An extension of P. Preda, A. Pogan, C. Preda, Timisoara’s theorems for the uniformly exponential stability of linear skew-product semiflows

by

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Abstract

In this paper, we shall extend P. Preda, A. Pogan, C. Preda, Timisoara’s theorems. This is done by employing skew-product semiflows technique and functionals on function spaces.

Key Words: Exponential stability, linear skew-product semiflows, sequence spaces.

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1 Introduction:

One of the most important results in the theory of stability for a strongly continuous semigroup of linear operators has been obtained by Datko [6] in 1970; it states that the semigroup \((T(t))_{t \geq 0}\) is uniformly exponentially stable if and only if for each \(x \in X\) the map \(t \to \|T(t)x\|\) lies in \(L^2(\mathbb{R}+)\). Later, A. Pazy proved that the result remains true even if we replace \(L^2(\mathbb{R}+)\) with \(L^p(\mathbb{R}+)\), where \(p \in [1, \infty)\). In 1973, R. Datko generalized the results above as follows.

\textbf{Theorem 1.1.} An evolution family \((U(t, s))_{t \geq s \geq 0}\) with exponential growth is uniformly exponentially stable if and only if there exists \(p \in [1, \infty)\) such that:

\[
\sup_{s \geq 0} \int_s^\infty \|U(\tau, s)x\|^p d\tau < \infty
\]

for all \(x \in X\).

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The result provided by Theorem 1.1 was extended to dichotomy by P. Preda and M. Megan [1] in 1985. The same result was generalized in 1986 by S. Rolewicz [4] in the following way.

**Theorem 1.2.** Let \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a continuous, non-decreasing function with \( \phi(0) = 0 \) and \( \phi(t) > 0 \) for each positive \( t \), and \( (U(t,s))_{t \geq s \geq 0} \) an evolution family, with exponential growth. If

\[
\sup_{s \geq 0} \int_{s}^{\infty} \phi(\|U(\tau,s)x\|) \, d\tau < \infty, \quad (x \in X)
\]

then \( (U(t,s))_{t \geq s \geq 0} \) is uniformly exponentially stable.

A new idea has been presented by P. Preda, A. Pogan, C. Preda, Timisoara in [2] in which the authors used functionals on function spaces to study the asymptotic behaviour of evolution family.

**Theorem 1.3.** An evolution family \( (U(t,s))_{t \geq s \geq 0} \) is uniformly exponentially stable if and only if there exists \( F \in \mathcal{F} \) such that the set

\[
A_F = \{ x : \sup_{t_0 \geq 0} F(\varphi(x,t_0,\cdot)) < \infty \}
\]

is of the second Baire category. Here \( \varphi(x,t_0,n) = \|U(n+t_0,t_0)x\| \).

In next section, the paper will consider the concept of linear skew-product semiflows. Then some conditions for uniformly exponential stability of linear skew-product semiflows by extending P. Preda, A. Pogan, C. Preda, Timisoara’s theorems will be given.

2 Notations and preliminaries:

2.1 Linear skew-product semiflows

Let us recall that basic notions of linear skew-product semiflows. \( X \) is a Banach space and \( (\odot, d) \) is a metric space. We denote by \( \mathcal{L}(X) \) be the Banach algebra of all bounded linear operators acting on \( X \).

**Definition 2.1.** \( \sigma : \odot \times \mathbb{R}_+ \to \odot \) is called a semiflow on \( \odot \) if

1. \( \sigma(\theta, 0) = \theta \) for all \( \theta \in \odot \).
2. \( \sigma(\theta, t + s) = \sigma(\sigma(\theta, s), t) \), for all \( (\theta, s, t) \in \odot \times \mathbb{R}_+^2 \).
3. \( \sigma \) is continuous.

**Definition 2.2.** A pair \( \pi = (\Phi, \sigma) \) is called a linear skew-product semiflow on \( \mathcal{E} = X \times \odot \) if \( \sigma \) is a semiflow on \( \odot \) and \( \Phi : \odot \times \mathbb{R}_+ \to \mathcal{L}(X) \) satisfies the following conditions:
1. $\Phi(\theta, 0) = I$ the identity operator on $X$, for all $\theta \in \ominus$.
2. $\Phi(\theta, t + s) = \Phi(\sigma(\theta, t), s)\Phi(\theta, t)$ for all $(\theta, t, s) \in \ominus \times \mathbb{R}^2_+$.
3. $\lim_{t \to 0} \Phi(\theta, t)x = x$, uniformly in $\theta$. This means that for every $x \in X$ and $\epsilon > 0$ there is $\delta = \delta(x, \epsilon) > 0$ such that $\|\Phi(\theta, t)x - x\| < \epsilon$ for all $\theta \in \ominus$ and $t \in [0, \delta]$.

**Proposition 2.1.** Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow on $E = X \times \ominus$. Then there exist constants $M \geq 1$ and $\omega > 0$ such that:

$$\|\Phi(\theta, t)\| \leq Me^{\omega t}$$

for all $(\theta, t) \in \ominus \times \mathbb{R}_+$.

**Proof:** See [9].

The mapping $\Phi$ given Definition 2.2 is called the cocycle associated to the linear skew-product semiflow $\pi$.

**Example 2.2.** Let $\ominus$ be a metric space, let $\sigma$ be a semiflow on $\ominus$ and $\{T(t)\}_{t \geq 0}$ be a $C_0$-semigroup on $X$. Then $\pi = (\Phi, \sigma)$, where $\Phi(\theta, t) = T(t)$ for all $(\theta, t) \in \ominus \times \mathbb{R}_+$ is a linear skew-product semiflow on $E = X \times \ominus$ which is called the linear skew-product semiflow generated by the $C_0$-semigroup $T$ and the semiflow $\sigma$.

**Example 2.3.** Let:

$$\ominus = \mathbb{R}_+, \sigma(\theta, t) = \theta + t$$

and let $\{U(t, s)\}_{t \geq s \geq 0}$ an evolution family on the Banach space $X$. We define:

$$\Phi(\theta, t) = U(t + \theta, \theta)$$

for all $(\theta, t) \in \mathbb{R}_+ \times \mathbb{R}_+$. Then $\pi = (\Phi, \sigma)$ is a linear skew-product semiflow on $E = X \times \ominus$ called the linear skew-product semiflow generated by the evolution family $\{U(t, s)\}_{t \geq s \geq 0}$.

The following example can be found in Chow and Leiva (see [7]):

**Example 2.4.** Let $\sigma$ be a semiflow on the compact Hausdorff space $\ominus$ and let $\{T(t)\}_{t \geq 0}$ be a $C_0$-semigroup on the Banach space $X$. For every strongly continuous mapping:

$$F : \ominus \to \mathcal{L}(X)$$

there is a linear skew-product semiflow $\pi_F = (\Phi_F, \sigma)$ on $E = X \times \ominus$ such that:

$$\Phi_F(\theta, t)x = T(t)x + \int_0^t T(t - s)F(\sigma(\theta, s))\Phi_F(\theta, s)x \, ds$$
The linear skew-product semiflow $\pi_F = (\Phi_F, \sigma)$ is called the linear skew-product semiflow generated by the triplet $(T, F, \sigma)$.

**Definition 2.3.** The linear skew-product semiflow $\pi = (\Phi, \sigma)$ is said to be uniformly exponentially stable if and only if there exist $K, \nu > 0$ such that:

$$\|\Phi(\theta, t)x\| \leq Ke^{-\nu t}\|x\|$$

for all $(\theta, t, x) \in \Theta \times \mathbb{R}_+ \times X$

**Definition 2.4.**

1. Let $\mathcal{N}$ be the set of all non-decreasing functions:

$$b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$$

with the property $b(t) > 0$ for all $t > 0$.

2. $\mathcal{N}^*$ is the set of functions $b \in \mathcal{N}$ and the property:

$$\lim_{t \to \infty} b(t) = \infty$$

3. For $b \in \mathcal{N}$, we put:

$$B(t) := \int_0^t b(\tau) \, d\tau$$

Then $B : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is non-decreasing, continuous and bijective.

### 2.2 Functionals on function and sequence spaces

In this section, we shall present some definitions, notations and results about normed function spaces. Let $(\Omega, \Sigma, \mu)$ be a measure space. By $\mathcal{M}(\Omega, \mathbb{R})$ we denote the linear space of measurable functions:

$$f : \Omega \rightarrow \mathbb{R}$$

identifying the functions which are equal $\mu$-a.e.

**Definition 2.5.** A function $N : \mathcal{M}(\Omega, \mathbb{R}) \rightarrow [0; \infty]$ is called a generalized norm on $\mathcal{M}(\Omega, \mathbb{R})$ if the following properties are satisfied:

1. $N(f) = 0$ if and only if $f = 0$ $\mu$-a.e.

2. If $|f| \leq |g|$ $\mu$-a.e. and $f, g \in \mathcal{M}(\Omega, \mathbb{R})$ then $N(f) \leq N(g)$

3. $N(af) = |a| N(f)$ for all $a \in \mathbb{R}$ and $f \in \mathcal{M}(\Omega, \mathbb{R})$ with $N(f) < \infty$

4. $N(f + g) \leq N(f) + N(g)$ for all $f, g \in \mathcal{M}(\Omega, \mathbb{R})$
Let $B = B_N$ be the set defined by:

$$B := \{ f \in \mathcal{M}(\Omega, \mathbb{R}) : N(f) < \infty \}$$

$B$ is normed function space with the norm $| f |_B := N(f)$.

**Definition 2.6.** If $\Omega = \mathbb{N}$ with the standard counting measure, we shall denote

1. $\mathcal{M}(\mathbb{N}, \mathbb{R}) = \mathcal{S}$ is the space of all real sequences and $\mathcal{M}^+(\mathbb{N}, \mathbb{R}) = \mathcal{S}^+$ the set of all positive sequences.
2. $\mathcal{S}_0^+$ is the space of all $s \in \mathcal{S}^+$ which satisfies the condition:

$$\max\{n \in \mathbb{N} \mid s(n) > 0\} < \infty$$

**Example 2.5.**

$$\chi_{\{0, \ldots, n\}} \in \mathcal{S}_0^+$$

**Definition 2.7.** $\mathcal{F}$ is the set of all functions:

$$F : \mathcal{S}^+ \to [0, \infty]$$

with the properties:

1. If $s_1, s_2 \in \mathcal{S}^+$ with $s_1 \leq s_2$ then $F(s_1) \leq F(s_2)$.
2. $F(\alpha \chi_{\{n\}}) \geq \alpha$ for all $(\alpha, n) \in \mathbb{R}_+^* \times \mathbb{N}^*$.
3. $\lim_{n \to \infty} F(\alpha \chi_{\{0, \ldots, n\}}) = \infty$ for all $\alpha \in \mathbb{R}_+^*$.

**Example 2.6.** The map $F : \mathcal{S}^+ \to [0, \infty]$ defined by:

$$F(s) = \sum_{n=0}^{\infty} e^n s(n)$$

belongs to $\mathcal{F}$.

**Definition 2.8.** Let $\mathcal{F}^*$ be the set of all functions $F \in \mathcal{F}$ which satisfies the condition:

$F(u) = F(v)$ for every $u, v \in \mathcal{S}_0^+$ provided that:

1. $\max\{n \in \mathbb{N} \mid u(n) > 0\} = \max\{n \in \mathbb{N} \mid v(n) > 0\}$
2. $u(j) = v(m - j)$ for all $j \in \{0, \ldots, m\}$, where $m := \max\{n \in \mathbb{N} \mid u(n) > 0\}$.

**Example 2.7.** The map $F : \mathcal{S}^+ \to [0, \infty]$ defined by:

1. $F(s) = \sum_{n=0}^{\infty} s(n)$
2. 

\[ F(s) = \sum_{n=0}^{\infty} e^{s(n)} \]

belongs to \( \mathcal{F}^* \).

**Proposition 2.8.** If \( F \in \mathcal{F} \) and \( L > 0 \) then:

\[ \lim_{n \to \infty} r(n) = \infty \]

where

\[ r(n) := \inf_{\alpha \in (0, L]} \frac{F(\alpha X \{0, \ldots, n\})}{\alpha^2} \]

**Proof:** See [2]. \( \Box \)

### 3 Main Results:

**Lemma 3.1.** If there are \( \tau > 0 \) and \( c \in (0; 1) \) such that:

\[ \|\Phi(\theta, \tau)x\| \leq c \|x\| \]

for all \( (\theta, x) \in \ominus \times X \)

then the linear skew-product semiflow \( \pi = (\Phi, \sigma) \) is uniformly exponentially stable.

**Proof:** See [2]. \( \Box \)

Let \( \pi \) be a linear skew-product semiflow. For every \( (x, \theta) \in X \times \ominus \), we denote:

\[ \varphi(x, \theta, \cdot) : \mathbb{N} \to \mathbb{R}_+ \]

the sequence defined by:

\[ \varphi(x, \theta, n) := \|\Phi(\theta, t_n)x\| \]

**Theorem 3.2.** The linear skew-product semiflow \( \pi \) is uniformly exponentially stable if and only if there exist \( F \in \mathcal{F} \) and a non-decreasing sequence \( (t_n) \subset \mathbb{R}_+ \) such that:

1. \( \delta := \sup_{n \in \mathbb{N}} |t_{n+1} - t_n| < \infty \)

2. The set

\[ A_F = \{ x \in X : \sup_{\theta \in \ominus} \sup_{n \in \mathbb{N}} F(X \{0, \ldots, n\} \varphi(x, \theta, \cdot)) < \infty \} \]

is of the second Baire category.
**Proof:** Necessity. This is a simple verification for:

\[
F(s) = \sum_{n=0}^{\infty} s(n)
\]

and \(t_n = n\).

Sufficiency.

**Case 1:**

\[
\sup_{n \in \mathbb{N}} t_n = T < \infty
\]

Put \(\alpha := Me^{\omega T}\) where \(M, \omega\) are given in Definition 2.2. It is clear that

\[
\|\Phi(\theta,t_n)x\| \leq Me^{\omega(T-t_n)}\|\Phi(\theta,t_n)x\| \leq \alpha \varphi(x,\theta,n)
\]

\[
X_{\{0,\ldots,n-1\}} \varphi(x,\theta,\cdot) \geq \left\| \Phi(\theta,T)\frac{x}{\alpha} \right\| X_{\{0,\ldots,n\}}
\]

\[
F(X_{\{0,\ldots,n\}} \varphi(x,\theta,\cdot)) \geq F \left( \left\| \Phi(\theta,T)\frac{x}{\alpha} \right\| X_{\{0,\ldots,n\}} \right)
\]

for all \(x \in A_F\). By Definition [2.7], it follows that \(\Phi(\theta,T)x = 0\).

**Case 2:**

\[
\sup_{n \in \mathbb{N}} t_n = \infty
\]

Having in mind that:

\[
A_F = \bigcup_{n=1}^{\infty} \{ A_F \cap \{ x \in X : \|x\| \leq n \} \}
\]

It follows that there exists \(n_0 \in \mathbb{N}\) such that the set:

\[
A_F \cap \{ x \in X : \|x\| \leq n_0 \}
\]

denoted by \(A_{n_0}\), is of the second Baire category.

**Step 1:** We prove that there exists

\[
K := \sup_{(\theta, n) \in \Theta \times \mathbb{N}} \|\Phi(\theta,t_n)\| < \infty
\]

Put \(\beta := Me^{-\delta}\). For each \(x \in A_{n_0}\) and \(n \geq 1\) we see that

\[
\|\Phi(\theta,t_n)x\| \leq Me^{\omega(t_n-t_{n-1})}\|\Phi(\theta,t_{n-1})x\| \leq Me^{-\delta}\|\Phi(\theta,t_{n-1})x\|
\]

\[
\varphi(x,\theta,n-1) \geq \left\| \Phi(\theta,t_n)\frac{x}{\beta} \right\|
\]

\[
X_{\{0,\ldots,n-1\}} \varphi(x,\theta,\cdot) \geq \left\| \Phi(\theta,t_n)\frac{x}{\beta} \right\| X_{\{n-1\}}
\]
\[
M(x) := \sup_{\theta \in \Theta} F(\xi_{(0,\ldots,n)} \varphi(x, \theta, .))
\]

\[
M(x) \geq F(\xi_{(0,\ldots,n-1)} \varphi(x, \theta, .)) \geq F\left(\left\| \Phi(\theta, t_n) \frac{x}{\beta} \right\| \xi_{(n-1)} \right) \geq \left\| \Phi(\theta, t_n) \frac{x}{\beta} \right\|
\]

Thus
\[
\left\| \Phi(\theta, t_n) \frac{x}{\beta} \right\| \leq M(x)
\]

Or
\[
\sup_{n \in \mathbb{N}, \theta \in \Theta} \|\Phi(\theta, t_n)x\| < \infty
\]

Using the fact that \(A_{n_0}\) is of the second Baire category by the Uniform Boundedness Principle (see [10], Theorem 2.5.5, page 26), we obtain
\[
\sup_{n \in \mathbb{N}, \theta \in \Theta} \|\Phi(\theta, t_n)\| < \infty
\]

**Step 2:** We prove that there exists
\[
l := \sup_{t \in \mathbb{R}, \theta \in \Theta} \|\Phi(\theta, t)\| < \infty
\]

If \(t \leq t_0\) then
\[
\|\Phi(\theta, t)\| \leq M e^{\omega t_0}
\]

If \(t \geq t_0\) then
\[
\|\Phi(\theta, t)\| \leq M \varepsilon
\]

There exists \(n \in \mathbb{N}\) such that \(t \in [t_n, t_{n+1}]\)
\[
\|\Phi(\theta, t)\| \leq M e^{\omega(t - t_n)} \|\Phi(\theta, t_n)x\| \leq M e^{\omega(t_{n+1} - t_n)} K \|x\| \leq M e^{\omega} K \|x\|
\]

Thus \(l = \max\{Me^{\omega t_0} , Me^{\omega}K\}\)

**Step 3:** We prove that there exist \(\tau\) and \(c\) as in Lemma 3.1. Indeed, we put
\[
r(n) := \inf_{\lambda \in (0, n)} \frac{F(\lambda \xi_{(0,\ldots,n)})}{\lambda^2}
\]

For \(x \in A_{n_0}\) and \(j \in \{0, \ldots, n\}\) we have
\[
\|\Phi(\theta, t_n)x\| \leq l \|\Phi(\theta, t_j)x\| = l \varphi(x, \theta, j)
\]
Linear skew-product semiflows

\[ \mathcal{X}_{\{0,\ldots,n\}} \varphi(x,\theta,.) \geq \frac{\|\Phi(\theta,t_n)x\|}{l} \mathcal{X}_{\{0,\ldots,n\}} \]

\[ F(\mathcal{X}_{\{0,\ldots,n\}} \varphi(x,\theta, .)) \geq F \left( \frac{\|\Phi(\theta,t_n)x\|}{l} \mathcal{X}_{\{0,\ldots,n\}} \right) \]

\[ \geq r(n) \left( \frac{\|\Phi(\theta,t_n)x\|}{l} \right)^2 \]

Thus

\[ \sup_{n \in \mathbb{N}} \sqrt{r(n)} \|\Phi(\theta,t_n)x\| < \infty \]

for all \( x \in A_{n_0} \). Using the fact that \( A_{n_0} \) is of the second Baire category by the Uniform Boundedness Principle (see [10], Theorem 2.5.5, page 26), we obtain

\[ L := \sup_{n \in \mathbb{N}} \sqrt{r(n)} \|\Phi(\theta,t_n)x\| < \infty \]

Or

\[ \|\Phi(\theta,t_n)x\| \leq L \|x\| \sqrt{r(n)} \]

By Proposition 2.7, there exists \( p \in \mathbb{N} \) such that:

\[ \frac{L}{\sqrt{r(p)}} \leq \frac{1}{2} \]

Or

\[ \|\Phi(\theta,t_p)x\| \leq \frac{\|x\|}{2} \]

By Lemma 3.1, the Theorem is proved. \qed

**Theorem 3.3.** The linear skew-product semiflow \( \pi \) is uniformly exponentially stable if and only if there exist a non-decreasing sequence \( (t_n) \subset \mathbb{R}_+ \), \( b \in \mathcal{N} \) and \( F \in \mathcal{F}^\ast \) such that:

1. \( \delta := \sup_{n \in \mathbb{N}} (t_{n+1} - t_n) < \infty \)

2. The set

\[ A_b = \{ x \in X : \sup_{\theta \in \mathcal{O}, n \in \mathbb{N}} F(\alpha \mathcal{X}_{\{0,\ldots,n\}} b(\varphi(x,\theta,.))) < \infty \text{ for all } \alpha \in \mathbb{R}_+ \} \]

is of the second Baire category.

Here \( b(\varphi(x,\theta,.))(j) = b(\varphi(x,\theta,j)) \).
Proof: Necessity. This is a simple verification for \( b(t) = t, \ t_n = n \) and

\[
F(s) = \sum_{n=0}^{\infty} s(n)
\]

Sufficiency. Take \( M, \omega \) as in Definition 2.2. Denote:

\[
M(x) := \sup_{\theta \in \ominus} F(\mathcal{X}_{\{0,\ldots,n\}} b(\varphi(x, \theta, .)))
\]

for each \( x \in A_b \). Next step we shall prove that:

\[
\sup_{n \in \mathbb{N}} \sup_{\theta \in \ominus} \| \Phi(\theta, t_n) x \| < \infty
\]

for all \( x \in A_b \). Indeed, we assume for a contradiction that there exists \( x_0 \in A_b \) such that:

\[
\sup_{n \in \mathbb{N}} \sup_{\theta \in \ominus} \| \Phi(\theta, t_n) x_0 \| = \infty
\]

We denote \( h(t) := e^{\omega \delta t} \). It is clear that:

\[
\mathcal{X}_{\{0,\ldots,n\}} b(h(.)) \geq \mathcal{X}_{\{0,\ldots,n\}} b(1)
\]

\[
\lim_{n \to \infty} F(\mathcal{X}_{\{0,\ldots,n\}} b(h(.))) = \infty
\]

It follows that there is \( n_0 \in \mathbb{N} \) such that:

\[
F(\mathcal{X}_{\{0,\ldots,n_0\}} b(h(.))) \geq M(x_0) + 1
\]

Taking into account that:

\[
\sup_{n \leq n_0} \sup_{\theta \in \ominus} \| \Phi(\theta, t_n) x_0 \| < \infty
\]

So:

\[
\sup_{n \geq n_0} \sup_{\theta \in \ominus} \| \Phi(\theta, t_n) x_0 \| = \infty
\]

Hence there exist \( n_1 \geq n_0; \ \beta \in \ominus \) such that:

\[
\varphi(x, \beta, n_1) = \| \Phi(\beta, t_{n_1}) x_0 \| \geq M h(n_0)
\]
However:

\[ M(x_0) \geq F(X_{0,\ldots,n_1}b(\varphi(x_0,\beta,.))) \]
\[ \geq F\left(X_{0,\ldots,n_1}b\left(\varphi(x_0,\beta,n_1)\right)\right) \]
\[ = F\left(X_{0,\ldots,n_1}b\left(\frac{\varphi(x_0,\beta,n_1)}{Mh(\cdot)}\right)\right) \]
\[ \geq F\left(X_{0,\ldots,n_0}b\left(\frac{Mh(n_0)}{Mh(\cdot)}\right)\right) \]
\[ = F\left(X_{0,\ldots,n_0}b\left(h(n_0-\cdot)\right)\right) \]
\[ = F\left(X_{0,\ldots,n_0}b(h(\cdot))\right) \]
\[ \geq M(x_0) + 1 \]

which is a contradiction. Hence, we have:

\[ \sup_{n \in \mathbb{N}} \sup_{\theta \in \text{null}} \|\Phi(\theta,t_n)x\| < \infty \]

for all \( x \in A_b \). By the Uniform Boundedness Principle (see [10], Theorem 2.5.5, page 26)

\[ L := \sup_{n \in \mathbb{N}} \sup_{\theta \in \text{null}} \|\Phi(\theta,t_n)\| < \infty \]

Let \( G : \mathcal{L}^+ \to [0,\infty] \) be the map defined by:

\[ G(s) = B^{-1}(F(B(s))) \]

Then \( G \in \mathcal{F} \) and \( A_b \subset \{ x \in X : \sup_{\theta \in \text{null}} G(X_{0,\ldots,n})\varphi(x,\theta,.)) < \infty \} \). By Theorem 3.2 we obtain that \( \pi \) is uniformly exponentially stable.

In what follows, we show that the result above remains true even if we replace the condition: \( F \in \mathcal{F}^* \) with \( F \in \mathcal{F} \).

**Theorem 3.4.** The linear skew-product semiflow \( \pi \) is uniformly exponentially stable if and only if there exist a non-decreasing sequence \((t_n) \subset \mathbb{R}_+\), \( b \in \mathbb{N}^* \) and \( F \in \mathcal{F} \) such that:

1. \( \delta := \sup_{n \in \mathbb{N}} (t_{n+1} - t_n) < \infty \)

2. The set

\[ A_b = \{ x \in X : \sup_{\theta \in \text{null}} F(\alpha X_{0,\ldots,n})b(\varphi(x,\theta,.)) < \infty \text{ for all } \alpha \in \mathbb{R}_+ \} \]

is of the second Baire category.
**Proof:** Necessity. It is a simple computation.

Sufficiency. For each \( x \in A_b \), we put:

\[
M(x) := \sup_{\theta \in \oplus, n \in \mathbb{N}} F(\mathcal{X}_{\{0, \ldots, n\}} b(\varphi(x, \theta., .)))
\]

Since \( b \) belongs to \( \mathcal{N}^* \), there exists \( \beta(x) \) such that:

\[
b(t) > M(x)
\]

for all \( t > \beta(x) \).

Thus

\[
\varphi(x, \theta, n) \leq \beta(x)
\]

Or

\[
\sup_{n \in \mathbb{N}} \| \Phi(\theta, t_n) x \| \leq \beta(x) < \infty
\]

for all \( x \in A_b \). By the Uniform Boundedness Principle (in [10], Theorem 2.5.5, page 26)

\[
L := \sup_{n \in \mathbb{N}} \| \Phi(\theta, t_n) \| < \infty
\]

Let \( G : \mathcal{L}^+ \to [0, \infty] \) be the map defined by:

\[
G(s) = B^{-1}(F(B(s.)))
\]

Then \( G \in \mathcal{F} \) and \( A_b \subset \{ x \in X : \sup_{\theta \in \oplus, n \in \mathbb{N}} G(\mathcal{X}_{\{0, \ldots, n\}} \varphi(x, \theta., .)) < \infty \} \). By Theorem 3.2 we obtain that \( \pi \) is uniformly exponentially stable.

In next section, we give a characterization for the uniformly exponential stability on Hilbert spaces. By using Theorem 3.4, we shall establish the connections between the uniformly exponential stability and some discrete equation.

### 4 Applications of the case real Hilbert spaces:

First, we give some definitions.
**Definition 4.1.** The map $H : \Theta \times \mathbb{N} \rightarrow \mathcal{L}(X)$ is called positive if:

$$<H(\theta, m)x, x> \geq 0$$

for all $m \in \mathbb{N}$ and $x \in X$; $\theta \in \Theta$.

**Definition 4.2.** The map $H : \Theta \times \mathbb{N} \rightarrow \mathcal{L}(X)$ is called uniformly positive if there exists the constant $a > 0$ such that:

$$<H(\theta, m)x, x> \geq a \|x\|^2$$

for all $m \in \mathbb{N}$ and $x \in X$; $\theta \in \Theta$.

**Remark 4.1.** If $a = 1$ then $H$ is called to be 1-uniformly positive.

**Theorem 4.2.** The linear skew-product semiflow $\pi = (\Phi, \sigma)$ is uniformly exponentially stable if and only if there exist $F \in \mathcal{F}$, $H : \Theta \times \mathbb{N} \rightarrow \mathcal{L}(X)$ is 1-uniformly positive and $W : \Theta \times \mathbb{N} \rightarrow \mathcal{L}(X)$ is positive such that:

1. $W := \sup_{\theta \in \Theta} \|W(\theta, 0)\| < \infty$
2. For all $(\theta, x) \in \Theta \times X$ and $n \geq 1$ we have

$$0 = <W(\theta, 0)x, x> - <\Phi^*(\theta, n)x, \Phi(\theta, n)x>$$

$$- F(\mathcal{X}_{\{0, \ldots, n-1\}} <\Phi^*(\theta, \cdot)H(\theta, n)\Phi(\theta, \cdot)x, x>)$$

where * denotes the conjugate transpose.

**Proof:** Necessity. It is easy to prove that:

$$F(s) = \sum_{n=0}^{\infty} s(n)$$

$$H(\theta, n) = I$$

$$W(\theta, m) = \sum_{k=0}^{\infty} \Phi^*(\sigma(\theta, m), k)\Phi(\sigma(\theta, m), k)$$

Sufficiency. For $(\alpha, x) \in \mathbb{R}_+ \times X$, we put $y := \sqrt{\alpha}x$. It is clear that:

$$F \left(\alpha \mathcal{X}_{\{0, \ldots, n-1\}} \|\Phi(\theta, \cdot)x\|^2\right)$$

$$F \left(\mathcal{X}_{\{0, \ldots, n-1\}} \|\Phi(\theta, \cdot)y\|^2\right)$$

$$\leq F \left(\mathcal{X}_{\{0, \ldots, n-1\}} <H(\theta, n)\Phi(\theta, \cdot)y, \Phi(\theta, \cdot)y>\right)$$

$$= F \left(\mathcal{X}_{\{0, \ldots, n-1\}} <\Phi^*(\theta, \cdot)H(\theta, n)\Phi(\theta, \cdot)y, y>\right)$$

$$= <W(\theta, 0)y, y> - <\Phi^*(\theta, n)W(\theta, n)\Phi(\theta, n)y, y>$$

$$= W\|y\|^2$$

$$= \alpha W\|x\|^2$$
Thus
\[ F \left( \alpha_{\mathcal{X}_{\{0,\ldots,n\}}} \| \Phi(\theta,.)x \|_2^2 \right) \leq \alpha W \| x \|_2^2 \]
for all \((n, \theta, x) \in \mathbb{N} \times \varnothing \times X\). Since \(X\) is a Banach space, by using Theorem 3.4, the Theorem is proved. □

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References


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