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Strong persistence property of generalized mixed product ideals

by
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Abstract

Let L be the generalized mixed product ideal induced by a monomial ideal I. The associated prime ideals of L are studied, including the stable set of associated prime ideals of this class of ideals. It is shown that I has the strong persistence property if and only if L has the strong persistence property.

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1 Introduction

A special class of monomial ideals, called mixed product ideals, was introduced by Restuccia and Villarreal [14]. They gave a complete classification of normal mixed product ideals, as well as applications in graph theory. In this paper we consider generalized mixed product ideals which were introduced by Herzog and Yassemi and which also include the so-called expansions of monomial ideals. The aim of this paper is to study the strong persistence property of generalized mixed product ideals.

Let K be a field and $K[x_1, \ldots, x_n]$ the polynomial ring in n variables over K with each x_i of degree 1. Let $I \subset S$ be a monomial ideal and let $G(I)$ be the set of unique monomial generators.

Let K be a field and $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be the polynomial ring over K in the variables x_i and y_j. Mixed product ideals are a special class of squarefree monomial ideals in S. They are of the form $(I_qJ_r + I_pJ_s)S$, where for integers a and b, the ideal I_a (resp. J_b) is the ideal generated by all squarefree monomials of degree a in the polynomial ring $K[x_1, \ldots, x_n]$ (resp. of degree b in the polynomial ring $K[y_1, \ldots, y_m]$), and where $0 < p < q \leq n$, $0 < r < s \leq m$. Thus the ideal $L = (I_qJ_r + I_pJ_s)S$ is obtained from the monomial ideal $I = (x^qy^r, x^py^s)$ by replacing x^q by I_q, x^p by I_p, y^r by J_r and y^s by J_s.

As mentioned above, Restuccia and Villarreal classified the normal mixed product ideals. In other words, they characterized the mixed product ideals whose Rees ring is normal. Rinaldo and Ionescu [9] studied the Castelnuovo-Mumford regularity, the depth and dimension of mixed product ideals and characterized when they are Cohen-Macaulay. Rinaldo [15] studied the Betti numbers of their finite free resolutions and Hoa and Tam [8] computed the regularity and some other algebraic invariants of mixed products of arbitrary graded ideals.

Together with Herzog and Yassemi [7] I introduced the generalized mixed product ideals, which are a far reaching generalization of the mixed product ideals introduced by Restuccia and Villarreal, and also generalizes the expansion construction by Bayati and Herzog [1].

For this new construction we choose for each i a set of new variables $x_{i1}, x_{i2}, \ldots, x_{im_i}$ and replace each of the factor $x_i^{a_i}$ in each minimal generator $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ of the monomial ideal I by a monomial ideal in $T_i = K[x_{i1}, x_{i2}, \ldots, x_{im_i}]$ generated in degree a_i. Indeed in the paper [1] a similar construction, called expansion, is made. There, however, each $x_i^{a_i}$ is replaced by $(x_{i1}, \ldots, x_{im_i})^{a_i}$, while in our generalized mixed product ideals each $x_i^{a_i}$ is replaced by an arbitrary monomial ideal of T_i generated in degree a_i. We also computed in [7] the minimal graded free resolution of generalized mixed product ideals and showed that a generalized mixed product ideal L induced by I has the same regularity as I, provided the ideals which replace the pure powers $x_i^{a_i}$ all have a linear resolution. As a consequence we obtained the result that under the above assumptions, L has a linear resolution if and only if I has a linear resolution. We also proved that the projective dimension of L can be expressed in terms of the multi-graded shifts in the resolution of I and the projective dimension of the ideals which replace the pure powers.

In [13] the author together with Tehranian computed powers of generalized mixed product ideals. We showed that L^k is again generalized mixed product ideal for all k and L^k induced by I^k, and we obtained the result that L^k has a linear resolution if and only if I^k has a linear resolution for all k, provided the ideals which replace the pure powers $x_i^{a_i}$ all have a linear resolution. Also we introduced the generalized mixed polymatroidal ideals. The class of generalized mixed polymatroidal ideals is a special class of generalized mixed product ideals which for each i we replace each factor $x_i^{a_i}$ in each minimal generator $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ of I by a polymatroidal ideal in T_i generated in degree a_i. In the paper with Tehranian we also proved that powers of a generalized mixed polymatroidal ideal is generalized mixed polymatroidal ideal and monomial localizations of a generalized mixed polymatroidal ideal at monomial prime ideals is again generalized mixed polymatroidal ideal.

The present paper is organized as follows. In Section 2 we study the associated prime ideals of generalized mixed product ideals, see Theorem 2.5. We show that the stable set of associated prime ideals of L can be compute by the stable set of associated prime ideals of I and $\operatorname{astab}(I) = \operatorname{astab}(L)$, see Corollary 2.7.

In Section 3 we study the strong persistence property of generalized mixed product ideals. In Theorem 3.3 we show that I has the strong persistence property if and only if L has the strong persistence property.

2 Associated primes

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over a field K in the variables $x_1, \ldots, x_n$ with the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$, and let $I \subset S$ be a monomial ideal with $I \neq S$ whose minimal set of generators is $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$. Here $\mathbf{x}^{\mathbf{a}} = x_1^{\mathbf{a}(1)}x_2^{\mathbf{a}(2)}\cdots x_n^{\mathbf{a}(n)}$ for $\mathbf{a} = (\mathbf{a}(1), \ldots, \mathbf{a}(n)) \in \mathbb{N}^n$. For a subset $A \subseteq S$, we define the *exponent set* of A by $E(A) := \{\mathbf{a} : \mathbf{x}^{\mathbf{a}} \in A\} \subseteq \mathbb{N}^n$.

The *Veronese ideal* of degree a is the ideal I_a of S which is generated by all the monomials in the variables $x_1, \ldots, x_n$ of degree a: $I_a = (x_1, \ldots, x_n)^a$. It is known that I_a is a normal ideal ([17, Proposition 12.3.9]).

Next we consider the polynomial ring T over K in the variables

$$x_{11}, \ldots, x_{1m_1}, x_{21}, \ldots, x_{2m_2}, \ldots, x_{n1}, \ldots, x_{nm_n}.$$

In [7] we introduced the generalized mixed product ideals. For $i = 1, \ldots, n$ and $j = 1, \ldots, m$ let $L_{i,\mathbf{a}_j(i)}$ be a monomial ideal in the variables $x_{i1}, x_{i2}, \ldots, x_{im_i}$ such that

$$L_{i,\mathbf{a}_j(i)} \subset L_{i,\mathbf{a}_k(i)} \quad \text{whenever} \quad \mathbf{a}_j(i) \geq \mathbf{a}_k(i). \tag{1}$$

Given these ideals we define for $j = 1, \ldots, m$ the monomial ideals

$$L_j = \prod_{i=1}^{n} L_{i,\mathbf{a}_j(i)} \subset T, \tag{2}$$

and set $L = \sum_{j=1}^{m} L_j$. The ideal L is called a *generalized mixed product ideal* induced by I.

Example 2.1. As mentioned in [14] mixed product ideals also appear as generalized graph ideals (called path ideals by Conca and De Negri [3]) of complete bipartite graphs. Let G a finite simple graph with vertices $x_1, \ldots, x_n$. A *path* of length t in G is sequence $x_{i_1}, \ldots, x_{i_t}$ of pairwise distinct vertices such that $\{x_{i_k}, x_{i_{k+1}}\}$ is an edge of G. Then the *path ideal* $I_t(G)$ is the ideal generated by all monomials $x_{i_1} \cdots x_{i_t}$ such that $x_{i_1}, \ldots, x_{i_t}$ is a path of length t.

Now let G be a complete n-partite graph with vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_n$ and $V_i = \{x_{i1}, \ldots, x_{im_i}\}$ for $i = 1, \ldots, n$. For this graph we have

$$I_t(G) = \sum_{0 \leq j_i \leq \min\{(t+1)/2, m_i\}, \sum_{i=1}^{n} j_i = t} I_{1j_1} I_{2j_2} \ldots I_{nj_n},$$

where the ideals I_{ij_i} are the monomial ideals generated by all squarefree monomials of degree j_i in the variables $\{x_{i1}, \ldots, x_{im_i}\}$. Thus $I_t(G)$ is induced by the ideal I of Veronese type generated by the monomials $x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n}$ with $\sum_{i=1}^{n} j_i = t$ and $0 \leq j_i \leq \min\{(t+1)/2, m_i\}$.

A generalized mixed product ideal depends not only on I but also on the family L_{ij}. Then we write $L(I; \{L_{ij}\})$ for the generalized mixed product ideal induced by a monomial ideal I. We use the notation $\mathbf{X}$ for the set $\{x_{11}, \ldots, x_{1m_1}, \ldots, x_{n1}, \ldots, x_{nm_n}\}$.

For $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_+^n, \mathbf{a} \preceq \mathbf{b}$ means that $\mathbf{a}(i) \leq \mathbf{b}(i)$ for all i. We write $\mathbf{a} \prec \mathbf{b}$ if $\mathbf{a} \preceq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$. For $\mathbf{a} = (\mathbf{a}(1), \ldots, \mathbf{a}(n)) \in \mathbb{Z}_+^n$, we define

$$GL(\mathbf{a}) = \{\mathbf{b} \in \mathbb{Z}^{m_1 + \cdots + m_n} \mid \mathbf{X}^{\mathbf{b}} \in G(L(\mathbf{x}^{\mathbf{a}}; \{L_{ij}\}))\}.$$

Suppose that $L(I; \{L_{ij}\})$ be the generalized mixed product ideal induced by I. For all $\mathbf{a} \in E(G(I))$, we write $\mathbf{X}^{GL(\mathbf{a})}$ for the set of monomials $\{\mathbf{X}^{\mathbf{b}} \mid \mathbf{b} \in GL(\mathbf{a})\}$. Then $L(I; \{L_{ij}\})$ is a monomial ideal of T generated by the monomials $\mathbf{X}^{\mathbf{b}}$, where $\mathbf{b} \in GL(\mathbf{a})$ for all $\mathbf{a} \in E(G(I))$.

We want to determine the associated prime ideals of generalized mixed product ideals. We denote the set of monomial prime ideals of S by $\mathcal{P}(S)$. A prime ideal $P \subseteq S$ is an *associated prime* of I if there exists an element $a \in S$ such that $I : (a) = P$. The set of associated primes of an ideal I in a ring S is to be denoted by $\mathrm{Ass}(S/I)$.

In the following, we observe how the generalized mixed product ideals commutes with the intersection of two monomial ideals. Given two monomials u and v, we denote by $\mathrm{lcm}(u, v)$ the least common multiple of u and v.

We define the K-algebra homomorphism $\pi^* : T \to S$ by $\pi^*(x_{ij}) = x_i$ for all i, j.

Lemma 2.2. *Let $L(I;\{L_{ij}\})=\sum_{k=1}^{r}\prod_{i=1}^{n}L_{i,\mathbf{a}_k(i)}$ and $L(J;\{L_{ij}\})=\sum_{l=1}^{s}\prod_{i=1}^{n}L_{i,\mathbf{b}_l(i)}$ be generalized mixed product ideals, respectively, induced by the monomial ideals I and J with $G(I)=\{\mathbf{x}^{\mathbf{a}_1},\dots,\mathbf{x}^{\mathbf{a}_r}\}$ and $G(J)=\{\mathbf{x}^{\mathbf{b}_1},\dots,\mathbf{x}^{\mathbf{b}_s}\}$. We assume that the ideals $L_{i,\mathbf{a}_k(i)}$ and $L_{i,\mathbf{b}_l(i)}$ are Veronese ideals of degree $\mathbf{a}_k(i)$ and $\mathbf{b}_l(i)$, respectively, in the variables $x_{i1},x_{i2},\dots,x_{im_i}$. Suppose that $G(I\cap J)=\{\mathbf{x}^{\mathbf{c}_1},\dots,\mathbf{x}^{\mathbf{c}_t}\}$. Then given $\mathbf{x}^{\mathbf{c}_j}$, there exist $\mathbf{x}^{\mathbf{a}_k}$ and $\mathbf{x}^{\mathbf{b}_l}$ such that $\mathbf{x}^{\mathbf{c}_j}=\mathrm{lcm}(\mathbf{x}^{\mathbf{a}_k},\mathbf{x}^{\mathbf{b}_l})$. We set*

$$L_{i,\mathbf{c}_j(i)}=L_{i,\max\{\mathbf{a}_k(i),\mathbf{b}_l(i)\}},$$

where the ideals $L_{i,\mathbf{c}_j(i)}$ are Veronese ideals of degree $\mathbf{c}_j(i)$ in the variables $x_{i1},x_{i2},\dots,x_{im_i}$. Furthermore, let

$$L(I\cap J;\{L_{ij}\})=\sum_{j=1}^{t}\prod_{i=1}^{n}L_{i,\mathbf{c}_j(i)}.$$

Then $L(I\cap J;\{L_{ij}\})$ is a generalized mixed product ideal, and

$$L(I\cap J;\{L_{ij}\})=L(I;\{L_{ij}\})\cap L(J;\{L_{ij}\}).$$

Proof. If $\mathrm{lcm}(\mathbf{x}^{\mathbf{a}_k},\mathbf{x}^{\mathbf{b}_l})=\mathrm{lcm}(\mathbf{x}^{\mathbf{a}_p},\mathbf{x}^{\mathbf{b}_q})$, then

$$L_{i,\max\{\mathbf{a}_k(i),\mathbf{b}_l(i)\}}=L_{i,\max\{\mathbf{a}_p(i),\mathbf{b}_q(i)\}}.$$

Thus the definition of $L_{i,\mathbf{c}_j(i)}$ is independent of the presentation of $\mathbf{x}^{\mathbf{c}_j}$ as a least common multiple of $\mathbf{x}^{\mathbf{a}_k}$ and $\mathbf{x}^{\mathbf{b}_l}$, and hence well defined. It also implies that

$$L_{i,\mathbf{c}_j(i)}\subset L_{i,\mathbf{c}_k(i)}\quad\text{whenever}\quad\mathbf{c}_j(i)\geq\mathbf{c}_k(i).$$

Thus shows that $L(I\cap J;\{L_{ij}\})$ is indeed a generalized mixed product ideal of $I\cap J$.

Next we show that $L(I;\{L_{ij}\})\cap L(J;\{L_{ij}\})=L(I\cap J;\{L_{ij}\})$. We have

$$\begin{aligned}L(I;\{L_{ij}\})\cap L(J;\{L_{ij}\}) &= (\sum_{k=1}^{r}\prod_{i=1}^{n}L_{i,\mathbf{a}_k(i)})\cap(\sum_{l=1}^{s}\prod_{i=1}^{n}L_{i,\mathbf{b}_l(i)})\\ &=\sum_{l=1}^{s}\sum_{k=1}^{r}(\prod_{i=1}^{n}L_{i,\mathbf{a}_k(i)})\cap(\prod_{i=1}^{n}L_{i,\mathbf{b}_l(i)}).\end{aligned}$$

The second equality holds because the summands are all monomial ideals. We assume that

$$L(I\cap J;\{L_{ij}\})=\sum_{j=1}^{t}\prod_{i=1}^{n}L_{i,\mathbf{c}_j(i)}.$$

If $\mathbf{x}^{\mathbf{c}_j}=\mathrm{lcm}(\mathbf{x}^{\mathbf{a}_k},\mathbf{x}^{\mathbf{b}_l})$, then $\prod_{i=1}^{n}L_{i,\mathbf{c}_j(i)}=\prod_{i=1}^{n}L_{i,\max\{\mathbf{a}_k(i),\mathbf{b}_l(i)\}}$. This shows that $L(I\cap J;\{L_{ij}\})\subset L(I;\{L_{ij}\})\cap L(J;\{L_{ij}\})$.

Conversely, take a summand $\prod_{i=1}^{n}(L_{i,\mathbf{a}_k(i)}\cap L_{i,\mathbf{b}_l(i)})$ of $L(I;\{L_{ij}\})\cap L(J;\{L_{ij}\})$. Since $I\cap J$ is generated by the elements $\mathrm{lcm}(\mathbf{x}^{\mathbf{a}_k},\mathbf{x}^{\mathbf{b}_l})$, $k=1,\dots,r$ and $l=1,\dots,s$, and since $\mathbf{x}^{\mathbf{c}_1},\dots,\mathbf{x}^{\mathbf{c}_t}$ is a minimal set of generators of $I\cap J$, there exists $\mathbf{c}_j$ such that $\mathbf{x}^{\mathbf{c}_j}\mid\mathrm{lcm}(\mathbf{x}^{\mathbf{a}_k},\mathbf{x}^{\mathbf{b}_l})$. Hence $\prod_{i=1}^{n}(L_{i,\mathbf{a}_k(i)}\cap L_{i,\mathbf{b}_l(i)})\subset\prod_{i=1}^{n}L_{i,\mathbf{c}_j(i)}$. This shows that $L(I;\{L_{ij}\})\cap L(J;\{L_{ij}\})\subset L(I\cap J;\{L_{ij}\})$. □

Lemma 2.3. *Let* $L(I;\{L_{ij}\}) = \sum_{k=1}^{r}\prod_{i=1}^{n} L_{i,\mathbf{a}_k(i)}$ *and* $L(J;\{L_{ij}\}) = \sum_{l=1}^{s}\prod_{i=1}^{n} L_{i,\mathbf{b}_l(i)}$ *be generalized mixed product ideals, respectively, induced by the monomial ideals* I *and* J *with* $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_r}\}$ *and* $G(J) = \{\mathbf{x}^{\mathbf{b}_1}, \ldots, \mathbf{x}^{\mathbf{b}_s}\}$*. We assume that the ideals* $L_{i,\mathbf{a}_k(i)}$ *and* $L_{i,\mathbf{b}_l(i)}$ *are Veronese ideals of degree* $\mathbf{a}_k(i)$ *and* $\mathbf{b}_l(i)$*, respectively, in the variables* $x_{i1}, x_{i2}, \ldots, x_{im_i}$*. Then* $L(I : J;\{L_{ij}\})$ *is a generalized mixed product ideal, and*

$$L(I : J;\{L_{ij}\}) = L(I;\{L_{ij}\}) : L(J;\{L_{ij}\}).$$

Proof. Let I and J be monomial ideals. It then follows from [5, Proposition 1.2.2] that

$$I : J = \bigcap_{\mathbf{x}^{\mathbf{b}_l}\in G(J)} I : \mathbf{x}^{\mathbf{b}_l}.$$

Then Lemma 2.2 implies that $L(I : J;\{L_{ij}\})$ is a generalized mixed product ideal induced by $I : J$.

Next we show that $L(I : J;\{L_{ij}\}) = L(I;\{L_{ij}\}) : L(J;\{L_{ij}\})$. By Lemma 2.2 it suffices to show that $L(I : v;\{L_{ij}\}) = L(I;\{L_{ij}\}) : L(v;\{L_{ij}\})$ for all monomials $v \in S$. If $v = \mathbf{x}^{\mathbf{c}} \in S$, then $L(\mathbf{x}^{\mathbf{c}};\{L_{ij}\}) = \prod_{i=1}^{n} L_{i,\mathbf{c}(i)} \subseteq T$, where the ideals $L_{i,\mathbf{c}(i)}$ are Veronese ideals of degree $\mathbf{c}(i)$ in the variables $x_{i1}, x_{i2}, \ldots, x_{im_i}$. By properties of ideal quotient, we have $I : J_1J_2 = (I : J_1) : J_2$ for all the ideals J_1, J_2 of S. Hence, we only need to show that for each variable $x_i \in S$ the equality $L(I : x_i;\{L_{ij}\}) = L(I;\{L_{ij}\}) : L_{i,1}$ holds where the ideals $L_{i,1}$ are Veronese ideals of degree one in $K[x_{i1}, x_{i2}, \ldots, x_{im_i}]$. Thus let $f \in T$. Then by [12, Lemma 2.2], one has $fx_{it} \in L(I;\{L_{ij}\})$ for all t if and only if $\pi^*(f)x_i = \pi^*(fx_{it}) \in I$. Hence $L(I : x_i;\{L_{ij}\}) = L(I;\{L_{ij}\}) : L_{i,1}$. □

In [11] we observed how the generalized mixed product ideal is related to the irredundant decomposition of a monomial ideal as an intersection of irreducible monomial ideals. To be precise, let $I \subset S$ be a monomial ideal with irredundant irreducible decomposition $I = \bigcap_{h=1}^{r} Q_h$. For each $h = 1, \ldots, r$ let $S_h \subseteq [n]$ be such that $Q_h = (x_i^{a_{ih}})_{i\in S_h}$. We set $L^* = \bigcap_{h=1}^{r}(L_{i,a_{ih}})_{i\in S_h}$. Therefore, [11, Theorem 1.2] yields $L = L^*$.

Let $P \in \mathcal{P}(S)$ be a monomial prime ideal. Since P is irreducible. We assume that $F \subseteq [n]$ be a non-empty subset of $[n]$ such that $P = (x_i)_{i\in F}$. Let $L(P;\{L_{ij}\})$ be the generalized mixed product ideal induced by P. Then [11, Theorem 1.2] implies that

$$L(P;\{L_{ij}\}) = (L_{i,1})_{i\in F}.$$

We set $P' = (L_{i,1})_{i\in F}$, where the ideals $L_{i,1} = (x_{i1}, \ldots, x_{im_i})$ for all $i \in F$. We denote the set of monomial prime ideals of T by $\mathcal{P}(T)$.

Lemma 2.4. *Let* $L(I;\{L_{ij}\}) = \sum_{j=1}^{m}\prod_{i=1}^{n} L_{i,\mathbf{a}_j(i)}$ *be the generalized mixed product ideal, induced by a monomial ideal* I *with* $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$*, where the ideals* $L_{i,\mathbf{a}_j(i)}$ *in* $K[x_{i1}, x_{i2}, \ldots, x_{im_i}]$ *are Veronese ideals of degree* $\mathbf{a}_j(i)$*. Then the monomial ideal* $L(I;\{L_{ij}\})$ *is* P'*-primary if* I *is* P*-primary.*

Proof. For a monomial ideal I with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$ let

$$L(I;\{L_{ij}\}) = \sum_{j=1}^{m}\prod_{i=1}^{n} L_{i,\mathbf{a}_j(i)}$$

be the generalized mixed product ideal induced by I. Let for two monomials $u, v \in T$, $uv \in L(I; \{L_{ij}\})$ but $u \notin L(I; \{L_{ij}\})$. Thus by [12, Lemma 2.2] we have $\pi^*(uv) = \pi^*(u)\pi^*(v) \in I$ and $\pi^*(u) \notin I$. On the other hand since I is P-primary ideal, we conclude that $\pi^*(v) \in \sqrt{I} = P$. Let P' be the generalized mixed product ideal induced by P, where the ideals $L_{i,1}$ in $K[x_{i1}, x_{i2}, \ldots, x_{im_i}]$ are Veronese ideals of degree 1. Hence [12, Lemma 2.2] and Lemma 2.3 imply that $v \in P'$ and $\sqrt{L(I; \{L_{ij}\})} = L(\sqrt{I}; \{L_{ij}\}) = P'$. Therefore $L(I; \{L_{ij}\})$ is P'-primary ideal. $\square$

Theorem 2.5. *Let L be the generalized mixed product ideal induced by a monomial ideal I with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$, where the ideals $L_{i,\mathbf{a}_j(i)}$ are Veronese ideals of degree $\mathbf{a}_j(i)$. Then*

$$\mathrm{Ass}(T/L) = \{P' : P \in \mathrm{Ass}(S/I)\}.$$

Proof. Let $L(I; \{L_{ij}\}) = \sum_{j=1}^{m} \prod_{i=1}^{n} L_{i,\mathbf{a}_j(i)}$ be the generalized mixed product ideal induced by a monomial ideal I with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$, and let $I = \bigcap_{h=1}^{r} Q_h$ be the irredundant primary decomposition of I. Hence, by applying Lemma 2.2 and 2.4 we obtain that

$$L(I; \{L_{ij}\}) = \bigcap_{h=1}^{r} L(Q_h; \{L_{ij}\})$$

is an irredundant primary decomposition of $T/L(I; \{L_{ij}\})$. Then the desired conclusion follows. $\square$

Example 2.6. Let $L = \sum_{j=1}^{2} L_{1,a_j} L_{2,b_j}$ where the ideals L_{1,a_j} in $K[x_{11}, x_{12}, \ldots, x_{1m_1}]$ and L_{2,b_j} in $K[x_{21}, x_{22}, \ldots, x_{2m_2}]$ are Veronese ideals of degree a_j and b_j, respectively. Assume further that $0 \leq a_1 < a_2 \leq m_1$ and $m_2 \geq b_1 > b_2 \geq 0$. The ideal L is a generalized mixed product ideal induced by the ideal

$$I = (x_1^{a_1} x_2^{b_1}, x_1^{a_2} x_2^{b_2}).$$

One easily checks that $I = (x_1^{a_1}) \cap (x_1^{a_2}, x_2^{b_1}) \cap (x_2^{b_2})$. This is the irredundant decomposition of I as an intersection of irreducible ideals. Then by [11, Theorem 1.2] we have

$$L = (L_{1,a_1}) \cap (L_{1,a_2}, L_{2,b_1}) \cap (L_{2,b_2}).$$

Therefore by Theorem 2.5, $\mathrm{Ass}(T/L)$ consists of all prime ideals of the form

$$(x_{11}, x_{12}, \ldots, x_{1m_1}), (x_{11}, x_{12}, \ldots, x_{1m_1}, x_{21}, x_{22}, \ldots, x_{2m_2}), (x_{21}, x_{22}, \ldots, x_{2m_2}).$$

Now we want to compute the stable set of the generalized mixed product ideals. Brodmann showed [2] that there exists an integer k_1 such that $\mathrm{Ass}(S/I^k) = \mathrm{Ass}(S/I^{k_1})$ for all $k \geq k_1$. The smallest such number is called *index of stability* of I. We denote this number by $\mathrm{astab}(I)$. The stable set $\mathrm{Ass}(S/I^k)$ is denoted by $\mathrm{Ass}^{\infty}(I)$.

Corollary 2.7. *Let L be the generalized mixed product ideal induced by a monomial ideal I with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$, where the ideals $L_{i,\mathbf{a}_j(i)}$ are Veronese ideals of degree $\mathbf{a}_j(i)$. Then*

$$\mathrm{Ass}^{\infty}(L) = \{P' : P \in \mathrm{Ass}^{\infty}(I)\}.$$

In particular $\mathrm{astab}(I) = \mathrm{astab}(L)$.

Proof. For a monomial ideal I with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$ let

$$L(I; \{L_{ij}\}) = \sum_{j=1}^{m} \prod_{i=1}^{n} L_{i,\mathbf{a}_j(i)}$$

be the generalized mixed product ideal induced by I. It follows from [13, Theorem 2.3] that $L(I; \{L_{ij}\})^k$ is a generalized mixed product ideal induced by I^k and $L(I; \{L_{ij}\})^k = L(I^k; \{L_{ij}\})$ for all $k \geq 1$. Hence

$$\operatorname{Ass}(T/L(I; \{L_{ij}\})^k) = \{P' : P \in \operatorname{Ass}(S/I^k)\}$$

by Theorem 2.5. □

3 On the strong persistence property of generalized mixed product ideals

The main goal of this section is to study the strong persistence property of generalized mixed product ideals. Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal as in Section 2 with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$, and L be defined as in (2).

We denote $I(P)$ the monomial localization of I, and by $S(P)$ the polynomial ring over K in the variables which belong to P. Recall that $I(P) \subset S(P)$ is the monomial ideal which is obtained from I as the image of the K-algebra homomorphism $\varphi : S \to S(P)$ which $\varphi(x_i) = x_i$ if $x_i \in P$, and $\varphi(x_i) = 1$, otherwise. The monomial localization $I(P)$ can also be described as the saturation $I : (\prod_{x_i \notin P} x_i)^{\infty}$.

Monomial localizations are compatible with products and intersections. In other words, if I_1 and I_2 are monomial ideals, then $(I_1 I_2)(P) = I_1(P) I_2(P)$ and $(I_1 \cap I_2)(P) = I_1(P) \cap I_2(P)$.

Now we want to study the persistence property of generalized mixed product ideals. We denote by $V(I)$ the set of prime ideals containing I and by $\mathfrak{m}_P$ the maximal ideal of the local ring $S(P)$. We say that $P \in V(I)$ is a *persistence prime ideal* of I, if whenever $P \in \operatorname{Ass}(S/I^k)$ for some exponent k, then $P \in \operatorname{Ass}(S/I^{k+1})$.

The ideal I is said to have the *persistence property* if all prime ideals $P \in \bigcup_k \operatorname{Ass}(S/I^k)$ are persistence prime ideals.

Theorem 3.1. *Let L be the generalized mixed product ideal induced by a monomial ideal I, where all $L_{i,a_{ih}} = (x_{i1}, x_{i2}, \ldots, x_{im_i})^{a_{ih}}$ for $h = 1, \ldots, r$. Then I has the persistence property if and only if L has the persistence property.*

Proof. For a monomial ideal I with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$ let

$$L(I; \{L_{ij}\}) = \sum_{j=1}^{m} \prod_{i=1}^{n} L_{i,\mathbf{a}_j(i)}$$

be the generalized mixed product ideal induced by I. Let $I = \bigcap_{h=1}^{r} Q_h$ be an irredundant irreducible decomposition of I. For each $h = 1, \ldots, r$ let $S_h \subseteq [n]$ be such that $Q_h =$

$(x_i^{a_{ih}})_{i\in S_h}$. Hence, by applying [11, Theorem 1.2] we obtain that

$$L(I;\{L_{ij}\}) = \bigcap_{h=1}^{r} (L_{i,a_{ih}})_{i\in S_h}.$$

For sufficiency let $P \in \operatorname{Ass}(S/I^k)$ for some $k > 0$. Due to Theorem 2.5, it follows that $P' \in \operatorname{Ass}(T/L(I^k;\{L_{ij}\}))$. According to [13, Theorem 2.3], this implies that $L(I;\{L_{ij}\})^k$ is a generalized mixed product ideal induced by I^k and $L(I;\{L_{ij}\})^k = L(I^k;\{L_{ij}\})$, and since $L(I;\{L_{ij}\})^k(P')$ is again a generalized mixed polymatroidal ideal induced by $I^k(P)$, see [13, Corollary 3.4]. Thus $P' \in \operatorname{Ass}(T/L(I;\{L_{ij}\})^k)$. By hypothesis, we conclude that $P' \in \operatorname{Ass}(T/L(I;\{L_{ij}\})^{k+1})$. We know that $P' \in \operatorname{Ass}(T/L(I;\{L_{ij}\})^{k+1})$ if and only if

$$(L(I;\{L_{ij}\})^{k+1}(P') : \mathfrak{m}_{P'}) \neq L(I;\{L_{ij}\})^{k+1}(P'),$$

where $\mathfrak{m}_{P'}$ is the graded maximal ideal of $T(P')$. Now we apply Lemma 2.3 and we obtain that $L(I^{k+1}(P) : \mathfrak{m}_P;\{L_{ij}\}) \neq L(I^{k+1}(P);\{L_{ij}\})$. Therefore $(I^{k+1}(P) : \mathfrak{m}_P) \neq I^{k+1}(P)$. This implies that $P \in \operatorname{Ass}(I^{k+1})$.

Necessity follows from in a similar way and the proof is complete. □

Example 3.2. Let I be the Stanley-Reisner ideal that corresponds to the natural triangulation of the projective plane. Then

$$I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6).$$

Let

$$\begin{aligned} L = (&L_{1,1}L_{2,1}L_{3,1}, L_{1,1}L_{2,1}L_{4,1}, L_{1,1}L_{3,1}L_{5,1}, L_{1,1}L_{4,1}L_{6,1}, L_{1,1}L_{5,1}L_{6,1}, L_{2,1}L_{3,1}L_{6,1},\\ &L_{2,1}L_{4,1}L_{5,1}, L_{2,1}L_{5,1}L_{6,1}, L_{3,1}L_{4,1}L_{5,1}, L_{3,1}L_{4,1}L_{6,1}),\end{aligned}$$

be the generalized mixed product ideal, induced by I in the polynomial ring

$$T = K[x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, x_{41}, x_{42}, x_{51}, x_{52}, x_{61}, x_{62}],$$

where the ideals $L_{i,1}$ are Veronese ideals of degree one in $T_i = K[x_{i1}, x_{i2}]$ for $i = 1, \ldots, 6$. According to [6, Proposition 1.1], it follows that I satisfies the persistence property. Therefore Theorem 3.1 implies that L satisfies the persistence property.

Let $I \subset S = K[x_1, \ldots, x_n]$ a graded ideal and P be a prime ideal with $I \subseteq P$. Recall that I satisfies the *strong persistence property with respect to* P if for all k and all $f \in (I^k(P) : \mathfrak{m}_P) \setminus I^k(P)$ there exists $g \in I(P)$ such that $fg \notin I^{k+1}(P)$. The ideal I is said to satisfy the *strong persistence property* if it satisfies the strong persistence property with respect to P for any prime ideal containing I.

The main result of this section is the following

Theorem 3.3. *Let L be the generalized mixed product ideal induced by a monomial ideal I, where all $L_{i,a_{ih}} = (x_{i1}, x_{i2}, \ldots, x_{im_i})^{a_{ih}}$ for $h = 1, \ldots, r$. Then I satisfies the strong persistence property if and only if L satisfies the strong persistence property.*

Proof. Let $L(I;\{L_{ij}\}) = \sum_{j=1}^{m}\prod_{i=1}^{n} L_{i,\mathbf{a}_j(i)}$ be the generalized mixed product ideal induced by a monomial ideal I with $G(I) = \{\mathbf{x}^{\mathbf{a}_1},\dots,\mathbf{x}^{\mathbf{a}_m}\}$. According to [13, Theorem 2.3] we have $L(I;\{L_{ij}\})^k$ is a generalized mixed product ideal induced by I^k and $L(I;\{L_{ij}\})^k = L(I^k;\{L_{ij}\})$ for all $k \geq 1$. Let $I = \bigcap_{h=1}^{r} Q_h$ be the irredundant irreducible decomposition of I. For each $h = 1,\dots,r$ let $S_h \subseteq [n]$ be such that $Q_h = (x_i^{a_{ih}})_{i\in S_h}$. Therefore, [11, Theorem 1.2] yields $L(I;\{L_{ij}\}) = \bigcap_{h=1}^{r}(L_{i,a_{ih}})_{i\in S_h}$.

Let I satisfies the strong persistence property. Thus by [4, Theorem 1.3], [13, Theorem 2.3] and Lemma 2.3 we have

$$\begin{aligned}
(L(I;\{L_{ij}\})^{k+1} : L(I;\{L_{ij}\})) &= (L(I^{k+1};\{L_{ij}\}) : L(I;\{L_{ij}\}))\\
&= L(I^{k+1} : I;\{L_{ij}\})\\
&= L(I^k;\{L_{ij}\})\\
&= L(I;\{L_{ij}\})^k
\end{aligned}$$

for all $k \geq 1$. Therefore [4, Theorem 1.3] implies that $L(I;\{L_{ij}\})$ satisfies the strong persistence property.

Conversely, assume that $L(I;\{L_{ij}\})$ satisfies the strong persistence property, but $(I^{k+1} : I) \neq I^k$ for some $k \geq 1$. Then by Lemma 2.3 and [13, Theorem 2.3] we have

$$(L(I;\{L_{ij}\})^{k+1} : L(I;\{L_{ij}\})) \neq L(I;\{L_{ij}\})^k$$

for some $k \geq 1$, contradicting the fact that $L(I;\{L_{ij}\})$ satisfies the strong persistence property. □

Example 3.4. Let

$$I = (x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_2x_3, x_2x_4, x_2x_5, x_2x_6, x_3x_5, x_3x_6, x_4x_5, x_4x_6)$$

be a matroidal ideal in the polynomial ring $S = K[x_1, x_2, x_3, x_4, x_5, x_6]$. Let

$$\begin{aligned}
L \;=\; &(L_{1,1}L_{3,1}, L_{1,1}L_{4,1}, L_{1,1}L_{5,1}, L_{1,1}L_{6,1}, L_{2,1}L_{3,1}, L_{2,1}L_{4,1}, L_{2,1}L_{5,1},\\
&L_{2,1}L_{6,1}, L_{3,1}L_{5,1}, L_{3,1}L_{6,1}, L_{4,1}L_{5,1}, L_{4,1}L_{6,1})
\end{aligned}$$

be the generalized mixed product ideal, induced by I in the polynomial ring

$$T = K[x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, x_{41}, x_{42}, x_{51}, x_{52}, x_{61}, x_{62}],$$

where the ideals $L_{i,1}$ are Veronese ideals of degree one in $T_i = K[x_{i1}, x_{i2}]$ for $i = 1,\dots,6$. Therefore Theorem 3.3 and [4, Proposition 2.4] imply that L satisfies the strong persistence property.

As a first application of Theorem 3.3, we study the strong persistence property of generalized mixed product ideals induced by a monomial ideal in $K[x_1, x_2]$.

Theorem 3.5. *Let $L = \sum_{j=1}^{m} L_{1,a_j}L_{2,b_j}$ be the generalized mixed product ideal induced by a monomial ideal $I = (x_1^{a_1}x_2^{b_1},\dots,x_1^{a_m}x_2^{b_m})$, where the ideals L_{1,a_j} in $K[x_{11}, x_{12},\dots,x_{1m_1}]$ and the ideals L_{2,b_j} in $K[x_{21}, x_{22},\dots,x_{2m_2}]$ are Veronese ideals of degree a_j and b_j, respectively. Assume that $0 \leq a_1 < \cdots < a_m \leq m_1$ and $m_2 \geq b_1 > \cdots > b_m \geq 0$. Furthermore, let I has a linear resolution. Then L satisfies the strong persistence property.*

Proof. The ideal L is a generalized mixed product ideal induced by the ideal

$$I = (x_1^{a_1}x_2^{b_1}, \dots, x_1^{a_m}x_2^{b_m}).$$

Suppose that I has a linear resolution. We denote by $\gcd(I)$ the greatest common divisor of the generator of I. Hence I is principal ideal or $I = uJ$, where $u = \gcd(I)$. If $I = uJ$, then J has a linear resolution and $\mathfrak{m}$-primary monomial ideal. Thus if I is principal or $I = uJ$, then I is polymatroidal. Therefore Theorem 3.3 and [4, Proposition 2.4] imply that L satisfies the strong persistence property. □

Next we study the strong persistence property of the generalized mixed product ideals induced by a Veronese ideal.

Theorem 3.6. *Let L be the generalized mixed product ideal induced by a Veronese I with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_m}\}$, where the ideals $L_{i,\mathbf{a}_j(i)}$ are Veronese ideals of degree $\mathbf{a}_j(i)$. Then L satisfies the strong persistence property.*

Proof. For a monomial ideal I with $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_m}\}$ let

$$L(I; \{L_{ij}\}) = \sum_{j=1}^{m} \prod_{i=1}^{n} L_{i,\mathbf{a}_j(i)}$$

be the generalized mixed product ideal induced by I. Hence, by applying [13, Theorem 2.3] we obtain that $L(I; \{L_{ij}\})^k$ is a generalized mixed product ideal induced by I^k and $L(I; \{L_{ij}\})^k = L(I^k; \{L_{ij}\})$ for all $k \geq 1$. Let I be the Veronese ideal of degree k. Then $I = (x_1, \dots, x_n)^k$, which is generated by all the monomials in the variables $x_1, \dots, x_n$ of degree k. Hence by [13, Theorem 2.3] we have

$$\begin{aligned} L(I; \{L_{ij}\}) = L((x_1, \dots, x_n)^k; \{L_{ij}\}) &= L((x_1, \dots, x_n); \{L_{ij}\})^k \\ &= (x_{11}, \dots, x_{1m_1}, \dots, x_{n1}, \dots, x_{nm_n})^k. \end{aligned}$$

This implies that $L(I; \{L_{ij}\})$ is a Veronese ideal of T of degree k. Therefore $L(I; \{L_{ij}\})$ satisfies the strong persistence property, by [4, Proposition 2.4]. □

As an application, we consider ideals arising from graph theory. Let G be a finite simple graph with vertex set $V(G) = [n]$ and edge set $E(G)$, and let $I(G)$ be its edge ideal in $S = K[x_1, \dots, x_n]$. We fix a vertex j of G. Then a new graph G' is defined by duplicating j, that is, $V(G') = V(G) \cup \{j'\}$ and

$$E(G') = E(G) \cup \{\{i, j'\} : \{i, j\} \in E(G)\}$$

where j' is new vertex. It follows that $I(G') = I(G) + (x_i x_{j'} : \{i, j\} \in E(G))$. This duplication can be iterated. We denote by $G^{(m_1, \dots, m_n)}$ the graph which is obtained from G by m_j duplications of j. Then edge ideal of $G^{(m_1, \dots, m_n)}$ can be described as follows: Let $S^{(m_1, \dots, m_n)}$ be the polynomial ring over K in the variables

$$x_{11}, \dots, x_{1m_1}, x_{21}, \dots, x_{2m_2}, \dots, x_{n1}, \dots, x_{nm_n},$$

and consider the monomial prime ideal $P_j = (x_{j1}, \ldots, x_{jm_j})$ in $S^{(m_1,\ldots,m_n)}$. Then

$$I(G^{(m_1,\ldots,m_n)}) = \sum_{\{v_i,v_j\}\in E(G)} P_iP_j.$$

We say the ideal $I(G^{(m_1,\ldots,m_n)})$ is obtained from $I(G)$ by expansion with respect to the n-tuple $(m_1, \ldots, m_n)$ with positive integer entries.

In the paper [1], Bayati et al introduced the expansion functor in the category of finitely generated multigraded S-modules and studied some homological behaviors of this functor.

Fix an order n-tuple $(m_1, \ldots, m_n)$ of positive integers. Whenever $I \subset S$ is a monomial ideal minimally generated by $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}$, the *expansion* of I with respect to the n-tuple $(m_1, \ldots, m_n)$, is defined by $I^{(m_1,\ldots,m_n)} = \sum_{j=1}^{m} \prod_{i=1}^{n} P_i^{\mathbf{a}_j(i)} \subset S^{(m_1,\ldots,m_n)}$ where P_i is the monomial prime ideal $(x_{i1}, \ldots, x_{im_i}) \subseteq S^{(m_1,\ldots,m_n)}$ and $\mathbf{a}_j(i)$ is the i-th component of the vector $\mathbf{a}_j$.

Lemma 3.7. *Let G be a graph on $[n]$. Then $I(G^{(m_1,\ldots,m_n)}) = I(G)^{(m_1,\ldots,m_n)}$.*

Proof. Let G be a simple graph on the vertex set $[n]$. Fix an order n-tuple $(m_1, \ldots, m_n)$ of positive integers. Then

$$I(G^{(m_1,\ldots,m_n)}) = \sum_{\{v_i,v_j\}\in E(G)} P_iP_j,$$

where $P_j = (x_{j1}, \ldots, x_{jm_j})$. According to [1, Lemma 1.1] we have

$$\begin{aligned} I(G^{(m_1,\ldots,m_n)}) &= \sum_{\{v_i,v_j\}\in E(G)} x_i^{(m_1,\ldots,m_n)} x_j^{(m_1,\ldots,m_n)} \\ &= \Big(\sum_{\{v_i,v_j\}\in E(G)} x_ix_j\Big)^{(m_1,\ldots,m_n)} = I(G)^{(m_1,\ldots,m_n)}, \end{aligned}$$

as desired. □

In the following, we study the strong persistence property of $I(G^{(m_1,\ldots,m_n)})$.

Corollary 3.8. *Let G be a simple graph on the vertex set $[n]$. Fix an order n-tuple $(m_1, \ldots, m_n)$ of positive integers. Then $I(G^{(m_1,\ldots,m_n)})$ satisfies the strong persistence property.*

Proof. Suppose that G is a graph with vertex set $V(G) = [n]$. Then by [4, Theorem 1.3] together with [10, Lemma 2.12] now yields $I(G)$ satisfies the strong persistence property. Then by Lemma 3.7, [13, Theorem 2.3] and Lemma 2.3 we have

$$\begin{aligned} (I(G^{(m_1,\ldots,m_n)})^{k+1} : I(G^{(m_1,\ldots,m_n)})) &= ((I(G)^{k+1})^{(m_1,\ldots,m_n)} : I(G)^{(m_1,\ldots,m_n)}) \\ &= (I(G)^{k+1} : I(G))^{(m_1,\ldots,m_n)} \\ &= (I(G)^k)^{(m_1,\ldots,m_n)} \\ &= I(G^{(m_1,\ldots,m_n)})^k \end{aligned}$$

for all $k \geq 1$. It then follows from [4, Theorem 1.3] that $I(G^{(m_1,\ldots,m_n)})$ satisfies the strong persistence property. □

The set of associated primes of all powers of $I(G^{(m_1,\dots,m_n)})$ can be described as follows:

Theorem 3.9. *Let G be a simple graph on the vertex set $[n]$. Fix an order n-tuple $(m_1,\dots,m_n)$ of positive integers. Then*

$$\mathrm{Ass}(S^{(m_1,\dots,m_n)}/I(G^{(m_1,\dots,m_n)}) = \{P' : P \in \mathrm{Ass}(S/I(G))\}.$$

Proof. Let G be a graph on $[n]$. Then [5, Corollary 1.3.6] implies that $I(G) = \bigcap_{P\in\mathrm{Ass}(I(G))} P$ is an irredundant primary decomposition of $I(G)$. Thus Lemma 2.2, Lemma 2.4 together with Lemma 3.7 yield

$$I(G^{(m_1,\dots,m_n)}) = \bigcap_{P\in\mathrm{Ass}(I(G))} P'$$

is an irredundant primary decomposition of $S^{(m_1,\dots,m_n)}/I(G^{(m_1,\dots,m_n)})$, as desired. □

Corollary 3.10. *Let G be a simple graph on the vertex set $[n]$. Fix an order n-tuple $(m_1,\dots,m_n)$ of positive integers. Then*

$$\mathrm{Ass}^\infty(I(G^{(m_1,\dots,m_n)})) = \{P' : P \in \mathrm{Ass}^\infty(I(G))\}.$$

In particular $\mathrm{astab}(I(G)) = \mathrm{astab}(I(G^{(m_1,\dots,m_n)}))$.

Proof. Lemma 3.7 with [13, Theorem 2.3] guarantees that $I(G^{(m_1,\dots,m_n)})^k$ is a generalized mixed product ideal induced by $I(G)^k$ for all $k \geq 1$. Therefore, Theorem 3.9 yields

$$\mathrm{Ass}(S^{(m_1,\dots,m_n)}/I(G^{(m_1,\dots,m_n)})^k) = \{P' : P \in \mathrm{Ass}(S/I(G)^k)\}.$$

Thus the desired conclusion follows. □

An ideal $I \subset S$ is called *normally torsion-free* if $\mathrm{Ass}(S/I^k) \subseteq \mathrm{Ass}(S/I)$ for all $k \in \mathbb{N}$. In the following, we study normally torsion-freeness of $I(G^{(m_1,\dots,m_n)})$.

Corollary 3.11. *Let G be a graph on $[n]$. Fix an order n-tuple $(m_1,\dots,m_n)$ of positive integers. Then $I(G^{(m_1,\dots,m_n)})$ is normally torsion-free if and only if $I(G)$ is.*

Proof. Let G be a simple graph on the vertex set $[n]$. Fix an order n-tuple $(m_1,\dots,m_n)$ of positive integers. Lemma 3.7 together with [13, Theorem 2.3] now yields $I(G^{(m_1,\dots,m_n)})^k$ is a generalized mixed product ideal induced by $I(G)^k$ for all $k \geq 1$. Let $P \in \mathrm{Ass}(S/I(G)^k)$ for an arbitrary $k \in \mathbb{N}$. Let P' be the generalized mixed product ideal induced by P, where the ideals $L_{i,1}$ in $K[x_{i1}, x_{i2}, \dots, x_{im_i}]$ are Veronese ideals of degree 1. Then Theorem 3.9, implies that $P' \in \mathrm{Ass}(S^{(m_1,\dots,m_n)}/I(G^{(m_1,\dots,m_n)})^k)$. By hypothesis, we conclude that $P' \in \mathrm{Ass}(S^{(m_1,\dots,m_n)}/I(G^{(m_1,\dots,m_n)}))$ and by Theorem 3.9 we have $P \in \mathrm{Ass}(S/I(G))$. Therefore $I(G)$ is a normally torsion-free of S.

Necessity follows in a similar way and the proof is complete. □

Example 3.12. Let G be a bipartite graph on the vertex set $V(G) = [n]$. Corollary 3.10 together with [16, Theorem 5.9] now yields

$$\mathrm{Ass}(S^{(m_1,\dots,m_n)}/I(G^{(m_1,\dots,m_n)})^k) = \mathrm{Ass}(S^{(m_1,\dots,m_n)}/I(G^{(m_1,\dots,m_n)}))$$

for all $k \geq 1$. Then $\mathrm{astab}(I(G^{(m_1,\dots,m_n)})) = 1$, hence

$$\mathrm{Ass}^\infty(S^{(m_1,\dots,m_n)}/I(G^{(m_1,\dots,m_n)})) = \mathrm{Ass}(S/I(G^{(m_1,\dots,m_n)})).$$

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