number l depending on the moving hyperplanes. It means that for each j, we suppose that there exists a positive number l_j such that $f_{i_1} \wedge \cdots \wedge f_{i_{l_j}} = 0$ on A_j for any l_j maps.

On the other hand, in 2017, L. N. Quynh [10] proposed a new technique, by which she studied the algebraic dependence of meromorphic maps sharing different family of moving hyperplanes regardless of multiplicities and obtained the results which are better than previous ones. Then, based on Quynh's technique, P. D. Thoan, N. H. Nam and N. V. An [22] and H. Cao and L. Duan [1] consider the case where the number l in Quynh's theorem may vary dependently on the moving hyperplanes like Quang's.

In Theorem 3 and 4 as follows, we will prove the similar results as in [1] for the case of holomorphic maps from disks with finite growth index. In concrete, we will prove algebraic dependence theorems for holomorphic maps from disks sharing moving hyperplanes, where all zeros with multiplicities more than a certain number are omitted and the number l depending on the moving hyperplanes.

Our theorems are stated as follows.

Theorem 3. *Let $f_1, \ldots, f_\lambda : \Delta(R) \to \mathbb{P}^n(\mathbb{C})$ be holomorphic maps with $c_{f^i} < +\infty$ $(i = 1, \ldots, \lambda)$. Let $\{a_i\}_{i=1}^q$ be slowly (with respect to $\{f_1, \ldots, f_\lambda\}$) moving hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Let $k_j (1 \le j \le q)$ be positive integers or $+\infty$. Suppose that $(f_i, a_j) \not\equiv 0$ and the following conditions hold*

i. $\operatorname{Supp} \nu^0_{(f_1,a_j),\le k_j} = \cdots = \operatorname{Supp} \nu^0_{(f_\lambda,a_j),\le k_j}$ *for each* $1 \le j \le q$.

ii. Let $l_1, \ldots, l_q$ be q positive integers with $2 \le l_i \le \lambda$ such that for any $1 \le i_1 < \cdots < i_{l_j} < q$, $f_{i_1}(z) \wedge \cdots \wedge f_{i_{l_j}}(z) = 0$ for each $z \in \cup_{j=1}^q \operatorname{Supp} \nu^0_{(f_1,a_j),\le k_j}$.

If $k + 1 = \max_{1 \le t \le \lambda}\{\operatorname{rank}_{\mathcal{R}}(f_t)\}$ and

$$\sum_{j=1}^q \frac{1}{k_j} < \frac{q(\lambda+1) - \sum_{j=1}^q l_j - 2(n-k)(\lambda-1) - d\lambda k(k+2)}{k(k+2)(\lambda-1)} - \frac{\lambda k(k+2)(n+1)\big(q(\lambda+1) - \sum_{j=1}^q l_j\big)}{4(n+2)(\lambda-1)} \min_{1\le i\le\lambda} c_{f_i}.$$

then $f_1, \ldots, f_\lambda$ are algebraically dependent over $\mathbb{C}$, i.e $f_i \wedge \ldots \wedge f_\lambda \equiv 0$ on $\Delta(R)$.

Remark.

1. When $R = +\infty$ and f_i is not constant, then $c_{f_i} = 0$, the theorem implies the result obtained in the Theorem 1.3 in [1].

2. Letting $\lambda = l_j = 2, d = 1$ and $k_j = +\infty$, we deduce the uniqueness theorem: If $q > \dfrac{2(n+2)(2k^2+2n+2k)}{2(n+2) - k^2(k+2)^2(n+1)\min_{1\le i\le\lambda} c_{f_i}}$, then $f_1 = f_2$.

When $d = 1$ and the number of hyperplanes is large enough $(q > (n-k)(k+1)+n+2)$, we will obtain the following algebraic dependence theorem.

Theorem 4. *Let $f_1, \ldots, f_\lambda : \Delta(R) \to \mathbb{P}^n(\mathbb{C})$ be holomorphic maps with $c_{f^i} < +\infty$ $(i = 1, \ldots, \lambda)$. Let $\{a_i\}_{i=1}^q$ be slowly (with respect to $\{f_1, \ldots, f_\lambda\}$) moving hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Let $k_j (1 \le j \le q)$ be positive integers or $+\infty$. Assume that $(f_i, a_j) \not\equiv 0$ $(1 \le i \le \lambda, 1 \le j \le q)$ and the following conditions are satisfied.*

i. $\left(f^1, a_i\right)^{-1}\{0\} \cap \left(f^1, a_j\right)^{-1}\{0\} = \emptyset \quad (1 \le i < j \le q)$, *ii.* $\min\left\{1, \nu^0_{(f_1,a_j),\le k_j}\right\} = \cdots = \min\left\{1, \nu^0_{(f_\lambda,a_j),\le k_j}\right\}$ *for each* $1 \le j \le q$.

iii. Let $l_1, \ldots, l_q$ *be* q *positive integers with* $2 \le l_i \le \lambda$, *such that for any increasing sequence* $1 \le i_1 < \cdots < i_{l_j} \le \lambda$, $f_{i_1}(z) \wedge \cdots \wedge f_{i_{l_j}}(z) = 0,\ z \in \cup_{j=1}^q \mathrm{Supp}\, \nu^0_{(f_1,a_j),\le k_j}$.

Put $\max\{\mathrm{rank}_{\mathcal{R}}(f_i), 1 \le i \le \lambda\} = k+1$, *where* k *is a positive integer. In addition, we suppose that* $q \ge (n-k+1)(k+2)(\lambda-1)$ *for* $k \le \frac{n-1}{2}$, *or* $q \ge \frac{(n+3)^2}{4}(\lambda-1)$ *for* $k > \frac{n-1}{2}$. *If*

$$\sum_{j=1}^{q} \frac{1}{k_j+1-k} < \frac{1}{k(k+2)} - \frac{\lambda}{q(\lambda+1) - \sum_{j=1}^q l_j + \lambda(k-1)} - \frac{k+1}{2(k+2)} \min_{1\le i\le\lambda} c_{f_i}$$

then $f_1, \ldots, f_\lambda$ *are algebraically dependent over* $\mathbb{C}$.

Remark.

1. When $\mathrm{rank}_{\mathcal{R}}(f_i) = k+1$ and $k_j = +\infty (1 \le i \le \lambda, 1 \le j \le q)$, we see that the above inequality becomes

$$q \ge \max\left\{(n-k+1)(k+2)(\lambda-1), \frac{1}{\lambda+1}\Big(\sum_{j=1}^q l_j - \lambda(k-1) + \frac{2\lambda k(k+2)}{2 - k(k+1)\min\limits_{1\le i\le\lambda} c_{f_i}}\Big)\right\}.$$

Further, when $R = +\infty$ and f_i is not constant, then $c_{f_i} = 0$, we deduce the result given in the Theorem 1.4 in [1].

2. Let $\lambda = l = 2$ when $k_j = +\infty$, if $\mathrm{rank}_{\mathcal{R}}(f_1) = \mathrm{rank}_{\mathcal{R}}(f_2) = k+1$ and

$$q > \max\Big\{(n-k+1)(k+2), \frac{4k(k+2)}{2 - k(k+1)\min\limits_{i=1,2} c_{f_i}} - 2(k-1)\Big\}$$

then $f_1 = f_2$.
We note that this statement is the result obtained in the uniqueness theorem (Theorem 1.3) in [8].

2 Basic notions and auxiliary results from Nevanlinna theory

We denote by $\Delta(R)$ a disk in $\mathbb{C}$, $\Delta(R) := \{z \in \mathbb{C}; |z| < R\}$, $(0 < R \le +\infty)$. Let ν be a divisor on $\Delta(R)$. We consider ν as a function on $\Delta(R)$ with values in $\mathbb{Z}$ such that $\mathrm{Supp}(\nu) := \{z; \nu(z) \ne 0\}$ is a discrete subset of $\Delta(R)$. Let k be a positive integer or $+\infty$. The truncated counting function of ν is defined by:

$$n^{[k]}(t) = \sum_{|z|\le t} \min\{k, \nu(z)\}\ (0 \le t \le R) \text{ and } N^{[k]}(r,\nu) = \int_{r_0}^{r} \frac{n^{[k]}(t) - n^{[k]}(0)}{t} dt.$$

We will omit the character ${}^{[k]}$ if $k = +\infty$.

Let $\varphi : \Delta(R) \to \mathbb{C} \cup \{\infty\}$ be a non-constant meromorphic function. We denote by ν^0_φ (resp. ν^∞_φ) the divisor of zeros (resp. divisor of poles) of φ and set $\nu_\varphi = \nu^0_\varphi - \nu^\infty_\varphi$. As usual, we will write $N^{[k]}_\varphi(r)$ and $N^{[k]}_{1/\varphi}(r)$ for $N^{[k]}(r, \nu^0_\varphi)$ and $N^{[k]}(r, \nu^\infty_\varphi)$ respectively. The proximity function of φ with respect to the point ∞ is defined by

$$m(r, \varphi) = \int_0^{2\pi} \log^+ |\varphi(re^{t\theta})| d\theta.$$

We consider φ as a holomorphic map into $\mathbb{P}^1(\mathbb{C})$ and denote by Ω_1 the Fubini-Study form on $\mathbb{P}^1(\mathbb{C})$. The characteristic function of φ is defined by

$$T_\varphi(r) = \int_0^r \frac{dt}{t} \int_{|z|<t} \varphi^*\Omega_1.$$

By Jensen's formula, we have

$$T_\varphi(r) = m(r, \varphi) + N_{1/\varphi}(r) + O(1).$$

Let $f : \Delta(R) \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map with a reduced representation $(f_0 : \cdots : f_n)$, where $f_0, \ldots, f_n$ are holomorphic functions on $\Delta(R)$ without common zeros. Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ defined by $H := \{(\omega_0 : \cdots : \omega_n); \sum_{i=0}^n a_i\omega_i = 0\}$, where a_i $(0 \le i \le n)$ are constants, not all zero. We define

$$(f, H) = \sum_{i=0}^n a_i f_i.$$

The function (f, H) depends on the choices of the reduced representation of f and the presentation of H, but the divisor $\nu_{(f,H)}$ is well-defined, not depending on these choices. The proximity function of f with respect to H is defined by

$$m_f(r, H) = \int_0^{2\pi} \log \frac{\|f\|(re^{t\theta}) \cdot \|H\|}{|(f, H)(re^{t\theta})|} d\theta,$$

where $\|f\| = (|f_0|^2 + \cdots + |f_n|^2)^{1/2}$ and $\|H\| = (|a_0|^2 + \cdots + |a_n|^2)^{1/2}$. The characteristic function of f (with respect to the Fubini-Study form Ω_n on $\mathbb{P}^n(\mathbb{C})$) is defined by

$$T_f(r) := \int_0^r \frac{dt}{t} \int_{|z|<t} f^*\Omega_n.$$

The first main theorem states that

$$T_f(r) = m_f(r, H) + N_{(f,H)}(r) + O(1).$$

Now we denote by $\mu_{f_1\wedge\cdots\wedge f_\lambda}$ the divisor associated to $f_1 \wedge \cdots \wedge f_\lambda$. According to [17], we have the first main theorem as follows.

Theorem 5 (The first main theorem for general position [17], p. 326)**.** *Let $f_i : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C}), 1 \le i \le k$ be meromorphic maps located in general position. Assume that $1 \le k \le n$. Then*

$$N_{\mu_{f_1\wedge\cdots\wedge f_\lambda}}(r) + m\left(r, f_1 \wedge \cdots \wedge f_\lambda\right) \le \sum_{1\le i\le\lambda} T_{f_i}(r) + O(1).$$

3 Proofs of the weighted second main theorems

Proof of Theorem 1

In order to show that the theorem holds, we need the following lemma which is a claim obtained in the proof of Theorem 1.1 in [8].

Lemma 6 ([8], Equation 2.10). *With the same assumption of $f, \{a_i\}(i = 1, \dots q)$ and $\gamma(r)$ as in Theorem 1, for every $\varepsilon > 0$, there exist a subset $J \subset \{1, \dots, 2n-k+2\}$ with $|J| = d+2$ $(\leq n+2)$ and a positive integer $n_0 \leq \dfrac{k(k+2)}{d+2}$ such that*

$$\Big\|_E \; T_f(r) \leq \sum_{j\in J} N^{[n_0]}_{(f,a_j)}(r) + \frac{n_0(\sharp J-1)}{2}\big((1+\varepsilon)\log\gamma(r) + \varepsilon\log r\big) + S(r),$$

where $S(r) = O\big(\log T_f(r) + \max_{1\leq i\leq q} T_{a_i}(r)\big)$.

We denote by $\mathcal{I}$ the set of all permutations of the set $\{1, \dots, q\}$. For each element $I = (i_1, \dots, i_q) \in \mathcal{I}$, we set

$$M_I = \{r \in (0,R) \mid N^{[1]}_{(f,a_{i_1})}(r) \leq \cdots \leq N^{[1]}_{(f,a_{i_q})}(r)\}.$$

We now consider an element $I = (i_1, \dots, i_q)$ of $\mathcal{I}$, for instance $I = (1, 2, \dots, q)$. Applying Lemma 6, there is a subset $J \subset \{1, 2, \dots, 2n-k+2\}$ with $|J| = d+2$ such that

$$\Big\|_E \; T_f(r) \leq \sum_{j\in J} N^{[n_0]}_{(f,a_j)}(r) + \frac{n_0(\sharp J-1)}{2}\big((1+\varepsilon)\log\gamma(r) + \varepsilon\log r\big) + S(r).$$

Put $J' = \{1, 2, \dots, 2n-k+2\}\backslash J$ and $J'' = \{1, 2, \dots, q\}\backslash\{1, 2, \dots, 2n-k+2\}$. We see that $|J'| = 2n-k-d$ and $J \cup J' \cup J'' = \{1, 2, \dots, q\}$.
Now we take $\eta \in \big[\max\limits_{1\leq i\leq q} \lambda_i, \dfrac{\sum_{i=1}^q \lambda_i}{2(n-k)}\big]$. Because

$$\sum_{i=1}^q \lambda_i - |J'|\eta = \sum_{i=1}^q \lambda_i - (2n-k+2-(d+2))\eta$$

and $k \leq d \leq n$, then we find that

$$0 < \sum_{i=1}^q \lambda_i - 2(n-k)\eta \leq \sum_{i=1}^q \lambda_i - |J'|\eta \leq \sum_{i=1}^q \lambda_i - (n-k)\eta. \tag{7}$$

This implies that

$$\begin{aligned}\Big\|_E \; \Big(\sum_{i=1}^q \lambda_i - |J'|\eta\Big)T_f(r) \leq{}& \Big(\sum_{i=1}^q \lambda_i - |J'|\eta\Big)\Big(\sum_{j\in J} N^{[n_0]}_{(f,a_{i_j})}(r)\Big)\\ &+ \frac{n_0(\sharp J-1)\big(\sum_{i=1}^q \lambda_i - |J'|\eta\big)}{2}\big((1+\varepsilon)\log\gamma(r) + \varepsilon\log r\big) + S(r).\end{aligned}$$

On the other hand, since $\sum_{j\in J'}\lambda_{i_j}-|J'|\eta\le 0$, we deduce that

$$\begin{aligned}
&\big(\sum_{i=1}^{q}\lambda_i-|J'|\eta\big)\sum_{t\in J}N^{[n_0]}_{(f,a_{i_t})}(r)\\
&=\big(\sum_{j\in J\cup J''}\lambda_{i_j}\big)\sum_{t\in J}N^{[n_0]}_{(f,a_{i_t})}(r)+\big(\sum_{j\in J'}\lambda_{i_j}-|J'|\eta)\big)\sum_{t\in J}N^{[n_0]}_{(f,a_{i_t})}(r)\\
&\le\big(\sum_{j\in J\cup J''}\lambda_{i_j}\big)\sum_{t\in J}N^{[n_0]}_{(f,a_{i_t})}(r).
\end{aligned}$$

In other words, we have the following estimations

$$\begin{aligned}
&\big(\sum_{j\in J\cup J''}\lambda_{i_j}\big)\sum_{t\in J}N^{[n_0]}_{(f,a_{i_t})}(r)=|J|\frac{\sum_{j\in J\cup J''}\lambda_{i_j}}{|J|}\sum_{j\in J}\lambda_{i_j}N^{[n_0]}_{(f,a_{i_j})}(r)\\
&=|J|\Big(\sum_{t\in J}\lambda_{i_t}N^{[n_0]}_{(f,a_{i_t})}(r)+\sum_{t\in J}\big(\frac{\sum_{j\in J\cup J''}\lambda_{i_j}}{|J|}-\lambda_{i_t}\big)N^{[n_0]}_{(f,a_{i_t})}(r)\Big)\\
&\le|J|\Big(\sum_{t\in J}\lambda_{i_t}N^{[n_0]}_{(f,a_{i_t})}(r)+\Big(\sum_{t\in J}\big(\frac{\sum_{j\in J\cup J''}\lambda_{i_j}}{|J|}-\lambda_{i_t}\big)\Big)N^{[n_0]}_{(f,a_{i_{2n-k+3}})}(r)\Big)\\
&=|J|\Big(\sum_{t\in J}\lambda_{i_t}N^{[n_0]}_{(f,a_{i_t})}(r)+\sum_{t\in J''}\lambda_{i_t}N^{[n_0]}_{(f,a_{i_{2n-k+3}})}(r)\Big)\\
&\le|J|\Big(\sum_{t\in J}\lambda_{i_j}N^{[n_0]}_{(f,a_{i_j})}(r)+\sum_{t\in J''}\lambda_{i_t}N^{[n_0]}_{(f,a_{i_t})}(r)\Big)\\
&\le|J|\sum_{j=1}^{q}\lambda_{i_j}N^{[n_0]}_{(f,a_{i_j})}(r)\\
&\le|J|n_0\sum_{j=1}^{q}\lambda_{i_j}N^{[1]}_{(f,a_{i_j})}(r)\\
&\le\frac{k(k+1)(d+2)}{d+2}\sum_{j=1}^{q}\lambda_{i_j}N^{[1]}_{(f,a_{i_j})}(r)\\
&\le k(k+1)\sum_{j=1}^{q}\lambda_{i_j}N^{[1]}_{(f,a_{i_j})}(r).
\end{aligned}$$

For $r\in M_I$, from these above estimations, we get

$$\begin{aligned}
\Big\|_E\ \big(\sum_{i=1}^{q}\lambda_i-|J'|\eta\big)T_f(r)\le k(k+1)\sum_{i=1}^{q}\lambda_iN^{[1]}_{(f,a_i)}(r)&\\
+\frac{n_0(\sharp J-1)\big(\sum_{i=1}^{q}\lambda_i-|J'|\eta\big)}{2}\big((1+\varepsilon)\log\gamma(r)+\varepsilon\log r\big)+S(r).&
\end{aligned}\tag{8}$$

In addition, we have

$$\frac{n_0(\sharp J-1)}{2} \le \frac{k(k+2)(d+1)}{2(d+2)} \le \frac{k(k+2)(n+1)}{2(n+2)}. \tag{9}$$

Combining (7), (9) and (8), we obtain

$$\begin{aligned} \|_E \ \frac{\sum_{i=1}^q \lambda_i - 2(n-k)\eta}{k(k+2)} T_f(r) &\le \sum_{i=1}^q \lambda_i N^{[1]}_{(f,a_i)}(r) \\ + \frac{k(k+2)(n+1)(\sum_{i=1}^q \lambda_i - (n-k)\eta)}{2(n+2)} &\big((1+\varepsilon)\log\gamma(r) + \varepsilon \log r\big) + S(r). \end{aligned} \tag{10}$$

We see that $\bigcup_{I\in\mathcal{I}} N_I = (0,R)$ and inequality (10) holds for every $r \in (0,R)$ outside a subset E with $\int_E \gamma(r)dr < +\infty$. Hence, the theorem is proved.

Proof of Theorem 2

The theorem will be proved based on the following lemma which is also a result given in the proof of Theorem 1.1.c in [8].

Lemma 11 ([8], Claim in the proof of Theorem 1.1.c)**.** *Let f, $\{a_i\}$ and $\gamma(r)$ be the same as in Theorem 2. Then for every $\varepsilon > 0$, we have*

$$\|_E \ T_f(r) \le \sum_{j=0}^{k+1} N^{[k]}_{(f,a_{i_j})}(r) + \frac{k(k+1)}{2}\left((1+\varepsilon)\log\gamma(r) + \varepsilon\log r\right) + S(r),$$

where $S(r) = O\big(\log T_f(r) + \max_{1\le i\le q} T_{a_i}(r)\big)$.

We denote by $\mathcal{I}$ the set of all permutations of the set $\{1,\dots,q\}$. For each element $I = (i_1,\dots,i_q) \in \mathcal{I}$, we set

$$N_I = \{r \in (0,R) \mid N^{[k]}_{(f,a_{i_1})}(r) \le \cdots \le N^{[k]}_{(f,a_{i_q})}(r)\}.$$

For all $r \in N_I$, we see that

$$\begin{aligned} &\sum_{j=0}^{k+1} N^{[k]}_{(f,a_{i_j})}(r) \le N^{[k]}_{(f,a_1)}(r) + \sum_{i=n-k+2}^{n+1} N^{[k]}_{(f,a_i)}(r) + N^{[k]}_{(f,a_{(n-k+1)(k+1)+1})}(r) \\ &\le \frac{1}{n-k+1}\Bigg(\sum_{i=1}^{n-k+1} N^{[k]}_{(f,a_i)}(r) + \sum_{i=n-k+2}^{(n-k+1)(k+1)} N^{[k]}_{(f,a_i)}(r) + \sum_{i=(n-k+1)(k+1)+1}^{(n-k+1)(k+2)} N^{[k]}_{(f,a_i)}(r)\Bigg) \\ &= \frac{1}{n-k+1}\sum_{j=1}^{(n-k+1)(k+2)} N^{[k]}_{(f,a_i)}(r). \end{aligned}$$

This follows that

$$
\begin{aligned}
(\sum_{i=1}^{q}\lambda_i)\sum_{j=0}^{k+1}N^{[k]}_{(f,a_{i_j})}(r) &\le \frac{\sum_{i=1}^{q}\lambda_i}{n-k+1}\sum_{j=1}^{(n-k+1)(k+2)}N^{[k]}_{(f,a_i)}(r)\\
&= \frac{(n-k+1)(k+2)}{n-k+1}\sum_{j=1}^{(n-k+1)(k+2)}\left(\frac{\sum_{i=1}^{q}\lambda_i}{(n-k+1)(k+2)}N^{[k]}_{(f,a_i)}(r)\right)\\
&\le (k+2)\left(\sum_{j=1}^{(n-k+1)(k+2)}\lambda_{i_j}N^{[k]}_{(f,a_i)}(r)\right)\\
&+ (k+2)\sum_{j=1}^{(n-k+1)(k+2)}\left(\frac{\sum_{i=1}^{q}\lambda_i}{(n-k+1)(k+2)}-\lambda_{i_j}\right)N^{[k]}_{(f,a_i)}(r)\\
&= (k+2)\sum_{i=1}^{q}\lambda_i N^{[k]}_{(f,a_i)}(r).
\end{aligned}
$$

Combining Lemma 11 together with these above estimations, we have

$$\Big\|_E \ \frac{\sum_{i=1}^{q}\lambda_i}{k+2}T_f(r) \le \sum_{i=1}^{q}\lambda_i N^{[k]}_{(f,a_i)}(r) + \frac{k(k+1)\sum_{i=1}^{q}\lambda_i}{2(k+2)}((1+\varepsilon)\log\gamma(r)+\varepsilon\log r)+S(r),$$

for all $r\in N_I$.
Repeating again the argument in the proof of Theorem 1, we have the desired conclusion in Theorem 2.

4 Proofs of algebraic dependence theorems

The following lemmas and claims obtained in [10] and [6] for the case of meromorphic maps from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. With the same arguments in these proofs, we can show that they also true for holomorphic maps from $\Delta(R)$ into $\mathbb{P}^n(\mathbb{C})$ in our situation. Therefore, here we just state results without presenting proofs. We also note that, in the case of $R=+\infty$, the Theorem 3 and 4 have already been proved in [1]. Then in the following proofs, we only consider the case where $R<+\infty$.

Proof of Theorem 3

Lemma 12 ([10], Claim 3.1). *For every $1\le i\le\lambda, 1\le j\le q$ and $1\le k\le n$, we have*

$$N^{[k]}_{(f,a_j),\le k_j}(r) \ge \frac{k_j+1}{k_j+1-k}N^{[k]}_{(f,a_j)}(r) - \frac{k}{k_j+1-k}T_f(r).$$

Lemma 13 ([6], Claim 4.2). *For every $1\le t\le\lambda$, we have*

$$\sum_{j=1}^{q}(\lambda-l_j+1)\min\left\{1,\nu_{(f_t,a_j)}(z)\right\} \le d\mu_{f_1\wedge\cdots\wedge f_\lambda}(z)+q(\lambda-1)\sum_{\beta}\mu_{a_{\beta(1)}\wedge\cdots\wedge a_{\beta(n+1)}}(z).$$

for each $z \notin A$, where the sum is taken over all injective map $\beta : \{1, 2, \ldots, n+1\} \to \{1, 2, \ldots, q\}$.

It suffices to prove the theorem in the case of $\lambda \leq n+1$. Suppose that $f_1 \wedge \cdots \wedge f_\lambda \not\equiv 0$. Since $\nu^{[1]}_{(f_i,a_j),\leq k_j}(z) \leq \nu^{[1]}_{(f_i,a_j)}(z)$, then it follows from Lemma 13 that

$$\begin{aligned}\sum_{j=1}^{q}(\lambda - l_j + 1)\, N^{[1]}_{(f_t,a_j),\leq k_j}(r) &\leq \sum_{j=1}^{q}(\lambda - l_j + 1)\, N^{[1]}_{(f_t,a_j)}(r)\\ &\leq dN_{\mu_{f_1\wedge\cdots\wedge f_\lambda}}(r) + q(\lambda-1)\sum_{\beta\in T[n+1,q]} N_{\mu_{a_{\beta(1)}\wedge\cdots\wedge a_{\beta(n+1)}}}(r)\\ &\leq d\sum_{i=1}^{\lambda} T_{f_i}(r) + q(\lambda-1)\sum_{\beta\in T[n+1,q]}\sum_{i=1}^{n+1} T_{a_{\beta(i)}}(r)\\ &= d\sum_{i=1}^{\lambda} T_{f_i}(r) + o\left(\max_{1\leq i\leq\lambda} T_{f_i}(r)\right).\end{aligned}$$

Using Lemma 12, we deduce that

$$\begin{aligned}d\sum_{i=1}^{\lambda} T_{f_i}(r) &\geq \sum_{j=1}^{q}(\lambda - l_j + 1)\, N^{[1]}_{(f_t,a_j),\leq k_j}(r) + o\left(\max_{1\leq i\leq\lambda} T_{f_i}(r)\right)\\ &\geq \sum_{j=1}^{q}(\lambda - l_j + 1)\left(\frac{k_j+1}{k_j} N^{[1]}_{(f_t,a_j)}(r) - \frac{1}{k_j} T_{f_i}(r)\right) + o\left(\max_{1\leq i\leq\lambda} T_{f_i}(r)\right)\\ &\geq \sum_{j=1}^{q}(\lambda - l_j + 1)\, N^{[1]}_{(f_t,a_j)}(r) - \sum_{j=1}^{q}\frac{\lambda - l_j + 1}{k_j} T_{f_i}(r) + o\left(\max_{1\leq i\leq\lambda} T_{f_i}(r)\right).\end{aligned}$$

Hence, by summing up both sides of these inequalities, we get

$$d\lambda\sum_{i=1}^{\lambda} T_{f_i}(r) + \sum_{j=1}^{q}\sum_{i=1}^{\lambda}\frac{\lambda - l_j + 1}{k_j} T_{f_i}(r) \geq \sum_{j=1}^{q}\sum_{i=1}^{\lambda}(\lambda - l_j + 1)\, N^{[1]}_{(f_t,a_j)}(r) + o\left(\max_{1\leq i\leq\lambda} T_{f_i}(r)\right).$$

Put $\lambda_i = \lambda - l_i + 1, i = 1, \ldots, q$, $k_t + 1 = \mathrm{rank}_{\mathcal{R}\{a_i\}}(f_t), t = 1, \ldots, \lambda$ and $k = \max_{1\leq t\leq\lambda} k_t$. We can easily see that

$$2(n - k_t)\max_{1\leq j\leq q}(\lambda - l_j + 1) \leq 2n(\lambda - 1) \leq q \leq \sum_{j=1}^{q}(\lambda - l_j + 1).$$

This follows that

$$2(n-k)\max\lambda_i \leq \sum_{j=1}^{q}\lambda_i.$$

From Theorem 1, for any $\eta \in \big[\max_{1\le i\le q} \lambda_i, \frac{\sum_{i=1}^q \lambda_i}{2(n-k)}\big]$, we have

$$\Big\|_E \frac{q(\lambda+1) - \sum_{j=1}^q l_j - 2\,(n-k_t)\,\eta}{k_t\,(k_t+2)} T_{f_t}(r) \le \sum_{j=1}^q (\lambda - l_j + 1)\, N^{[1]}\left(r, \nu_{(f_t,a_j)}\right) + \delta(k_t)\big((1+\varepsilon)\log(\gamma_t) + \varepsilon \log r\big) + S(r),$$

where $\delta(k_t) = \dfrac{k_t(k_t+2)(n+1)(\sum_{i=1}^q \lambda_i - (n-k_t)\eta)}{2(n+2)}$.

On the other hand,

$$\frac{q(\lambda+1) - \sum_{j=1}^q l_j - 2n\eta + 2k\eta}{k(k+2)} \le \frac{q(\lambda+1) - \sum_{j=1}^q l_j - 2n\eta + 2k_t\eta}{k_t\,(k_t+2)}$$

and

$$\frac{k_t(k_t+2)(n+1)(\sum_{i=1}^q \lambda_i - (n-k_t)\eta)}{2(n+2)} \le \frac{k(k+2)(n+1)(\sum_{i=1}^q \lambda_i - (n-k)\eta)}{2(n+2)}.$$

Moreover, we take $\gamma_t(r) = \exp\{\big(\min_{1\le i\le \lambda} c_{f_i} + \varepsilon\big) \sum_{i=1}^\lambda T_{f_i}(r)\}$. Then we have

$$\begin{aligned}
&\Big\|_E\ \frac{q(\lambda+1) - \sum_{j=1}^q l_j - 2(n-k)\eta}{k(k+2)} \sum_{i=1}^\lambda T_{f_i}(r) \\
&\le d\lambda \sum_{i=1}^\lambda T_{f_i}(r) + \sum_{i=1}^\lambda \sum_{j=1}^q \frac{\lambda - l_j + 1}{k_j} T_{f_i}(r) \\
&+ \lambda\delta(k)\Big((1+\varepsilon)\big(\min_{1\le i\le\lambda} c_{f_i} + \varepsilon\big) \sum_{i=1}^\lambda T_{f_i}(r) + \varepsilon \log r\Big) + S(r).
\end{aligned}$$

Letting $r \to +\infty$ ($r \notin E$) and letting $\varepsilon \to 0^+$, we get

$$\begin{aligned}
\sum_{j=1}^q \frac{\lambda-1}{k_j} &\ge \sum_{j=1}^q \frac{\lambda - l_j + 1}{k_j} \\
&\ge \frac{q(\lambda+1) - \sum_{j=1}^q l_j - 2(n-k)\eta}{k(k+2)} - d\lambda \\
&- \frac{\lambda k(k+2)(n+1)\big(q(\lambda+1) - \sum_{j=1}^q l_j - (n-k)\eta\big)}{2(n+2)} \min_{1\le i\le\lambda} c_{f_i}.
\end{aligned} \tag{14}$$

In other words, we see that

$$\lambda - 1 \le \max_{1\le i\le q} \lambda_i \le \eta \le \frac{\sum_{j=1}^q \lambda_j}{2(n-k)} = \frac{q(\lambda+1) - \sum_{j=1}^q l_j}{2(n-k)}.$$

It follows from (14) that

$$\begin{aligned}\sum_{j=1}^{q}\frac{\lambda-1}{k_j} \geq \sum_{j=1}^{q}\frac{\lambda-l_j+1}{k_j} &\geq \frac{q(\lambda+1)-\sum_{j=1}^{q}l_j-2(n-k)(\lambda-1)}{k(k+2)}-d\lambda\\ &\quad-\frac{\lambda k(k+2)(n+1)\big(q(\lambda+1)-\sum_{j=1}^{q}l_j\big)}{4(n+2)}\min_{1\leq i\leq\lambda}c_{f_i}.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{j=1}^{q}\frac{1}{k_j} \geq &\frac{q(\lambda+1)-\sum_{j=1}^{q}l_j-2(n-k)(\lambda-1)-d\lambda k(k+2)}{k(k+2)(\lambda-1)}\\ &-\frac{\lambda k(k+2)(n+1)\big(q(\lambda+1)-\sum_{j=1}^{q}l_j\big)}{4(n+2)(\lambda-1)}\min_{1\leq i\leq\lambda}c_{f_i}.\end{aligned}$$

It is a contradiction. The proof of Theorem 1.3 is finished.

Proof of Theorem 4

Lemma 15 ([10], Claim 3.3)**.** *Let $h_i:\Delta(R)\to\mathbb{P}^n(\mathbb{C})(1\leq i\leq p\leq n+1)$ be meromorphic maps with reduced representations $h_i:=(h_{i0}:\cdots:h_{in})$, $a_i:=(a_{i0}:\cdots:a_{in})$. Put $\widetilde{h}:=((h_i,a_1):\cdots:(h_i,a_{n+1}))$, and assume that $a_1,\ldots,a_{n+1}$ are located in general position such that $(h_i,a_j)\neq 0$ $(1\leq i\leq p, 1\leq j\leq n+1)$. Let z_0 be a point of $\Delta(R)$ such that $(a_1\wedge\cdots\wedge a_{n+1})(z_0)\neq 0$. Then $(h_1\wedge\cdots\wedge h_p)(z_0)=0$ if and only if $(\widetilde{h}_1\wedge\cdots\wedge\widetilde{h}_p)(z_0)=0$.*

Now we put $J=\{j_1,\ldots,j_\lambda\}$, $J^c=\{1,\ldots,q\}\backslash J$ and

$$B_J=\begin{pmatrix}(f_1,a_{j_1}) & \cdots & (f_\lambda,a_{j_1})\\ (f_1,a_{j_2}) & \cdots & (f_\lambda,a_{j_2})\\ \vdots & \vdots & \vdots\\ (f_1,a_{j_\lambda}) & \cdots & (f_\lambda,a_{j_\lambda})\end{pmatrix},$$

then we have the following lemma.

Lemma 16 ([10], Claim 3.4)**.** *If B_J is nondegenerate, i.e., $\det B_J\not\equiv 0$, then*

$$\begin{aligned}\nu_{\widetilde{f}_1\wedge\cdots\wedge\widetilde{f}_\lambda}\geq&\sum_{j\in J}\left(\min_{1\leq i\leq\lambda}\left\{\nu^0_{(f_i,a_j),\leq k_j}\right\}-\min\left\{1,\nu^0_{(f_i,a_j),\leq k_j}\right\}\right)\\ &+\sum_{j=1}^{q}(\lambda-l_j+1)\min\left\{1,\nu^0_{(f_i,a_j),\leq k_j}\right\}\end{aligned}$$

on the set $\Delta(R)\backslash\left(a_{j_1\wedge\ldots\wedge a_{j_\lambda}}\right)^{-1}(0)$, where $\widetilde{f}_i:=((f_i,a_{j_1}):\cdots\ :((f_i,a_{j_\lambda})$.

It suffices to prove the theorem in the case of $\lambda \le n+1$. Suppose that $f_1 \wedge \cdots \wedge f_\lambda \not\equiv 0$. For $\varepsilon > 0$, we put

$$\gamma(r) = \exp\{\big(\min_{1\le i\le \lambda} c_{f_i} + \varepsilon\big) \sum_{i=1}^{\lambda} T_{f_i}(r)\}.$$

We see that $\int_0^R \gamma(r)dr = \infty$.
For each $j, 1 \le j \le q$, we set

$$N_j(r) = \sum_{i=1}^{\lambda} N^{[k]}_{(f_i,a_j),\le k_j}(r) - ((\lambda-1)k+1)N^{[1]}_{(f_i,a_j),\le k_j}(r).$$

and for each permutation $I = (j_1, \ldots, j_q)$ of $(1, \ldots, q)$, we also put

$$T_I = \left\{ r \in (0,R) \mid N_{j_1}(r) \ge \cdots \ge N_{j_q}(r) \right\}.$$

It is clear that $\bigcup_I T_I = (0,R)$. Therefore, there exists a permutation, for instance, it is $I_1 = (1, \ldots, q)$ such that $\int_{T_{I_1}} \gamma(r)dr = \infty$. Then, we have

$$N_1(r) \ge N_2(r) \ge \cdots \ge N_q(r).$$

By the assumption that $f_1 \wedge \cdots \wedge f_\lambda \not\equiv 0$, there exists ordered set of indices $J = \{j_1, \cdots, j_\lambda\}$ with $1 = j_1 < \cdots < j_\lambda \le n+1$ such that $\det B_J \not\equiv 0$. We note that

$$N_1(r) = N_{j_1}(r) \ge N_{j_2}(r) \ge \cdots \ge N_{j_\lambda}(r) \ge N_{n+1}(r),$$

for each $r \in T_{I_1}$. We see that $\min_{1\le i\le\lambda} b_i \ge \sum_{i=1}^{\lambda} \min\{k, b_i\} - (\lambda-1)k$, for every non-negative integers λ and $b_1, \ldots, b_\lambda$. Lemma 16 implies that

$$\begin{aligned}
&\sum_{j\in J}\left(\sum_{i=1}^{\lambda} \min\left\{k, \nu^0_{(f_i,a_j),\le k_j}\right\} - ((\lambda-1)k+1)\min\left\{1, \nu^0_{(f_i,a_j),\le k_j}\right\}\right)\\
&+\sum_{j=1}^{q}(\lambda - l_j + 1)\min\left\{1, \nu^0_{(f_i,a_j),\le k_j}\right\} \le \mu_{\widetilde{f}_1\wedge\cdots\wedge\widetilde{f}_\lambda}.
\end{aligned}$$

on the set $\Delta(R)\backslash(a_{j_1} \wedge \cdots \wedge a_{j_\lambda})^{-1}(0)$. Integrating both sides of the inequality, we get

$$\begin{aligned}
&\sum_{j\in J}\left(\sum_{i=1}^{\lambda} N^{[k]}_{(f_i,a_j),\le k_j}(r) - ((\lambda-1)k+1)N^{[1]}_{(f_1,a_j),\le k_j}(r)\right)\\
&+\sum_{j=1}^{q}(\lambda - l_j + 1)\, N^{[1]}_{(f_1,a_j),\le k_j}(r) \le N_{\widetilde{f}_1\wedge\cdots\wedge\widetilde{f}_\lambda}(r) = N_{\det B_J}(r).
\end{aligned} \tag{17}$$

On the other hand, by Jensen's formula, we obtain

$$N_{\det B_J}(r) \le \int_{S(r)} \log|\det B_J|\,\sigma_n + O(1) \le \sum_{i=1}^{\lambda} T_{f_i}(r) + o\left(\max_{1\le i\le\lambda} T_{f_i}(r)\right). \tag{18}$$

Set $T(r)=\sum_{i=1}^{\lambda} T_{f_i}(r)$. Combining (17) and (18), we get for all $r\in I_1$,

$$\begin{aligned}
T(r) \geq & \sum_{i=1}^{\lambda} N_{j_i}(r) + \sum_{j=1}^{q} (\lambda - l_j + 1)\, N^{[1]}_{(f_1,a_j),\leq k_j}(r) + o\left(\max_{1\leq i\leq \lambda} T_{f_i}(r)\right)\\
\geq & \frac{\lambda}{q}\sum_{j=1}^{q} N_j(r) + \sum_{j=1}^{q} (\lambda - l_j + 1)\, N^{[1]}_{(f_1,a_j),\leq k_j}(r) + o\left(\max_{1\leq i\leq \lambda} T_{f_i}(r)\right)\\
= & \sum_{j=1}^{q}\left(\lambda - l_j + 1 - \frac{\lambda((\lambda-1)k+1)}{q}\right) N^{[1]}_{(f_1,a_j),\leq k_j}(r)\\
& + \sum_{j=1}^{q}\sum_{i=1}^{\lambda} \frac{\lambda}{q} N^{[k]}_{(f_i,a_j),\leq k_j}(r) + o\left(\max_{1\leq i\leq \lambda} T_{f_i}(r)\right)\\
\geq & \sum_{j=1}^{q}\sum_{i=1}^{\lambda}\left(\frac{\lambda - l_j + 1}{\lambda k} - \frac{(\lambda-1)k+1}{qk} + \frac{\lambda}{q}\right) N^{[k]}_{(f_i,a_j),\leq k_j}(r)\\
& + o\left(\max_{1\leq i\leq \lambda} T_{f_i}(r)\right).
\end{aligned}$$

From Lemma 12 and the above inequality, we have

$$\begin{aligned}
T(r) \geq & \sum_{j=1}^{q}\sum_{i=1}^{\lambda}\left(\frac{q\,(\lambda - l_j + 1) + \lambda(k-1)}{q\lambda k}\right) N^{[k]}_{(f_i,a_j)}(r)\\
& + \sum_{j=1}^{q}\left(\frac{q\,(\lambda - l_j + 1) + \lambda(k-1)}{q\lambda k}\right)\frac{k}{k_j+1-k} T(r) + o\left(\max_{1\leq i\leq \lambda} T_{f_i}(r)\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
& q\lambda k + \sum_{j=1}^{q}\frac{k}{k_j+1-k}\big(q\,(\lambda - l_j + 1) + \lambda(k-1)\big) T(r)\\
& \geq \sum_{j=1}^{q}\sum_{i=1}^{\lambda}\big(q\,(\lambda - l_j + 1) + \lambda(k-1)\big) N^{[k]}_{(f_i,a_j)}(r) + o\left(\max_{1\leq i\leq \lambda} T_{f_i}(r)\right).
\end{aligned} \tag{19}$$

For each $1\leq j\leq q$, put $\lambda_j = q\,(\lambda - l_j + 1) + \lambda(k-1)$, we see that

$$\frac{\sum_{i=1}^{q}\lambda_i}{\max\lambda_i} \geq \frac{q^2 + q\lambda(k-1)}{q(\lambda-1)+\lambda(k-1)} \geq \frac{q}{\lambda-1}.$$

In other words,

- If $k\leq \dfrac{n-1}{2}, q\geq (n-k+1)(k+2)(\lambda-1)$, we get

$$\frac{\sum_{i=1}^{q}\lambda_i}{\max\lambda_i} \geq (n-k+1)(k+2) \geq (n-k_t+1)\,(k_t+2)\,.$$

- or if $k > \frac{n-1}{2}, q \geq \frac{(n+3)^2}{4}(\lambda - 1)$, then

$$\frac{\sum_{i=1}^{q} \lambda_i}{\max \lambda_i} \geq \frac{(n+3)^2}{4} \geq (n - k_t + 1)(k_t + 2).$$

In these cases, for $1 \leq t \leq \lambda$, taking $\gamma_t(r) = \exp\{\big(\min_{1\leq i\leq\lambda} c_{f_i} + \varepsilon\big)\sum_{i=1}^{\lambda} T_{f_i}(r)\}$ and applying Theorem 2, we get

$$\begin{aligned}
&\Big\|_E \sum_{j=1}^{q}\big(q(\lambda - l_j + 1) + \lambda(k-1)\big)N^{[k]}_{(f_t,a_j)}(r) + \\
&\frac{k(k+1)\sum_{i=1}^{q}\big((q(\lambda - l_j + 1) + \lambda(k-1)\big)}{2(k+2)}\left((1+\varepsilon)\big(\min_{1\leq i\leq\lambda} c_{f_i} + \varepsilon\big)T(r) + \varepsilon\log r\right) + S(r) \\
&\geq \frac{\sum_{j=1}^{q}\left(q(\lambda - l_j + 1) + \lambda(k-1)\right)}{k+2}T_{f_t}(r).
\end{aligned} \tag{20}$$

Combining inequalities (19) and (20), we have

$$\begin{aligned}
&\Big\|_E\ q\lambda k + \sum_{j=1}^{q}\frac{k}{k_j + 1 - k}\left(q(\lambda - l_j + 1) + \lambda(k-1)\right)T(r) \\
&\frac{k(k+1)\sum_{i=1}^{q}\big(q(\lambda - l_j + 1) + \lambda(k-1)\big)}{2(k+2)}\left((1+\varepsilon)\big(\min_{1\leq i\leq\lambda} c_{f_i} + \varepsilon\big)T(r) + \varepsilon\log r\right) + S(r) \\
&\geq \sum_{t=1}^{\lambda}\frac{\sum_{j=1}^{q}\left((q(\lambda - l_j + 1) + \lambda(k-1)\right)}{k+2}T_{f_t}(r).
\end{aligned}$$

Letting $r \to R$ ($r \in T_{I_1}, r \notin E$) and $\varepsilon \to 0^+$, we obtain

$$\begin{aligned}
&\frac{q\big(q(\lambda+1) - \sum_{j=1}^{q} l_j + \lambda(k-1)\big)}{k+2} \leq q\lambda k + \sum_{j=1}^{q}\frac{k}{k_j + 1 - k}\big(q(\lambda - l_j + 1) + \lambda(k-1)\big) \\
&+ \frac{k(k+1)q\big(q(\lambda+1) - \sum_{j=1}^{q} l_j + \lambda(k-1)\big)}{2(k+2)}\min_{1\leq i\leq\lambda} c_{f_i}.
\end{aligned}$$

Thus

$$\sum_{j=1}^{q}\frac{1}{k_j + 1 - k} \geq \frac{1}{k(k+2)} - \frac{\lambda}{q(\lambda+1) - \sum_{j=1}^{q} l_j + \lambda(k-1)} - \frac{k+1}{2(k+2)}\min_{1\leq i\leq\lambda} c_{f_i}.$$

This is a contradiction. Hence, we have $f_1 \wedge \cdots \wedge f_\lambda \equiv 0$.

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