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Fully invariant submodules for constructing dual Rickart modules and dual Baer modules

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Abstract

Fully invariant submodules play an important designation in studying the structure of some known modules such as (dual) Rickart and (dual) Baer modules. In this work, we introduce F-dual Rickart (Baer) modules via the concept of fully invariant submodules. It is shown that M is F-dual Rickart if and only if $M = F \oplus L$ such that F is a dual Rickart module. We prove that a module M is F-dual Baer if and only if M is F-dual Rickart and M has $SSSP$ for direct summands of M contained in F. We present a characterization of right I-dual Baer rings where I is an ideal of R. Some counter-examples are provided to illustrate new concepts.

Key Words: Fully invariant submodule, dual Rickart module, F-dual Rickart module, F-dual Baer module.
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1 Introduction

All rings considered in this paper will be associative with an identity element and all modules will be unitary right modules unless otherwise stated. Let R be a ring and M an R-module. $S = End_R(M)$ will denote the ring of all R-endomorphisms of M. We will use the notation $N \ll M$ to indicate that N is small in M (i.e. $\forall L \lneq M, L + N \neq M$). A module M is called *hollow* if every proper submodule of M is small in M. The notation $N \leq^{\oplus} M$ denotes that N is a direct summand of M. $N \trianglelefteq M$ means that N is a fully invariant submodule of M (i.e., $\forall \phi \in End_R(M),\ \phi(N) \subseteq N$). $Rad(M)$ and $Soc(M)$ denote the radical and the socle of a module M, respectively.

Let $L \subseteq K \leq M$. We say that K *lies above* L in M if $K/L \ll M/L$. A module M is called *lifting* if every submodule A of M lies above a direct summand D of M ([3]).

Let M be a module. Following [6], M is called *(dual) Rickart* in case for every endomorphism φ of M, $(Im\varphi)$ $Ker\varphi$ is a direct summand of M. For the study of (dual) Rickart modules, idempotents of endomorphism rings of modules are important. In particular as an interesting result, a module M is Rickart and dual Rickart if and only if $End_R(M)$ is a von Neumann regular ring. Amouzegar in [1] introduced a generalization of both lifting modules and dual Rickart modules as $\mathcal{I}$-lifting modules. The author showed that a projective $\mathcal{I}$-lifting module is a direct sum of cyclic modules. She also present a characterization of $\mathcal{I}$-lifting rings in terms of finitely supplemented modules. Although the class of $\mathcal{I}$-lifting modules is larger than the class of dual Rickart modules, studying and investigating them seem to have more difficulties.

In [2], it is introduced a various of $\mathcal{I}$-lifting modules via a fixed fully invariant submodule of a given module. By the way, they call a module M, $\mathcal{I}_F$-lifting (where F is a fully invariant submodule of M) provided for every endomorphism φ of M, the submodule $\varphi(F)$ lies above a direct summand of M. It is obvious that a module M is $\mathcal{I}$-lifting if and only if M is $\mathcal{I}_M$-lifting. Various properties of such modules have been also investigated in [2]. As a continuoution of the last work, also Moniri and Amouzegar in [8] tried to study H-supplemented modules via the same approach as in [2]. A module M is called $\mathcal{I}_F$-H-supplemented provided for every $\varphi \in End_R(M)$ there exists a direct summand D of M such that $\varphi(F) + X = M$ if and only if $D + X = M$, for all submodules X of M. Some conditions to ensure that a $\mathcal{I}_F$-H-supplemented module is $\mathcal{I}_F$-lifting, were presented in [8]. The relation with the other similar classes of modules was also investigated. The authors also studied direct sums of $\mathcal{I}_F$-H-supplemented modules.

Motivating by mentioned works we are interested to study on dual Rickart modules via fully invariant submodules. In fact, in the definition of a dual Rickart module, one can replaced M by a fully invariant submodule of M. We call M, F-dual Rickart provided for every endomorphism φ of M the submodule $\varphi(F)$ is a direct summand of M. In what follows by F we mean a fully invariant submodule of M.

Any undefined terminologies not defined in the manuscript can be found in [3, 7].

2 F-dual Rickart modules and F-dual Baer modules

Recently dual Rickart modules and their various generalizations have been extensively studied and investigated. In particular, in [2] it is introduced a new generalization of both dual Rickart modules and $\mathcal{I}$-lifting modules via fully invariant submodules. A module M is called $\mathcal{I}_F$-lifting provided for every endomorphism φ of M, the submodule $\varphi(F)$ of M lies above a direct summand of M. So, it will be of interest for us to change "lying above a direct summand" to "be a direct summand" as well.

Definition 1. Let M be a module and F a fully invariant submodule of M. We say M is F-dual Rickart if for every $\varphi \in End_R(M)$, the submodule $\varphi(F)$ is a direct summand of M.

It is clear that an arbitrary module is 0-dual Rickart and M is dual Rickart if and only if M is M-dual Rickart. It can be worth to say that a dual Rickart module M may not be F-dual Rickart for a fully invariant submodule. For instance, the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is dual Rickart while it is not a $Soc(\mathbb{Z}_{p^\infty})$-dual Rickart $\mathbb{Z}$-module (see Example 1). It is clear by definitions that any F-dual Rickart module is $\mathcal{I}_F$-lifting while the other side may not hold.

Example 1. Let M be a module and F a nontrivial fully invariant submodule of M (that is, F will be different from 0 and M). If F is small in M, then M is $\mathcal{I}_F$-lifting (note that in this case for every φ in $End_R(M)$, the submodule $\varphi(F)$ is a small submodule of M)(see [2, Example 2.2(1)]). So that $\varphi(F)$ can not be a direct summand of M. It follows that M is not a F-dual Rickart module. In particular, every hollow module M is $\mathcal{I}_F$-lifting for every nontrivial fully invariant submodule F of M while M is not F-dual Rickart. For example the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$ is $\mathcal{I}_{<1/p+\mathbb{Z}>}$-lifting. Note that $Soc(M) =< 1/p + \mathbb{Z} >$.

The following provides an important characterization of F-dual Rickart modules which will be used freely throughout the paper.

Theorem 1. *Let M be a module and F be a fully invariant submodule of M. Then the following conditions are equivalent:*

(1) *M is F-dual Rickart;*

(2) *$M = F \oplus L$ where F is a dual Rickart module.*

Proof. (1) ⇒ (2) Let M be F-dual Rickart. Then it is clear that F is a direct summand of M. Set $M = F \oplus L$ for a submodule L of M. Suppose that g is an endomorphism of F. Then $h = j \circ g \circ \pi$ is an endomorphism of M such that j is the inclusion from F to M and π is the projection of M on F. Being M a F-dual Rickart module implies $h(F) = Img$ is a direct summand of M and hence a direct summand of F as $h(F)$ is contained in F.

(2) ⇒ (1) Let $M = F \oplus L$ such that F is dual Rickart. Suppose that φ is an endomorphism of M. Then $\lambda = \pi \circ \varphi \circ j$ will be an endomorphism of F where $j : F \to M$ is the inclusion and $\pi : M \to F$ is the projection on F. As $\lambda(F) = \varphi(F)$ and F is a dual Rickart module, then $\varphi(F)$ is a direct summand of F and consequently of M, as required. □

Example 2. *([2, Example 2.8])* (1) Let F be a field and $R = \prod_{i=1}^{\infty} F_i$ where $F_i = F$ for each $i \in \mathbb{N}$. Then R is a von Neumann regular V-ring. Take $M = R$ and K be any finitely generated ideal of R. So that K is a direct summand of M. It is well-known that M is a dual Rickart module (see [6, Remark 2.2]) and hence K as a direct summand is also dual Rickart (see [6, Proposition 2.8]). Now $M = K \oplus L$. Hence, M is a K-dual Rickart module by Theorem 1.

(2) Let L be an V-ring and K be a field. Then $S = K \times L$ is an V-ring as well. Consider the central idempotent $e = (1, 0)$ of S. Then $Se = eS \cong K$ as both left S-module and right S-module. Let R be the ring $M_n(S)$ (the ring of all $n \times n$ matrices with entries from S). As R is Morita-equivalent to S, it should be also an V-ring. Now, R has a central idempotent, $f = eI$ where I is the identity matrix of R. Then $fR = Rf$ is isomorphic to $M_n(Se)$ so that $fR = Rf \cong M_n(K)$. Note that $F = Rf$ is a two-sided ideal of R and also is a direct summand of R. Being K a field implies that $M_n(K)$ and hence F is semisimple and so is dual Rickart. It follows from Theorem 1 that R is a F-dual Rickart module.

Remark 1. *Let M be an indecomposable module and F a nonzero fully invariant submodule of M. Then M is F-dual Rickart if and only if $F = M$ is dual Rickart. In other words, if F is a nontrivial fully invariant submodule of M. Then M can not be F-dual Rickart. For instance, a local module M with $Rad(M) \neq 0$ is not a $Rad(M)$-dual Rickart module.*

Proposition 1. *Let M be a module, F a fully invariant submodule of M and N a direct summand of M. If M is F-dual Rickart, then N is $F \cap N$-dual Rickart.*

Proof. Set $M = N \oplus K$. By [2, Lemma 2.9(1)], $F \cap N$ is a fully invariant submodule of N. Consider an arbitrary endomorphism λ of N. Then $f = j \circ \lambda \circ \pi$ will be an endomorphism of M, so that $f(F) = \lambda(F \cap N)$ is a direct summand of M as M is F-dual Rickart. Note that $j : N \to M$ is the inclusion and $\pi : M \to N$ is the projection of M on N. It follows that $\lambda(F \cap N)$ is a direct summand of N, which completes the proof. □

Definition 2. *Let M be a module and F a fully invariant submodule of M. We say that M is F-dual Baer provided for every right ideal I of $End_R(M)$ the submodule $IF = \sum_{\varphi \in I} \varphi(F)$ is a direct summand of M.*

Theorem 2. *Let M be a module and F a fully invariant submodule of M. Then the following are equivalent:*

(1) *M is F-dual Baer;*

(2) *F is a dual Baer direct summand of M;*

(3) *M is F-dual Rickart and M has $SSSP$ for direct summands of M contained in F;*

(4) *For every subset B of $End_R(M)$, the submodule $\sum_{\varphi\in B}\varphi(F)$ is a direct summand of M.*

Proof. (1) $\Rightarrow$ (2) Consider S as a right ideal of S. Then by (1), $SF=\sum_{\varphi\in S}\varphi(F)=F$ is a direct summand of M. Now, let I be a right ideal of $End_R(F)$ and consider the inclusion $j:F\to M$ and the projection $\pi_F:M\to F$. Consider the subset $I_0=\{j\circ\lambda\circ\pi_F\mid\lambda\in I\}$ of S. Then $J=I_0S$ is a right ideal of S. As $IF=\sum_{\varphi\in I}\varphi(F)=\sum_{\varphi\in J}\varphi(F)=JF$ and M is a F-dual Baer module, we conclude that $IF=JF$ is a direct summand of M and consequently is a direct summand of F, as well. It follows from [5, Theorem 2.1], F is a dual Baer module.

(2) $\Rightarrow$ (1) Let I be a right ideal of S and $B=\{\pi_F\circ(\varphi\mid_F)\mid\varphi\in I\}$. Note that $J=BEnd_R(F)$ is a right ideal of $End_R(F)$. Since $JF=IF$ and F is a dual Baer module, we conclude that JF is a direct summand of F and hence a direct summand of M.

(1) $\Rightarrow$ (3) Let $\varphi\in S$. As M is F-dual Baer and $<\varphi>F=\varphi(F)$, then $\varphi(F)$ is a direct summand of M. Let $\{e_\gamma\mid\gamma\in\Gamma\}$ be a set of idempotents of S such that $Ime_\gamma\subseteq F$ for each $\gamma\in\Gamma$. Suppose $I=<\sum_{\gamma\in\Gamma}e_\gamma>$ that is a right ideal of S. Now, $IF=\sum_{\varphi\in I}\varphi(F)\subseteq\sum_{\gamma\in\Gamma}e_\gamma(M)$. As $e_\gamma(M)$ is contained in $\sum_{\varphi\in I}\varphi(F)$, it follows that $\sum_{\gamma\in\Gamma}e_\gamma(M)=\sum_{\varphi\in I}\varphi(F)=IF$ is a direct summand of M (note that M is F-dual Baer).

(3) $\Rightarrow$ (4) It follows from the fact that F is fully invariant in M.

(4) $\Rightarrow$ (1) It is obvious. □

By Theorem 2, every F-dual Baer module is F-dual Rickart. Consider any von Neumann regular ring R that is not a semisimple ring (for instance $R=\prod_{i\in\mathbb{N}}K_i$, where $K_i=K$ is a field). Then R is R-dual Rickart while R is not R-dual Baer (see [5, Corollary 2.9]).

Proposition 2. *Let M be a regular module and F a fully invariant submodule of M. If M satisfies $SSSP$ on direct summands of M contained in F, then M is F-dual Baer.*

Proof. Let φ be an arbitrary endomorphism of M. As $\varphi(F)=\sum_{x\in\varphi(F)}xR$, and M is regular, it follows that $\varphi(F)$ is a direct summand of M. □

As a consequence of Theorem 2 and Proposition 2, if M is a regular F-dual Baer module then F is a semisimple module.

In the light of Theorem 2, we have the following remark.

Remark 2. *Let M be an indecomposable module and F a nonzero fully invariant submodule of M. Then M is F-dual Baer if and only if $F=M$ is dual Baer.*

Example 3. (1) Consider $\mathbb{Z}$ as an $\mathbb{Z}$-module. If there exists a fully invariant submodule F of $\mathbb{Z}$ such that $\mathbb{Z}$ is F-dual Baer, then $F=0$ since $\mathbb{Z}$ is not dual Baer by [5, Corollary 3.5].

(2) If there exists a fully invariant submodule F of $\mathbb{Q}$ as an $\mathbb{Z}$-module such that $\mathbb{Q}$ is F-dual Baer, then $F=0$ or $F=\mathbb{Q}$.

(3) For a prime integer p, consider the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$. If there exists a fully invariant submodule F of $\mathbb{Z}_{p^\infty}$ such that $\mathbb{Z}_{p^\infty}$ is F-dual Baer, then $F = 0$ or $F = \mathbb{Z}_{p^\infty}$.

Theorem 3. *Let M be a module and F a fully invariant submodule of M. Then M is F-dual Baer if and only if for every direct summand N of M, we have N is $F \cap N$-dual Baer.*

Proof. Let M be F-dual Baer and $M = N \oplus N'$ for a submodule N' of M. Then $F = (F \cap N) \oplus (F \cap N')$ as F is a fully invariant submodule of M. Suppose that A is a subset of $End_R(N)$. Then $B = \{j \circ \varphi \circ \pi_N \mid \varphi \in A\}$ in which $\pi_N : M \to N$ is the projection of M on N and j is the inclusion from N to M, is a subset of $End_R(M)$. It is straightforward to check that $A(F \cap N) = \sum_{\varphi \in A} \varphi(F \cap N) = \sum_{g \in B} g(F)$. Being M, a F-dual Baer module implies that $A(F \cap N)$ is a direct summand of M and hence a direct summand of N. The result follows from Theorem 2. The converse is clear.

□

One can easy prove the following lemma.

Lemma 1. *Let M and M' be modules and $f : M \to M'$ an isomorphism. If M is F-dual Baer, then M' is $f(F)$-dual Baer.*

Corollary 1. *Let M be a module, P a projective module and $f : M \to P$ be an epimorphism such that $Ker\, f$ is contained in a fully invariant submodule F of M. Then, if M is F-dual Baer, then P is E-dual Baer where $E \cong \frac{F}{Ker\, f}$.*

Proof. It is clear by Theorem 3 and Lemma 1. □

Proposition 3. *Let M be a module. Then*

(1) *If M is a finitely generated $Rad(M)$-dual Baer module, then $Rad(M) = 0$.*

(2) *If M is a finitely cogenerated $Soc(M)$-dual Baer module, then M is semisimple.*

Proof. (1) Since M is finitely generated, $Rad(M)$ is small in M. By Theorem 2, $Rad(M)$ is a direct summand of M. Hence $Rad(M) = 0$.

(2) Since M is finitely cogenerated, $Soc(M)$ is essential in M and, by Theorem 2, $Soc(M)$ is a direct summand of M. Hence $Soc(M) = M$ and so M is semisimple. □

Corollary 2. *Let M be a module. Then*

(1) *If M is a Noetherian $Rad(M)$-dual Baer module, then $Rad(M) = 0$.*

(2) *If M is an Artinian $Soc(M)$-dual Baer module, then M is semisimple.*

3 Relatively F-dual Rickart modules

In this section we shall define relative F-dual Rickart modules and we will apply this concept to study finite direct sums of F-dual Rickart modules.

Definition 3. Let M and N be R-modules and F be a fully invariant submodule of M. We say M is *N-F-dual Rickart* if for every homomorphism $\phi : M \to N$, the submodule $\phi(F)$ is a direct summand of N.

It is clear that a right module M is F-dual Rickart if and only if M is M-F-dual Rickart. We provide an equivalent condition for relatively F-dual Rickart modules.

Theorem 4. *Let M and N be right R-modules and F be a fully invariant submodule of M. Then M is N-F-dual Rickart if and only if for every direct summand L of M and every submodule K of N, L is K-$F \cap L$-dual Rickart.*

Proof. Let M be N-F-dual Rickart. Suppose that $L = eM$ for some $e^2 = e \in End_R(M)$ and let K be a submodule of N. Assume that $\psi \in Hom(L, K)$. Since $\psi eM = \psi L \subseteq K \subseteq N$ and M is N-F-dual Rickart, $\psi e(F)$ is a direct summand of N. As $\psi e(F)$ is contained in K, we conclude that $\psi e(F)$ is a direct summand of K. We shall prove that $\psi(F \cap L)$ is a direct summand of K. Suppose that $M = L \oplus L'$. Being F a fully invariant submodule of M implies that $F = (F \cap L) \oplus (F \cap L')$. Then $e(F) = e(F \cap L) = F \cap L$. Now $\psi e(F) = \psi(F \cap L)$ combining with M is F-dual Rickart relative to N, we come to a conclusion that $\psi(F \cap L)$ is a direct summand of K.

The converse is clear. □

Corollary 3. *The following conditions are equivalent for a module M and a fully invariant submodule F of M:*

(1) *M is F-dual Rickart;*

(2) *For any submodule N of M, every direct summand L of M is N-$F \cap L$-dual Rickart;*

(3) *If L and N are direct summands of M, then for any $\psi \in Hom_R(L, N)$, the submodule $\psi |_L (F \cap L)$ is a direct summand of N.*

Proposition 4. *Let M be a F-dual Rickart module and F a fully invariant submodule of M. Then*

(1) *If L and K are direct summands of M with $L \subseteq F$, then $L + K$ is a direct summand of M.*

(2) *M has SSP for direct summands of M that are contained in F.*

Proof. (1) Let $K = eM$ and $L = fM$ for some $e^2 = e \in End_R(M)$ and $f^2 = f \in End_R(M)$. Since $M = fM \oplus (1 - f)M$, $L = fM \subseteq F$ and F is a fully invariant submodule of M, we have $F = fM \oplus (F \cap (1 - f)M)$. Then $((1 - e)f)(F) = (1 - e)fM$. As M is a F-dual Rickart module, $((1 - e)f)(F) = (1 - e)fM$ is a direct summand of M. Since $(1 - e)fM = (fM + eM) \cap (1 - e)M$, $M = ((fM + eM) \cap (1 - e)M) \oplus T$ for some $T \leq M$. Hence $(1 - e)M = ((fM + eM) \cap (1 - e)M) \oplus (T \cap (1 - e)M)$. So $M = eM \oplus (1 - e)M = eM + ((fM + eM) \cap (1 - e)M) \oplus (T \cap (1 - e)M) = (fM + eM) + (T \cap (1 - e)M)$. Since $(fM + eM) \cap (T \cap (1 - e)M) = 0$, $M = (eM + fM) \oplus (T \cap (1 - e)M)$. Hence $K + L$ is a direct summnd of M.

(2) It is clear by (1). □

Theorem 5. *Let M be a module and F a fully invariant submodule of M. Then M is F-dual Rickart if and only if $\sum_{\phi \in I} \phi(F)$ is a direct summand of M for every finitely generated right ideal I of $End_R(M)$.*

Proof. Assume that I is a finitely generated right ideal of $End_R(M)$ generated by $\phi_1, \ldots, \phi_n$. As M is F-dual Rickart, $\phi_i(F)$ is a direct summand of M for each $1 \leq i \leq n$. By Proposition 4, M has SSP for direct summands which are contained in F. Since $\phi_i(F) \subseteq F$, $\sum_{\phi \in I} \phi(F) = \phi_1(F) + \cdots + \phi_n(F)$ is a direct summand of M. The converse is obvious. □

4 Applications of F-dual Baer modules to rings

In this section, we provide the applications of F-dual Baer modules to rings. It is clear that I is a fully invariant submodule of the right R-module R if and only if it is an ideal of R.

Definition 4. *Let I be an ideal of a ring R. Then R is called a right I-dual Baer ring if it is I-dual Baer as a right R-module.*

A left I-dual Baer ring R is defined similarly for an ideal I of R. The property of being a I-dual Baer ring is not left-right symmetric as the following example shows.

Example 4. Let $R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$ where K is a field. Consider the ideal $I = \begin{bmatrix} K & K \\ 0 & 0 \end{bmatrix}$ of R. Note that $R = I \oplus J$ where $J = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix}$ is a right ideal of R. It is easy to see that I is dual Baer as an R-module. Hence R is right I-dual Baer by Theorem 2. Moreover, since I is essential in R as a left ideal, it can not be a direct summand of the left R-module ${}_RR$. Therefore, R is not left I-dual Baer.

It is clear that every semisimple ring R is right I-dual Baer for any ideal I of R. In the following, we present a characterization of right I-dual Baer rings using semisimple direct summands.

Theorem 6. *Let R be a ring and I an ideal of R. Then the following are equivalent:*
(1) *R is right I-dual Baer;*
(2) *$R = I \oplus K$ for some right ideal K of R and I is dual Baer as an R-module;*
(3) *$R = I \oplus K$ for some right ideal K of R and I is semisimple as an R-module.*

Proof. (1) $\Leftrightarrow$ (2) By Theorem 2.

(1) $\Rightarrow$ (3) The ring R has a decomposition $R = I \oplus K$ where K is a right ideal of R. Assume that B is a submodule of I. We claim that B is a direct summand of I. Since B has the form $\sum_{b \in B} bR$ and R is I-dual Baer, $\sum_{b \in B} bI$ is a direct summand of R. Therefore, BI is a direct summand of R. Hence $B = BI$ is a direct summand of I since $B \subseteq I$. Therefore I is semisimple.

(3) $\Rightarrow$ (1) Suppose that $R = I \oplus K$ with a right ideal K of R and I is semisimple. Since I is semisimple, I is dual Baer. Therefore, R is I-dual Baer by Theorem 2. □

Theorem 7. *The following are equivalent for a ring R:*
(1) *There exists an ideal I of R such that R is right I-dual Baer;*
(2) *For every cyclic projective R-module M, there exists a fully invariant submodule F of M such that M is F-dual Baer.*

Proof. (1) $\Rightarrow$ (2) Suppose that M is a cyclic projective R-module. Then, $M = mR \cong R/r_R(m)$ for some $m \in M$. Therefore, $r_R(m)$ is a direct summand of R. Hence, $R = r_R(m) \oplus J$ where J is a right ideal of R. Assume that g is an isomorphism from J to M. In view of Proposition 1, J is $(J \cap I)$-dual Baer. Hence M is $g(J \cap I)$-dual Baer by Lemma 1.

(2) $\Rightarrow$ (1) It is obvious. □

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Remark 3. *Let R be a ring with $J(R) \neq 0$. Then R is not $J(R)$-dual Baer. For if, suppose that R is $J(R)$-dual Rickart. Then $\sum_{\phi \in I} \phi(J(R))$ is a direct summand of R for any finitely generated right ideal I of R by Theorem 5. Since $J(R)$ is small in R, $\sum_{\phi \in I} \phi(J(R))$ is small in R. Therefore, $IJ(R) = \sum_{a \in I} aJ(R) = 0$. Set $I = R$, so $J(R) = 0$. Therefore, R can not be a $J(R)$-dual Baer module since R is not $J(R)$-dual Rickart.*

5 Direct sum of F-dual Rickart modules and direct sum of F-dual Baer modules

In this section, we study direct sums of F-dual Rickart modules and direct sums of F-dual Baer modules. The following example shows that a direct sum of F-dual Rickart modules is not F-dual Rickart, in general.

Let R be a ring, M be an R-module and let $\mathcal{S}$ denotes the class of all small right R-modules (a right R-module U is small in case U is a small submodule of a right R-module V). Recall from [9] that M is said to be *(non)cosingular* in case ($\overline{Z}(M) = M$) $\overline{Z}(M) = 0$ where $\overline{Z}(M) = \cap\{Kerf \mid f : M \to U, U \in \mathcal{S}\}$. Note that $\overline{Z}^2(M)$ is defined to be $\overline{Z}(\overline{Z}(M))$.

Example 5. *([4, Example 4.2])* Let K be a field and $R = \prod_{i=1}^{\infty} K_i$ where $K_i = K$ for each $i \in \mathbb{N}$. Then R is a von Neumann regular V-ring. Take $M_1 = R$ and $M_2 = \oplus_{i=1}^{\infty} K_i$. By [6, Example 5.1], M_1 and M_2 are dual Rickart and $M_1 \oplus M_2$ is not dual Rickart. Since R is a V-ring, by [9, Proposition 2.5], every R-module is noncosingular. So by [4, Proposition 3.4], M_i is $\overline{Z}^2(M_i)$-dual Rickart while $M_1 \oplus M_2$ is not $\overline{Z}^2(M_1 \oplus M_2)$-dual Rickart.

In the following, we show that when a direct sum of F-dual Rickart modules is also F-dual Rickart.

Proposition 5. *Let $M = \oplus_{i=1}^{n} M_i$ and N be modules and $F \trianglelefteq M$. If N has SSP for direct summands which are contained in $N \cap F$, then M is N-F-dual Rickart if and only if M_i is N-$F \cap M_i$-dual Rickart for all $1 \leq i \leq n$.*

Proof. The sufficiency is obvious from Theorem 4. For the necessity, let ϕ be a homomorphism from M to N. Then $\phi = (\phi_i)_{i=1}^{n}$ where ϕ_i is a homomorphism from M_i to N for each $1 \leq i \leq n$. By hypothesis, $\phi_i(F \cap M_i)$ is a direct summand of N for each $1 \leq i \leq n$. Since F is a fully invariant submodule of M and N has SSP for direct summands which are contained in $N \cap F$, we have

$\phi(F) = \phi(\oplus_{i=1}^{n} F \cap M_i) = \phi_1(F \cap M_1) + \phi_2(F \cap M_2) + \cdots + \phi_n(F \cap M_n) \leq^{\oplus} N$. Therefore M is N-F-dual Rickart. □

Corollary 4. *Let $M = \oplus_{i=1}^{n} M_i$ be a module and F a fully invariant submodule of M. Then M is F-dual Rickart relative to M_j $(1 \leq j \leq n)$ if and only if M_i is $F \cap M_i$-dual Rickart relative to M_j for each $1 \leq i \leq n$.*

Theorem 8. *Let $\{M_i\}_{i=1}^{n}$ and N be modules and F be a fully invariant submodule of N. Assume that for each $i \geq j$ with $1 \leq i, j \leq n$, M_i is M_j-projective. Then N is $\oplus_{i=1}^{n} M_i$-F-dual Rickart if and only if N is M_j-F-dual Rickart for all $1 \leq j \leq n$.*

Proof. The sufficiency is obvious from Theorem 4. For the necessity, suppose that N is M_j-F-dual Rickart for all $1 \leq j \leq n$. We prove by induction on n. Assume that $n = 2$ and N is F-dual Rickart relative to M_1 and M_2. Let ϕ be a homomorphism from N to $M_1 \oplus M_2$. Then $\phi = \pi_1\phi + \pi_2\phi$, where π_i is the natural projection from $M_1 \oplus M_2$ to M_i $(i = 1, 2)$. As N is M_2-F-dual Rickart, $\pi_2\phi(F)$ is a direct summand of M_2. Let $M_2 = \pi_2\phi(F) \oplus M_2'$ for some $M_2' \leq M_2$. Hence $M_1 \oplus M_2 = M_1 \oplus \pi_2\phi(F) \oplus M_2'$. As M_2 is M_1-projective, $\pi_2\phi(F)$ is M_1-projective. Since $M_1 + \phi(F) = M_1 \oplus \pi_2\phi(F)$ is a direct summand of $M_1 \oplus M_2$, there exists $T \subseteq \phi(F)$ such that $M_1 + \phi(F) = M_1 \oplus T$, by [7, Lemma 4.47]. Thus $\phi(F) = (\phi(F) \cap M_1) \oplus T$. Since N is M_1-F-dual Rickart, $\pi_1\phi(F) = M_1 \cap (M_2 + \phi(F)) = M_1 \cap \phi(F)$ is a direct summand of M_1. Therefore $\phi(F)$ is a direct summand of $M_1 \oplus T$. Since $M_1 \oplus T = M_1 \oplus \phi(F) \leq^{\oplus} M_1 \oplus M_2$, $\phi(F)$ is a direct summand of $M_1 \oplus M_2$. Thus N is F-dual Rickart relative to $M_1 \oplus M_2$. Now, assume that N is F-dual Rickart relative to $\oplus_{i=1}^{n} M_i$. We show that N is F-dual Rickart relative to $M_{n+1} \oplus (\oplus_{i=1}^{n} M_i)$. Since M_{n+1} is M_j-projective for each $1 \leq j \leq n$, M_{n+1} is $\oplus_{i=1}^{n} M_i$-projective. As N is M_{n+1}-F-dual Rickart, N is $\oplus_{i=1}^{n+1} M_i$-F-dual Rickart by a similar argument for the case n = 2. □

We mention that in the above theorem we use ideas of the proof of [6, Theorem 5.5].

Corollary 5. *Let $\{M_i\}_{i=1}^{n}$ be modules and F be a fully invariant submodule of $\oplus_{i=1}^{n} M_i$. Assume that for each $i \geq j$ with $1 \leq i, j \leq n$, M_i is M_j-projective. Then $\oplus_{i=1}^{n} M_i$ is F-dual Rickart if and only if M_i is M_j-$F \cap M_i$-dual Rickart for all $1 \leq i, j \leq n$.*

Proof. The sufficiency is obvious from Theorem 4. For the necessity, assume that M_i is M_j-$F \cap M_i$-dual Rickart for all $1 \leq j \leq n$. Now $\oplus_{i=1}^{n} M_i$ is M_j-F-dual Rickart for all $1 \leq j \leq n$ by Corollary 4. Therefore, by Theorem 8, $\oplus_{i=1}^{n} M_i$ is F-dual Rickart. □

Theorem 9. *Let $M = \oplus_{i=1}^{n} M_i$ be a module, $F \trianglelefteq M$ and $M_i \trianglelefteq M$ for all $i \in \{1, \ldots, n\}$. Then M is a F-dual Rickart module if and only if M_i is $F \cap M_i$-dual Rickart for all $i \in \{1, \ldots, n\}$.*

Proof. The necessity follows from Proposition 1. Conversely, let M_i be a $F \cap M_i$-dual Rickart module for all $i \in \{1, \ldots, n\}$. Since $F \trianglelefteq M$, $F = \oplus_{i=1}^{n}(F \cap M_i)$. Let $\phi = (\phi_{ij})_{i,j \in \{1,\ldots,n\}} \in End_R(M)$ be arbitrary, where $\phi_{ij} \in Hom(M_j, M_i)$. Since $M_i \trianglelefteq M$ for all $i \in \{1, \ldots, n\}$ and $F = \oplus_{i=1}^{n}(F \cap M_i)$, $\phi(F) = \oplus_{i=1}^{n} \phi_{ii}(F \cap M_i)$. As M_i is $F \cap M_i$-dual Rickart, $\phi_{ii}(F \cap M_i)$ is a direct summand of M_i and so $\phi(F)$ is a direct summand of M. Therefore M is a F-dual Rickart module. □

In the following we present an example which shows that direct sums of F-dual Baer modules need not be F-dual Baer.

Example 6. Let p be a prime integer. Then, $\mathbb{Z}_p$ and $\mathbb{Z}_{p^\infty}$ are dual Baer $\mathbb{Z}$-modules. Hence $\mathbb{Z}_p$ is $\mathbb{Z}_p$-dual Baer and $\mathbb{Z}_{p^\infty}$ is $\mathbb{Z}_{p^\infty}$-dual Baer. However, $\mathbb{Z}_p \oplus \mathbb{Z}_{p^\infty}$ is not a dual Baer module by [5, Corollary 3.5]. Therefore, $\mathbb{Z}_p \oplus \mathbb{Z}_{p^\infty}$ is not a $\mathbb{Z}_p \oplus \mathbb{Z}_{p^\infty}$-dual Baer $\mathbb{Z}$-module.

In the following we study some conditions that ensure us direct sums of F-dual Baer modules inherit the property.

Theorem 10. *Let $M = \oplus_{i=1}^n M_i$ be a module, $F \trianglelefteq M$ and $M_i \trianglelefteq M$ for all $i \in \{1, \ldots, n\}$. Then M is a F-dual Baer module if and only if M_i is $F \cap M_i$-dual Baer for all $i \in \{1, \ldots, n\}$.*

Proof. The necessity follows from Theorem 3. Conversely, let M_i be a $F \cap M_i$-dual Baer module for all $i \in \{1, \ldots, n\}$ and I be a subset of $End_R(M)$. Since $F \trianglelefteq M$, $F = \oplus_{i=1}^n (F \cap M_i)$. Let $\phi = (\phi_{ij})_{i,j \in \{1,\ldots,n\}} \in End_R(M)$ be arbitrary, where $\phi_{ij} \in Hom(M_j, M_i)$. Since $M_i \trianglelefteq M$ for all $i \in \{1, \ldots, n\}$ and $F = \oplus_{i=1}^n (F \cap M_i)$, we have $\phi(F) = \oplus_{i=1}^n \phi_{ii}(F \cap M_i)$. Hence $\sum_{\phi \in I} \phi(F) = \sum_{\phi \in I_i} \oplus_{i=1}^n \phi_{ii}(F \cap M_i) = \oplus_{i=1}^n \sum_{\phi \in I_i} \phi_{ii}(F \cap M_i)$ where $I_i = \{\phi|_{M_i} : \phi \in I\} \subseteq End_R(M_i)$. As M_i is $F \cap M_i$-dual Baer for all $i \in \{1, \ldots, n\}$, $\sum_{\phi \in I_i} \phi_{ii}(F \cap M_i)$ is a direct summand of M_i and so $\sum_{\phi \in I} \phi(F)$ is a direct summand of M. Therefore M is a F-dual Baer module. □

We can prove the following proposition similar to the proof of Theorem 10.

Proposition 6. *Let $\{M_i\}_{i \in \mathcal{I}}$ be a class of R-modules for an index set $\mathcal{I}$. If for every $i \in \mathcal{I}$, F_i and M_i are fully invariant submodules of $\bigoplus_{i \in \mathcal{I}} M_i$, then $\bigoplus_{i \in \mathcal{I}} M_i$ is $\bigoplus_{i \in \mathcal{I}} F_i$-dual Baer if and only if M_i is F_i-dual Baer for every $i \in \mathcal{I}$.*

We now define relatively F-dual Baer modules and then we study direct sums of F-dual Baer modules applying this definition.

Definition 5. Let M and N be R-modules and F a fully invariant submodule of M. Then, M is called *N-F-dual Baer* if for every subset I of $Hom_R(M, N)$, $\sum_{\phi \in I} \phi(F)$ is a direct summand of N.

It is clear that a module M is F-dual Baer if and only if it is M-F-dual Baer.

Theorem 11. *Let $M = M_1 \oplus M_2$ and N be R-modules and F fully invariant in M. If M is N-F-dual Baer, then for any direct summand K of N, M_i is K-$(F \cap M_i)$-dual Baer for $i = 1, 2$.*

Proof. Since F is a fully invariant submodule of M, $F = (F \cap M_1) \oplus (F \cap M_2)$. Suppose that A is a subset of $Hom_R(M_1, K)$. Then $B = \{j \circ \varphi \circ \pi_{M_1} \mid \varphi \in A\}$ in which $\pi_{M_1} : M \to M_1$ is the projection of M on M_1 and j is the inclusion from K to N, is a subset of $Hom_R(M, N)$. It is easy to check that $A(F \cap M_1) = \sum_{\varphi \in A} \varphi(F \cap M_1) = \sum_{g \in B} g(F)$. As M is a N-F-dual Baer module, $A(F \cap M_1)$ is a direct summand of N and hence a direct summand of K. □

Proposition 7. *Let $\{M_i\}_{i \in \mathcal{J}}$ be a class of R-modules for an index set $\mathcal{J}$, N an R-module and F be a fully invariant submodule of $\bigoplus_{i \in \mathcal{J}} M_i$. Then, the following hold.*

(1) *Let N have the SSP for direct summands which are contained in $N \cap F$, and $\mathcal{J}$ be finite. Then, $\bigoplus_{i\in\mathcal{J}} M_i$ is N-F-dual Baer if and only if M_i is N-$F \cap M_i$-dual Baer for all $i \in \mathcal{J}$.*

(2) *Let N have the $SSSP$ for direct summands which are contained in $N \cap F$, and $\mathcal{J}$ be arbitrary. Then, $\bigoplus_{i\in\mathcal{J}} M_i$ is N-F-dual Baer if and only if M_i is N-$F \cap M_i$-dual Baer for all $i \in \mathcal{J}$.*

Proof. (1) The sufficiency is obvious from Theorem 11. For the necessity, suppose that A is a subset of $Hom_R(\bigoplus_{i\in\mathcal{J}} M_i, N)$. Then $B_i = \{\phi j_i \mid \phi \in A\}$ in which j_i is the inclusion from M_i to $\bigoplus_{i\in\mathcal{J}} M_i$, is a subset of $Hom_R(M_i, N)$.

Assume that ϕ is a homomorphism from $\bigoplus_{i\in\mathcal{J}} M_i$ to N. Then $\phi = (\phi_i)_{i\in\mathcal{J}}$ where $\phi_i = \phi j_i$ is a homomorphism from M_i to N for each $i \in \mathcal{J}$. By hypothesis, $\sum_{\phi_i\in B_i} \phi_i(F \cap M_i)$ is a direct summand of N for each $i \in \mathcal{J}$. Since F is a fully invariant submodule of M and N has SSP for direct summands which are contained in $N \cap F$, we have

$$\textstyle\sum_{\phi\in A} \phi(F) = \sum_{\phi\in A} \phi(\oplus_{i=1}^n (F \cap M_i)) = \sum_{i\in\mathcal{J}} \sum_{\phi_i\in B_i} \phi_i(F \cap M_i) \leq^{\oplus} N.$$

Therefore $\bigoplus_{i\in\mathcal{J}} M_i$ is N-F-dual Baer.

(2) Similar to (1). □

Corollary 6. *Let $\{M_i\}_{i\in\mathcal{J}}$ be a class of R-modules for an index set $\mathcal{J}$ and F be a fully invariant submodule of $\bigoplus_{i\in\mathcal{J}} M_i$. Then, for each $j \in \mathcal{J}$, $\bigoplus_{i\in\mathcal{J}} M_i$ is M_j-F-dual Baer if and only if M_i is M_j-$F \cap M_i$-dual Baer for all $i \in \mathcal{J}$.*

Proof. It follows from Proposition 7 and Theorem 2. □

Similar to the proof of Theorem 8, one can prove the following theorem.

Theorem 12. *Let $\{M_i\}_{i=1}^n$ and N be modules and F be a fully invariant submodule of N. Assume that for each $i \geq j$ with $1 \leq i, j \leq n$, M_i is M_j-projective. Then N is $\oplus_{i=1}^n M_i$-F-dual Baer if and only if N is M_j-F-dual Baer for all $1 \leq j \leq n$.*

Corollary 7. *Let $\{M_i\}_{i=1}^n$ be modules and F be a fully invariant submodule of $\oplus_{i=1}^n M_i$. Assume that for each $i \geq j$ with $1 \leq i, j \leq n$, M_i is M_j-projective. Then $\oplus_{i=1}^n M_i$ is F-dual Baer if and only if M_i is M_j-$F \cap M_i$-dual Baer for all $1 \leq i, j \leq n$.*

Proof. The sufficiency is obvious from Theorem 11. For the necessity, assume that M_i is M_j-$F \cap M_i$-dual Rickart for all $1 \leq j \leq n$. Now $\oplus_{i=1}^n M_i$ is M_j-F-dual Rickart for all $1 \leq j \leq n$ by Corollary 6. Therefore, by Theorem 12, $\oplus_{i=1}^n M_i$ is F-dual Rickart. □

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