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Discretization of Springer fibres

by
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Abstract

Consider a nilpotent element e of a simple complex Lie algebra. The Springer fibre corresponding to e admits a discretization (discrete analogue) introduced by the author in 1999. In this paper we propose a conjectural description of that discretization which is more amenable to computation. We also propose a conjectural PBW basis of that discretization.

Key Words: Springer fibre, K-group, PBW basis.
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0 Introduction

0.1. Let G be an almost simple simply connected algebraic group over $\mathbf{C}$ with Lie algebra $\mathfrak{g}$. Let $e \in \mathfrak{g}$ be a fixed nilpotent element and let $\mathcal{B}_e$ be the variety of Borel subalgebras of $\mathfrak{g}$ that contain e (a Springer fibre). We fix a homomorphism of algebraic groups $\zeta : SL_2(\mathbf{C}) \to G$ whose differential carries $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ to e. Let F be the centralizer in G of the image of ζ (a reductive group). Let $\bar{F} = F/(F^0 \mathcal{Z}_G)$. (For any algebraic group $\mathcal{G}$ we denote by $\mathcal{G}^0$ the identity component of $\mathcal{G}$; $\mathcal{Z}_G$ is the centre of G.) Following [4] we view $\mathcal{B}_e$ as a variety with $\mathbf{C}^*$-action given by $\lambda : \mathfrak{b} \mapsto \mathrm{Ad}(\zeta\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right))\mathfrak{b}$.

Let W be the (extended) affine Weyl group corresponding to the dual of G. Let c be the two-sided cell of W associated to the G–conjugacy class of $u = \exp(e) \in G$ in [6, 4.8]. In this paper we consider the following four sets associated to e.

(a) The subset $\underline{\mathbf{B}}^{\pm}_{\mathcal{B}_e}$ of $K_{\mathbf{C}^*}(\mathcal{B}_e)$ (the K-group of $\mathbf{C}^*$-equivariant coherent sheaves on $\mathcal{B}_e$) introduced in [8, 5.15].

(b) The set $R(c)$ of right cells of W that are contained in c.

(c) The set Ξ_e of connected components of the fixed point set $\mathcal{B}_e^{\mathbf{C}^*}$ of the $\mathbf{C}^*$-action on $\mathcal{B}_e$.

(d) The set $\bar{\Xi}_e$ of orbits of the $\bar{F}$-action on Ξ_e induced by the conjugation action of F on $\mathcal{B}_e^{\mathbf{C}^*}$.

In the rest of this paper $\underline{\mathbf{B}}^{\pm}_{\mathcal{B}_e}$ is renamed as $\mathbf{B}^{\pm}_e$. Note that in [8] it is conjectured (and in [1] it is proved) that

(e) $\mathbf{B}^{\pm}_e$ is a signed basis of the $K_{\mathbf{C}^*}(point)$-module $K_{\mathbf{C}^*}(\mathcal{B}_e)$.

One of the themes of this paper is a conjectural diagram involving the sets (a)-(d).

$$\begin{array}{ccc} \mathbf{B}_e & \xrightarrow{\rho} & R(c) \\ {\scriptstyle\sigma}\downarrow & & {\scriptstyle\sigma'}\downarrow \\ \Xi_e & \xrightarrow{\rho'} & \bar{\Xi}_e \end{array}$$

Here $\mathbf{B}_e$ is the set of orbits of multiplication by $\{1, -1\}$ on $\mathbf{B}_e^{\pm}$;

(f) ρ is the (conjectural) map in [8, 17.1(c)] which identifies $R(c)$ with the set of $\bar{F}$-orbits on $\mathbf{B}_e$ (for the action of $\bar{F}$ on $\mathbf{B}_e$ induced by the conjugation action of F on $\mathcal{B}_e$); σ is a (conjectural) surjective map (compatible with the actions of $\bar{F}$) discussed in Section 1; ρ' is the obvious orbit map; σ' is the unique (surjective) map which makes the diagram commutative.

In this paper we introduce a new (conjectural) signed basis $\tilde{\mathbf{B}}_e^{\pm}$ of (a localization of) $K_{\mathbf{C}^*}(\mathcal{B}_e)$ which is in natural bijection with $\mathbf{B}_e^{\pm}$ and is such that $\mathbf{B}_e^{\pm}$ can be reconstructed from the knowledge of $\tilde{\mathbf{B}}_e^{\pm}$ and from the bar-involution of $K_{\mathbf{C}^*}(\mathcal{B}_e)$ in a way similar (but more intricate) to the way the canonical basis of the + part of a quantum group can be reconstructed from a PBW basis of that + part. Thus we can think of $\tilde{\mathbf{B}}_e^{\pm}$ as being something like a PBW (signed) basis. The set $\tilde{\mathbf{B}}_e^{\pm}$ is naturally partitioned into subsets indexed by Ξ_e in (c); this can be viewed as a surjective map $\mathbf{B}_e^{\pm} \to \Xi_e$ which factors through a surjective map $\mathbf{B}_e \xrightarrow{\sigma} \Xi_e$ appearing in the diagram above.

0.2. The set $\mathbf{B}_e$ is a *discretization* (or discrete analogue) of $\mathcal{B}_e$ in the sense that it is a finite set with a number of elements equal to the sum of Betti numbers (or equivalently the sum of Betti numbers in even degrees) of $\mathcal{B}_e$. (This follows from 0.1(e).)

0.3. The set $\mathbf{B}_e$ indexes the simple objects in a certain block of unrestricted representations of the analogue of $\mathfrak{g}$ over a field of positive, large characteristic (this has been conjectured in [7, §14] and proved in [1]).

0.4. In section 2 we state some conjectures which, if true, would describe completely the finite set $\mathbf{B}_e$ with action of $\bar{F}$ (that is, they describes which isotropy groups appear and how many points have isotropy groups in a fixed conjugacy class).

1 The maps $\mathbf{B}_e \to \Xi_e$, $R(c) \to \bar{\bar{\Xi}}_e$

1.1. Let $\mathcal{B}$ be the variety of Borel subalgebras of $\mathfrak{g}$. We have $\mathcal{B}_e = \{\mathfrak{b} \in \mathcal{B}; e \in \mathfrak{b}\}$. As in 0.1 we consider $K_{\mathbf{C}^*}(\mathcal{B}_e)$, the K-theory of $\mathbf{C}^*$-equivariant coherent sheaves on $\mathcal{B}_e$; we denote it by K_e. We regard K_e as a module over $\mathcal{A} := \mathbf{Z}[v, v^{-1}]$ (the representation ring of $\mathbf{C}^*$) in the usual way. Here v is an indeterminate representing the identity homomorphism $\mathbf{C}^* \to \mathbf{C}^*$.

In [8, 5.15] we have defined an involution $\tilde{\beta} : K_e \to K_e$, a symmetric $\mathcal{A}$-bilinear pairing $(||) : K_e \times K_e \to \mathcal{A}$ and the subset

$$\underline{\mathbf{B}}_{\mathcal{B}_e}^{\pm} = \{\xi \in K_e; \tilde{\beta}(\xi) = \xi, (\xi||\xi) \in 1 + v^{-1}\mathbf{Z}[v^{-1}]\}$$

of $K_e - \{0\}$ (now denoted by $\mathbf{B}_e^{\pm}$). We will also write $^-$ instead of $\tilde{\beta}$.

1.2. Let $'\mathcal{A}$ be the subring of $\mathbf{Q}(v)$ consisting of quotients f/g where $g \in \mathbf{Z}[v]$ has constant term 1 and $f \in \mathcal{A}$; let $''\mathcal{A}$ be the subring of $\mathbf{Q}(v)$ consisting of quotients f/g where $g \in \mathbf{Z}[v]$ has constant term 1 and $f \in \mathbf{Z}[v]$. We have $\mathcal{A} \subset {'\mathcal{A}}, {''\mathcal{A}} \subset {'\mathcal{A}}$.

For any $m \in \mathbf{Z}$ let

$$\mathfrak{g}_m = \{x \in \mathfrak{g}; \mathrm{Ad}(\zeta \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right))x = \lambda^m x \quad \forall \lambda \in \mathbf{C}^*\}.$$

Then $\mathfrak{p} := \sum_{m\in\mathbf{N}} \mathfrak{g}_m$ is the Lie algebra of a parabolic subgroup P of G containing F. Let $\mathcal{M}$ be the (finite) set of orbits of P on $\mathcal{B}$ (for the conjugation action). Let $\mathcal{M}_e = \{\omega \in \mathcal{M}; \mathcal{B}_e \cap \omega \neq \emptyset\}$. Let $\tilde{\mathcal{M}}_e$ be the set of all subvarieties $X \subset \mathcal{B}_e$ such that X is a connected component of $\mathcal{B}_e \cap \omega$ for some $\omega \in \mathcal{M}_e$.

If $X \in \tilde{\mathcal{M}}_e$ is a connected component of $\mathcal{B}_e \cap \omega$ with $\omega \in \mathcal{M}_e$, we set $\mathcal{B}_e^{<X} = \cup_{\omega'}(\mathcal{B}_e \cap \omega')$ where $\omega' \in \mathcal{M}_e$ is subject to $\omega' \subset \bar{\omega}$ (closure in $\mathcal{B}$) and $\omega' \neq \omega$; we set $\mathcal{B}_e^{\le X} = X \cup \mathcal{B}_e^{<X}$.

By arguments in [8, Section 1] (based on results in [3]) we see that the $\mathcal{A}$-linear maps $K_{\mathbf{C}^*}(\mathcal{B}_e^{<X}) \to K_e$, $K_{\mathbf{C}^*}(\mathcal{B}_e^{\le X}) \to K_e$ induced by the closed imbedding $\mathcal{B}_e^{<X} \subset \mathcal{B}_e$, $\mathcal{B}_e^{\le X} \subset \mathcal{B}_e$, are injective; hence the $'\mathcal{A}$-linear maps $'\mathcal{A} \otimes_{\mathcal{A}} K_{\mathbf{C}^*}(\mathcal{B}_e^{<X}) \to {}'\mathcal{A} \otimes_{\mathcal{A}} K_e$, $'\mathcal{A} \otimes_{\mathcal{A}} K_{\mathbf{C}^*}(\mathcal{B}_e^{\le X}) \to {}'\mathcal{A} \otimes_{\mathcal{A}} K_e$ obtained by extension of scalars are injective. Hence $'\mathcal{A} \otimes_{\mathcal{A}} K_{\mathbf{C}^*}(\mathcal{B}_e^{<X})$ and $'\mathcal{A} \otimes_{\mathcal{A}} K_{\mathbf{C}^*}(\mathcal{B}_e^{\le X})$ can be identified with their image $'K_e^{<X}$ and $'K_e^{\le X}$ in $'K_e := {}'\mathcal{A} \otimes_{\mathcal{A}} K_e$.) The same arguments show that we have an exact sequence

$$0 \to K_{\mathbf{C}^*}(\mathcal{B}_e^{<X}) \to K_{\mathbf{C}^*}(\mathcal{B}_e^{\le X}) \to K_{\mathbf{C}^*}(X) \to 0$$

associated to the inclusions $\mathcal{B}_e^{<X} \subset \mathcal{B}_e^{\le X}$, $X \subset \mathcal{B}_e^{\le X}$; from this we deduce an exact sequence $0 \to {}'K_e^{<X} \to {}'K_e^{\le X} \xrightarrow{t} {}'K_{\mathbf{C}^*}(X)$ where $'K_{\mathbf{C}^*}(X) = {}'\mathcal{A} \otimes_{\mathcal{A}} K_{\mathbf{C}^*}(X)$.

We have naturally $K_e \subset {}'K_e$.

Now $(||) : K_e \times K_e \to \mathcal{A}$ extends to a symmetric $'\mathcal{A}$-bilinear pairing $'K_e \times {}'K_e \to {}'\mathcal{A}$. For any $X \in \tilde{\mathcal{M}}_e$ let $'K_e^X$ be the set of all $\xi \in {}'K_e^{\le X}$ such that $(\xi||\xi') = 0$ for any $\xi' \in {}'K_e^{<X}$. Restricting $t : {}'K_e^{\le X} \xrightarrow{t} {}'K_{\mathbf{C}^*}(X)$ to $'K_e^X$ we obtain a map

(a) $'K_e^X \to {}'K_{\mathbf{C}^*}(X)$.

Let $''K_e$ be the $''\mathcal{A}$-submodule of $'K_e$ generated by $\mathbf{B}_e^{\pm}$.

1.3. We now state some conjectural properties of the submodules $'K_e^X$ of $'K_e$.

(i) We have a direct sum decomposition $'K_e = \oplus_{X\in\tilde{\mathcal{M}}_e} {}'K_e^X$. Hence for $b \in \mathbf{B}_e^{\pm}$ we can write uniquely $b = \sum_{X\in\tilde{\mathcal{M}}_e} b^X$ where $b^X \in {}'K_e^X$. Moreover, the maps 1.2(a) are isomorphisms, hence they convert the direct sum decomposition above into $'K_e = \oplus_{X\in\tilde{\mathcal{M}}_e} {}'K_{\mathbf{C}^*}(X)$.

(ii) Let $b \in \mathbf{B}_e^{\pm}$. There is a unique $X_b \in \mathcal{M}_e$ such that $b^X \in v(''K_e)$ for all $X \in \tilde{\mathcal{M}}_e - \{X_b\}$ and $b^{X_b} - b \in v(''K_e)$ (so that $b^{X_b} \notin v(''K_e)$). The map $\mathbf{B}_e^{\pm} \to \tilde{\mathcal{M}}_e$, $b \mapsto X_b$ is surjective.

In this and the next subsection (but not in other subsections) we identify $\mathbf{B}_e$ with a subset of $\mathbf{B}_e^{\pm}$ by choosing one element in each orbit of multiplication by $\{1,-1\}$ on $\mathbf{B}_e^{\pm}$. Setting $\tilde{b} = b^{X_b}$ for any $b \in \mathbf{B}_e$ we have $\tilde{b} = \sum_{b'\in\mathbf{B}_e} c_{b,b'} b'$ where $c_{b,b'} \in {}''\mathcal{A}$ satisfy $c_{b,b} \in 1 + v(''\mathcal{A})$, $c_{b,b'} \in v(''\mathcal{A})$ for $b \neq b'$. It follows that the square matrix $(c_{b,b'})$ indexed by $\mathbf{B}_e \times \mathbf{B}_e$ has determinant in $1 + v(''\mathcal{A})$ hence is invertible in $''\mathcal{A}$. Since $\{b; b \in \mathbf{B}_e\}$ is an $''\mathcal{A}$-basis of $''K_e$, it follows that

(a) $\tilde{\mathbf{B}}_e := \{\tilde{b}; b \in \mathbf{B}_e\}$ is again an $''\mathcal{A}$-basis of $''K_e$.

1.4. We show that the $\mathcal{A}$-basis $\mathbf{B}_e$ can be reconstructed from the $'\mathcal{A}$-basis $\tilde{\mathbf{B}}_e$ of $'K_e$ (assuming 1.3(i),(ii)).

We shall indicate a number of steps which start with $\tilde{\mathbf{B}}_e$ and end with $\mathbf{B}_e$ (the definition of these steps does not involve $\mathbf{B}_e$, but the verification of their correctness does).

Step 1. We note that $''K_e$ is defined purely in terms of $\tilde{\mathbf{B}}_e$ (it is the $''\mathcal{A}$-submodule of $'K_e$ generated by $\tilde{\mathbf{B}}_e$).

Step 2. We set ${}^{+}K_e = K_e \cap {}''K_e$.

Step 3. Let ${}^{-}K_e$ be the image of ${}^{+}K_e$ under^{-}: $K_e \to K_e$.

Step 4. We form ${}^{+}K_e \cap {}^{-}K_e$.

Step 5. We have a map ${}^{+}K_e \cap {}^{-}K_e \xrightarrow{\iota} {}^{+}K_e/v{}^{+}K_e$ (restriction of the obvious map ${}^{+}K_e \to {}^{+}K_e/v{}^{+}K_e$).

Step 6. We have a map ${}^{+}K_e/v{}^{+}K_e \xrightarrow{\iota'} {}''K_e/v{}''K_e$ induced by the obvious inclusion ${}^{+}K_e \subset {}''K_e$.

Step 7. For any $\mathfrak{b} \in \tilde{\mathbf{B}}_e$ there is a unique element $\tau(\mathfrak{b}) \in {}^{+}K_e \cap {}^{-}K_e$ such that $\iota'\iota(\tau(\mathfrak{b}))$ is the image of $\mathfrak{b}$ in ${}''K_e/v{}''K_e$.

Step 8. The elements $\{t(\mathfrak{b}); \mathfrak{b} \in \tilde{\mathbf{B}}_e\}$ form a $\mathbf{Z}$-basis of ${}^{+}K_e \cap {}^{-}K_e$ and an $\mathcal{A}$-basis of K_e. This is $\mathbf{B}_e$.

We now justify Step 7. Note that ${}^{+}K_e$ is the set of all $\sum_{b\in\mathbf{B}_e} c_b b$ where for any b we have $c_b \in \mathcal{A} \cap {}''\mathcal{A}$ or equivalently $c_b \in \mathbf{Z}[v]$. (If $a \in \mathbf{Z}[v, v^{-1}]$ is of the form f/g where $g \in \mathbf{Z}[v]$ has constant term 1 and $f \in \mathbf{Z}[v]$, then $a \in \mathbf{Z}[v]$. Indeed, we have $a = \sum_{i\in\mathbf{Z}} a_i v^i$ where $a_i \in \mathbf{Z}$ satisfies $a_i = 0$ for $i \gg 0$ and for $i \ll 0$, since $a \in \mathcal{A}$, and $a_i = 0$ for $i < 0$, since $a \in {}''\mathcal{A}$.) It follows that ${}^{-}K_e$ is the set of all $\sum_{b\in\mathbf{B}_e} c_b b$ where for any b we have $c_b \in \mathbf{Z}[v^{-1}]$. Hence ${}^{+}K_e \cap {}^{-}K_e$ is the set of all $\sum_{b\in\mathbf{B}_e} c_b b$ where for any b we have $c_b \in \mathbf{Z}$. The map ι' in Step 6 is an isomorphism. (We use that the map $\mathbf{Z}[v]/v\mathbf{Z}[v] \to {}''\mathcal{A}/v({}''\mathcal{A})$ induced by the inclusion $\mathbf{Z}[v] \to {}''\mathcal{A}$ is an isomorphism.) Moreover the map ι in Step 5 is an isomorphism. Now Step 7 holds in view of Steps 5 and 6. We now justify Step 8. Under the isomorphism $\iota'\iota$, the $\mathbf{Z}$-basis $\mathbf{B}_e$ of ${}^{+}K_e \cap {}^{-}K_e$ corresponds to the $\mathbf{Z}$-basis of ${}''K_e/v({}''K_e)$ formed by the image of $\mathbf{B}_e$ or equivalently by the image of $\tilde{\mathbf{B}}_e$. This justifies Step 8.

We note that we can reconstruct $\mathbf{B}_e$ from slightly less than the knowledge of $\tilde{\mathbf{B}}_e$: it is enough to have ${}''K_e$ and the image of $\tilde{\mathbf{B}}_e$ under ${}''K_e \to {}''K_e/v({}''K_e)$.

1.5. In this subsection we assume that G is of type A_2 and $e \in \mathfrak{g}$ is subregular nilpotent. Using [10, Sec.5], we see that $\mathbf{B}_e^{\pm}$ consists of $\pm$ three elements b_1, b_2, b_3 satisfying $(b_i||b_i) = 1 + v^{-2}$ and $(b_i||b_j) = -v^{-1}$ if $i \neq j$. The set $\tilde{\mathcal{M}}_e$ has three elements which can be denoted by X_1, X_2, X_3 so that ${}'K_e^{X_3} = {}'K_e^{\le X_3}$ has basis $\{b_3 + vb_1 + vb_2\}$, ${}'K_e^{\le X_i}$ has basis $\{b_3 + vb_1 + vb_2, b_i\}$ for $i = 1, 2$. It follows that for $i = 1, 2$, ${}'K_e^{X_i}$ has basis $\{b_i - \delta^{-1}(v^3 - v^2)(b_3 + vb_1 + vb_2)\}$ where $\delta = 1 - v^2 - 2v^3 + 2v^4$.

We have

$$b_i = \delta^{-1}(v^3 - v^2)(b_3 + vb_1 + vb_2) + (b_i - \delta^{-1}(v^3 - v^2)(b_3 + vb_1 + vb_2)),$$

for $i = 1, 2$,

$$\begin{aligned} b_3 \quad = \quad & \delta^{-1}(1 - v^2)(b_3 + vb_1 + vb_2) - v(b_1 - \delta^{-1}(v^3 - v^2)(b_3 + vb_1 + vb_2)) \\ & -v(b_2 - \delta^{-1}(v^3 - v^2)(b_3 + vb_1 + vb_2)) \end{aligned}$$

Hence

$$\tilde{b}_i = b_i - \delta^{-1}(v^3 - v^2)(b_3 + vb_1 + vb_2) \text{ for } i = 1, 2,$$
$$\tilde{b}_3 = \delta^{-1}(1 - v^2)(b_3 + vb_1 + vb_2).$$

The map $\mathbf{B}_e^{\pm} \to \tilde{\mathcal{M}}_e$ is $\pm b_i \mapsto X_i$ for $i = 1, 2, 3$. We see that 1.3(i),(ii) hold in this case.

1.6. In this subsection we assume that G is of type D_4 or G_2 and $e \in \mathfrak{g}$ is subregular nilpotent. Using [9], [10], we see that $\mathbf{B}_e^{\pm}$ consists of $\pm$ five elements b_0, b_1, b_2, b_3, b_4 satisfying $(b_i||b_i) = 1 + v^{-2}$ for $i = 0,1,2,3,4$, $(b_i||b_j) = 0$ if $i \neq j$ in $1,2,3,4$, $(b_0||b_i) = -v^{-1}$ for $i = 1,2,3,4$. The set $\tilde{\mathcal{M}}_e$ has four elements which can be denoted by X_0, X_1, X_2, X_3 so that $'K_e^{X_0} = {}'K_e^{\le X_0}$ has basis $\{b_0, b_4 + v^2(b_1 + b_2 + b_3)\}$. $'K_e^{\le X_i}$ has basis $\{b_4 + v^2(b_1 + b_2 + b_3), b_0, b_i\}$ for $i = 1,2,3$. It follows that for $i = 1,2,3$, $'K_e^{X_i}$ has basis

$$\{b_i + (v + v^3)\epsilon^{-1} b_0 + v^4 \epsilon^{-1}(b_4 + v^2(b_1 + b_2 + b_3))\}$$

where $\epsilon = 1 + 2v^2 - 3v^6$. We have

$$\begin{aligned} b_4 &= \epsilon^{-1}(1+2v^2)(b_4 + v^2(b_1+b_2+b_3)) + 3\epsilon^{-1}(v^3+v^5)b_0 \\ &\quad - \sum_{i \in \{1,2,3\}} v^2 \epsilon^{-1}(v^4(b_4 + v^2(b_1+b_2+b_3)) + (v+v^3)b_0 + \epsilon b_i), \end{aligned}$$

$$\begin{aligned} b_i &= -\epsilon^{-1} v^4 (b_4 + v^2(b_1+b_2+b_3)) - \epsilon^{-1}(v+v^3)b_0 \\ &\quad + \epsilon^{-1}(v^4(b_4 + v^2(b_1+b_2+b_3)) + (v+v^3)b_0 + \epsilon b_i) \end{aligned}$$

for $i = 1,2,3$. Hence

$$\tilde{b}_0 = b_0,$$

$$\tilde{b}_4 = \epsilon^{-1}(1+2v^2)(b_4 + v^2(b_1+b_2+b_3)) + 3\epsilon^{-1}(v^3+v^5)b_0,$$

$$\tilde{b}_i = \epsilon^{-1}(v^4(b_4 + v^2(b_1+b_2+b_3)) + (v+v^3)b_0 + \epsilon b_i) \text{ for } i = 1,2,3.$$

The map $\mathbf{B}_e^{\pm} \to \tilde{\mathcal{M}}_e$ is $\pm b_i \mapsto X_i$ for $i = 0,1,2,3$ and $\pm b_4 \mapsto X_0$. We see that 1.3(i),(ii) hold in this case.

1.7. If $\omega \in \mathcal{M}_e$, then F acts on $\mathcal{B}_e \cap \omega$ by conjugation. This induces an action of $\bar{F}$ on the set of connected components of $\mathcal{B}_e \cap \omega$. By [3], this action of $\bar{F}$ is transitive. Thus, $\bar{F}$ acts naturally on $\tilde{\mathcal{M}}_e$ and the map $\tilde{\mathcal{M}}_e \to \mathcal{M}_e$ (with $X \mapsto \omega$ when $X \subset \mathcal{B}_e \cap \omega$) has fibres given precisely by the $\bar{F}$-orbits on $\tilde{\mathcal{M}}_e$.

By [3], if $X \in \tilde{\mathcal{M}}^e$, then $X^{\mathbf{C}^*} = X \cap \mathcal{B}_e^{\mathbf{C}^*}$ is a connected component $\mathcal{B}_e^{\mathbf{C}^*}$ that is an element of Ξ_e; moreover, $X \mapsto X^{\mathbf{C}^*}$ is a bijection $\tilde{\mathcal{M}}_e \xrightarrow{\sim} \Xi_e$. Thus we may identify $\tilde{\mathcal{M}}_e$ with Ξ_e and $\mathcal{M}_e$ with $\bar{\Xi}_e$ (see 0.1).

Using the identification $\tilde{\mathcal{M}}_e = \Xi_e$, the map $\mathbf{B}_e^{\pm} \to \tilde{\mathcal{M}}_e$ in 1.3(ii) can be identified with a map $\mathbf{B}_e^{\pm} \to \Xi_e$, which factors through a (surjective) map $\sigma : \mathbf{B}_e \to \Xi_e$. Thus all maps in the diagram in 0.1 are defined.

1.8. One can define a (non-conjectural) direct sum decomposition $\mathbf{Q}(v) \otimes_{\mathcal{A}} K_e = \oplus_{X \in \tilde{\mathcal{M}}_e} K(X)$ into $\mathbf{Q}(v)$-vector subspaces $K(X)$ indexed by $X \in \tilde{\mathcal{M}}_e$ by noting that by a known localization property we have $\mathbf{Q}(v) \otimes_{\mathcal{A}} K_e = \mathbf{Q}(v) \otimes_{\mathcal{A}} K_{\mathbf{C}^*}(\mathcal{B}_e^{\mathbf{C}^*})$ and then using the direct sum decomposition of the last vector space coming from the decomposition of $\mathcal{B}_e^{\mathbf{C}^*}$ into connected components (which are indexed by $\tilde{\mathcal{M}}_e$). One can project any $b \in \mathbf{B}_e^{\pm}$ to the summands in this decomposition and one can ask whether these projections behave as in 1.3(ii). It appears that this is not the case.

2 $\mathbf{B}_e$ and the Burnside group of $\bar{F}$

2.1. Let H be a finite group. Let $\Omega(H)$ be the Burnside group of H that is, the free abelian group with generators the various conjugacy classes of subgroups of H. To any finite set X with an H-action (or H-set) we can associate an element $(X) \in \Omega(H)$ by the requirement that $(X \sqcup X') = (X) + (X')$ for two finite H-sets and $(H/H') = H'$ for any subgroup H' of H where H/H' is an H-set under left translation.

Let $M(H)$ be the set of all pairs (s, ρ) where $s \in H$ and ρ is an irreducible representation over $\mathbf{C}$ (up to isomorphism) of the centralizer $Z_H(s)$ of s in H; the pairs (s, ρ) are taken modulo H-conjugacy. Let $\mathbf{C}[M(H)]$ be the $\mathbf{C}$-vector space with basis $M(H)$.

Now let X be a finite H-set. For any $(s, \rho) \in M(H)$ the fixed point set X^s has an action of $Z_H(s)$ (restriction of the H-action on X) hence we can consider the multiplicity $N_{s,\rho}$ of ρ in the permutation representation of $Z_H(s)$ on X^s. We set $[X] = \sum_{(s,\rho)\in M(H)} N_{s,\rho}(s, \rho) \in \mathbf{C}[M(H)]$. Now $(X) \mapsto [X]$ for any finite H-set defines a homomorphism

(a) $\Omega(H) \to \mathbf{C}[M(H)]$.

2.2. We choose a Borel subgroup B of F^0 and a maximal torus T of B. Let $F' = \{g \in F; gBg^{-1} = B, gTg^{-1} = T\}$. Then $F'^0 = T$ and the obvious map $F'/T\mathcal{Z}_G \to F/F^0\mathcal{Z}_G = \bar{F}$ is an isomorphism. Let $\mathcal{B}_e^T = \{\mathfrak{b} \in \mathcal{B}_e; \mathrm{Ad}(t)\mathfrak{b} = \mathfrak{b} \text{ for all } t \in T\}$. Now F' acts on $\mathcal{B}_e^T$ by $g : \mathfrak{b} \mapsto \mathrm{Ad}(g)\mathfrak{b}$. This action is trivial on $T\mathcal{Z}_G$ hence it induces an action of $F'/T\mathcal{Z}_G = \bar{F}$ on $\mathcal{B}_e^T$.

Let $s \in \bar{F}$. Let $\mathcal{B}_e^{T,s}$ be the fixed point set of the action of s on $\mathcal{B}_e^T$. Note that $Z_{\bar{F}}(s)$ acts on $\mathcal{B}_e^{T,s}$ as the restriction of the $\bar{F}$-action on $\mathcal{B}_e^T$. Hence for any i there is an induced action of $Z_{\bar{F}}(s)$ on $H^i(\mathcal{B}_e^{T,s}, \mathbf{C})$. We define an element $\phi_e \in \mathbf{C}[M(\bar{F})]$ in which the coefficient of $(s, \rho) \in M(\bar{F})$ is:

(a) $\sum_i (-1)^i$(multiplicity of ρ in the $Z_{\bar{F}}(s)$-module $H^i(\mathcal{B}_e^{T,s}, \mathbf{C})$).

The following is a strengthening of the statement 0.2 that $\mathbf{B}_e$ is a discretization of $\mathcal{B}_e$.

Conjecture 2.3. *We have* $[\mathbf{B}_e] = \phi_e \in \mathbf{C}[M(\bar{F})]$.

2.4. Let W' be the affine Weyl group corresponding to the dual of the adjoint group of G. We have $W' \subset W$. We can find a finite parabolic subgroup W'' of W' and a two-sided cell c'' of W'' such that $c'' \subset c$ (see [6, 4.8(d)]); moreover, by [11, 1.5(b2)], we can assume that the finite group $\mathcal{G}_{c''}$ associated to c'' in [5, 3.5] coincides with $\bar{F}$. Let $\mathcal{F}_e$ be the set of subgroups of $\bar{F} = \mathcal{G}_{c''}$ attached in [5, 3.8] to the various left cells of W'' contained in c'' (or rather one such subgroup in each $\bar{F} = \mathcal{G}_{c''}$-conjugacy class). From [12] we see that:

(a) The elements $[\bar{F}/H] \in \mathbf{C}[M(\bar{F})]$ for various $H \in \mathcal{F}_e$ are linearly independent.

For $b \in \mathbf{B}_e$ let $\bar{F}_b \subset \bar{F}$ be the stabilizer of b for the $\bar{F}$-action on $\mathbf{B}_e$.

Conjecture 2.5. *$\mathcal{F}_e$ (see 2.4) is a set of representatives for the $\bar{F}$-conjugacy classes of subgroups of $\bar{F}$ of the form $\bar{F}_b$ for some $b \in \mathbf{B}_e$.*

2.6. Assuming that 2.3 and 2.5 hold, we see that the element $(\mathbf{B}_e)$ of the Burnside group $\Omega(\bar{F})$ is explicitly determined. Indeed, the element $\phi_e \in \mathbf{C}[M(\bar{F})]$ can be explicitly computed from the knowledge of Green functions for G and its subgroups. Using 2.3 we see that $[\mathbf{B}_e] \in \mathbf{C}[M(\bar{F})]$ is explicitly determined. Using 2.5 we see that $(\mathbf{B}_e)$ is determined by $[\mathbf{B}_e]$ hence is also explicitly determined.

2.7. Assuming 2.5 and that ρ is as in 0.1(f) we see that to any $\Gamma \in R(c)$ one can attach a subgroup $H_\Gamma \in \mathcal{F}_e$ characterized by the condition that H_Γ is conjugate to $\bar{F}_b$ for some/any $b \in \rho^{-1}(\Gamma)$. We note that the subgroups $H_\Gamma \subset \bar{F}$ associated to the various $\Gamma \in R(c)$ can be regarded as affine analogues of the finite groups associated in [5] to the right cells (or left cells) inside a two-sided cell of a finite Weyl group.

2.8. For $\xi \in \Xi_e$ we denote by $\bar{F}_\xi$ the stabilizer of ξ in the $\bar{F}$-action on Ξ_e. Assuming 2.5 and the truth of the conjectures in 1.3, we note that the map $\sigma : \mathbf{B}_e \to \Xi_e$ in 1.7 is $\bar{F}$-equivariant. Hence if $b \in \mathbf{B}_e$ then

(a) $\bar{F}_b \subset \bar{F}_{\sigma(b)}$.

This seems to be an equality in many (but not all) cases. Assume for example that G is of type E_8 and e is such that $\bar{F} = S_5$. In this case the subgroups $\{\bar{F}_\xi; \xi \in \Xi_e\}$ of $\bar{F}$ are exactly the conjugates of the subgroups in $\mathcal{F}_e$ (a result of [3]); we expect that in this case (a) is an equality.

Assume now that G is of type E_8 and e is of type $E_8(b_6)$ (notation as in [2, p. 407]). In this case we have $\bar{F} = S_3$ and for $\xi \in \Xi_e$, $\bar{F}_\xi$ is one of the subgroups S_2, S_3 or a cyclic group of order 3 of S_3 (this can be deduced from [3, 4.1]); if in (a), $\bar{F}_{\sigma(b)}$ is cyclic of order 3, we expect to have $\bar{F}_b = \{1\}$ so that (a) is not an equality.

3 The bijection 0.3(a); an example

In this section we consider the example where G is of type G_2 and that e is a subregular nilpotent element. Let W be as in 0.1. The simple reflections in W are s_0, s_1, s_2 where $s_0 s_1$ has order 3, $s_1 s_2$ has order 6 and $s_0 s_2 = s_2 s_0$. In this case c is the two-sided cell of W containing s_0, s_1, s_2. It is known [4] that c consists of all non-identity elements of W with a unique reduced expression. We write $i_1 i_2 i_3$ instead of $s_{i_1} s_{i_2} s_{i_3}$. The elements of c are

0	01	012	0121	01212	012121	0121210
			01210			

10	1	12	121	1212	12121	121210
			1210			

2	21	212	2121	21212
	210		21210	

Note the apparition of two Coxeter graph of affine type E_7 and one of affine type D_6. We write the elements of $\mathbf{B}_e$ as $[0], [1], [2], [2'], [2'']$. where the action of $F = \bar{F} = S_3$ on $\mathbf{B}_e$ keeps $[0]$ and $[1]$ fixed and permutes cyclically $[2], [2'], [2'']$. The irreducible representations of S_3 are denoted by $1, r, \epsilon$ where r is 2-dimensional and ϵ is the sign. The irreducible representations of S_2 are denoted by $1, \epsilon$ where ϵ is the sign. The unit representation of S_1 is denoted by 1.

We show that c is in natural bijection with the set of irreducible F-vector bundles on $\mathbf{B}_e \times \mathbf{B}_e$ (up to isomorphism).

To an element of c we associate the irreducible F-vector bundle on $\mathbf{B}_e \times \mathbf{B}_e$ which appears in the same position in the following list.

$$\begin{array}{ccccccc} ([0][0];1) & ([0][1];1) & ([0][2];1) & ([0][1];r) & ([0][2];\epsilon) & ([0][1];\epsilon) & ([0][0];\epsilon) \\ & & & ([0][0];r) & & & \end{array}$$

$$\begin{array}{ccccccc} ([1][0];1) & ([1][1];1) & ([1][2];1) & ([1][1];r) & ([1][2];\epsilon) & ([1][1];\epsilon) & ([1][0];\epsilon) \\ & & & ([1][0];r) & & & \end{array}$$

$$\begin{array}{ccccc} ([2][2];1) & ([2][1];1) & ([2][2'];1) & ([2][1];\epsilon) & ([2][2];\epsilon) \\ & ([2][0];1) & ([2][0];\epsilon) & & \end{array}$$

Here a symbol $([?][?];?)$ represents a vector bundle on $\mathbf{B}_e \times \mathbf{B}_e$: the first two components give a point in the support of the vector bundle, the third component is the representation of the stabilizer of that point in the fibre at that point.

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