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Harmonic morphisms from Minkowski space and hyperbolic numbers

by
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To Professor S. Ianuş on the occasion of his 70th Birthday

Abstract

We show that all harmonic morphisms from 3-dimensional Minkowski space with values in a surface have a Weierstrass representation involving the complex numbers or the hyperbolic numbers depending on the signature of the codomain. We deduce that there is a non-trivial *globally defined* submersive harmonic morphism from Minkowski 3-space to a surface, in contrast to the Riemannian case. We show that a *degenerate* harmonic morphism on a Minkowski space is precisely a null real-valued solution to the wave equation, and we find all such.

Key Words: harmonic morphism, harmonic map, wave equation, hyperbolic number

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1 Introduction

A C^2 map $\varphi : (M,g) \to (N,h)$ between Riemannian manifolds is called a *harmonic morphism* if, for every harmonic function $f : V \to \mathbb{R}$ from an open subset V of N with $\varphi^{-1}(V)$ non-empty, the composition $f \circ \varphi : \varphi^{-1}(V) \to \mathbb{R}$ is harmonic. It is a fundamental result of Fuglede and Ishihara [7, 10], that φ is a harmonic morphism if and only if it is both a *harmonic map* and *horizontally weakly conformal.* If we allow the metrics g and h to be indefinite, the situation becomes more subtle due to the three possible types of tangent vector that can occur: *spacelike, timelike* or *null.* However, provided sufficient care is taken over the definitions, the same characterization applies [8, 4].

In this more general setting, we say that a C^1-map $\varphi : (M,g) \to (N,h)$ between semi-Riemannian manifolds is *horizontally (weakly) conformal* or *semiconformal* at $x \in M$ with *square dilation* $\Lambda(x)$ if

$$g\big(\mathrm{d}\varphi_x^*(U), \mathrm{d}\varphi_x^*(V)\big) = \Lambda(x)\, h(U,V) \qquad (U, V \in T_{\varphi(x)}N) \tag{1.1}$$

for some $\Lambda(x) \in \mathbb{R}$, where $\mathrm{d}\varphi_x^* : T_{\varphi(x)}N \to T_xM$ denotes the adjoint of $\mathrm{d}\varphi_x$. If φ is horizontally weakly conformal at every point, then we shall simply say that φ is *horizontally weakly conformal.* Note that, contrary to the Riemannian case, the function $\Lambda : M \to \mathbb{R}$ can take on nonpositive values. In fact, recall that a subspace W of T_xM is called *degenerate* if there exists a non-zero vector $v \in W$ such that $g(v,w) = 0$ for all $w \in W$, and *null* if $g(v,w) = 0$ *for all* $v, w \in W$; then we have three types of points, as follows (see [4, Proposition 14.5.4]).

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Proposition 1.1. *Let $\varphi : (M, g) \to (N, h)$ be a C^1 horizontally weakly conformal map. Then, for each $x \in M$, precisely one of the following holds*:

(i) $\mathrm{d}\varphi_x = 0$, *thus* $\mathrm{d}\varphi$ *has* rank 0 *at* x;

(ii) $\Lambda(x) \neq 0$. *Then φ is submersive at x and $\mathrm{d}\varphi_x$ maps the* horizontal space $\mathcal{H}_x := (\ker \mathrm{d}\varphi_x)^{\perp}$ *conformally onto $T_{\varphi(x)}N$ with square conformality factor $\Lambda(x)$, i.e., $h(\mathrm{d}\varphi_x(X), \mathrm{d}\varphi_x(Y)) = \Lambda(x)\, g(X, Y)$ $(X, Y \in \mathcal{H}_x)$, we call x a* regular point *of φ*;

(iii) $\Lambda(x) = 0$ *but* $\mathrm{d}\varphi_x \neq 0$. *Then the* vertical space $\mathcal{V}_x := \ker \mathrm{d}\varphi_x$ *is degenerate and $\mathcal{H}_x \subseteq \mathcal{V}_x$; equivalently, $\mathcal{H}_x$ is null and non-zero. We say that x is a* degenerate point *of φ, or that φ is* degenerate at x.

We call φ *non-degenerate* if it has no degenerate points, i.e., all points are of type (i) or (ii) above; this is always the case when the domain is Riemannian. Points that are not regular, i.e. points of type (i) or (iii), are called *critical points*.

Recall that a C^2 map $\varphi : (M, g) \to (N, h)$ is harmonic if it satisfies the *harmonicity equation* $\tau(\varphi) = 0$ where $\tau(\varphi) = \mathrm{Tr}\, \nabla \mathrm{d}\varphi$ is the tension field of φ, see [4, Chapters 3 and 14] for an account adapted to our needs. When the domain is of Riemannian signature, the harmonicity equation is elliptic; in particular, for maps between Euclidean spaces, it is Laplace's equation. On the other hand, when (M, g) is of Lorentzian signature, the harmonicity equation is hyperbolic. In particular, recall that m-dimensional *Minkowski space* $\mathbb{M}^m = \mathbb{R}^m_1$ is defined to be $\mathbb{R}^m$ endowed with the metric of signature $(1, m-1)$ given in standard coordinates $(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ by $g = -\mathrm{d}x_1^{\,2} + \mathrm{d}x_2^{\,2} + \ldots \mathrm{d}x_m^{\,2}$. Then a map $\varphi : \mathbb{M}^m \to \mathbb{R}$ or $\mathbb{C}$ is harmonic if and only if it satisfies the wave equation (1.2a) below.

Harmonic morphisms to *surfaces* are particularly nice; from the definition it is clear that the composition of such a map with a conformal or weakly conformal map of surfaces is again a harmonic morphism. In particular, the concept of harmonic morphism depends only on the conformal class of the metric on the surface; hence, when it is of Riemannian signature and oriented, we can take it to be a *Riemann surface*. A map $\varphi : \mathbb{M}^m \to N^2$ from Minkowski m-space to a Riemann surface is a harmonic morphism if and only if, in any local complex coordinate on N^2, it satisfies

$$\begin{cases} \text{(a)} \quad \Box\varphi \equiv -\dfrac{\partial^2 \varphi}{\partial {x_1}^2} + \displaystyle\sum_{i=2}^{m} \dfrac{\partial^2 \varphi}{\partial {x_i}^2} = 0, \\ \text{(b)} \quad \langle \operatorname{grad}\varphi \,,\, \operatorname{grad}\varphi \rangle_1 \equiv -\left(\dfrac{\partial \varphi}{\partial x_1}\right)^2 + \displaystyle\sum_{i=2}^{m} \left(\dfrac{\partial \varphi}{\partial x_i}\right)^2 = 0, \end{cases} \tag{1.2}$$

for $(x_1, \ldots, x_m) \in U$; the second equation being the condition of horizontal weak conformality. Here $\langle\ ,\ \rangle_1$ denotes the standard Lorentzian inner product defined for $\boldsymbol{v} = (v_1, v_2, \ldots, v_m)$, $\boldsymbol{w} = (w_1, w_2, \ldots, w_m) \in \mathbb{R}^m$ by

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle_1 = -v_1 w_1 + v_2 w_2 + \ldots + v_m w_m\,. \tag{1.3}$$

Harmonic morphisms from domains of *Euclidean* 3-space into a Riemann surface have a particularly elegant description in terms of holomorphic data [3] which we called a *Weierstrass representation* as the data coincides with that well-known representation of minimal surface in $\mathbb{R}^3$. More precisely, the fibres of a harmonic morphism $\varphi : U \to N^2$ from a domain U of $\mathbb{R}^3$ with values in a Riemann surface form a foliation by line segments which determines a holomorphic curve in the *mini-twistor space* of all lines in $\mathbb{R}^3$, a complex surface. Conversely, such a curve determines a foliation by line segments, and so a harmonic morphism, on some open subset of $\mathbb{R}^3$. A detailed account of this correspondence is given in [4, Chapter 1].

From the Weierstrass representation and some geometrical arguments, one can deduce a *Bernstein Theorem* that the only harmonic morphism defined globally on $\mathbb{R}^3$ with values in a surface is orthogonal projection onto a two-dimensional subspace, followed by a weakly conformal map [3].

In the semi-Riemannian case, there are harmonic morphisms all of whose fibres are degenerate. For maps from a Minkowski space, we show that these are precisely null *real-valued* solutions of the wave equation; in Section 4, we show how to find these by the method of Collins [6].

As for higher dimensions, see [4, §6.8] for the Riemannian case. Note also that (semi-)Riemannian submersions with minimal or totally geodesic fibres are harmonic morphisms; this is a subject close to Stere Ianuş's heart, for example, see [1, 11] for classifications of such maps from pseudo-hyperbolic spaces. For a study of the foliations which give rise to harmonic morphisms, see [4, 9].

2 Harmonic morphisms from Minkowski 3-space to a Riemann surface

We begin by characterizing those submersive (and so non-degenerate) harmonic morphisms defined on open subsets of Minkowski 3-space $\mathbb{M}^3 = \mathbb{R}^3_1$ with values in a Riemann surface. All manifolds and tensors defined on them are assumed to be smooth (C^∞).

Let $\varphi : U \to N^2$ be a C^2 mapping from an open subset U of $\mathbb{R}^3_1$ onto a 2-dimensional Riemannian manifold. Let (u, v) be isothermal coordinates on a domain of N^2; then $u + \mathrm{i}v$ gives a local complex coordinate with respect to which we write $\varphi(x_1, x_2, x_3) = \varphi_1(x_1, x_2, x_3) + \mathrm{i}\,\varphi_2(x_1, x_2, x_3)$. Then, φ is a harmonic morphism if and only if it satisfies the pair of equations (1.2) with $m = 3$. As before, the *pair* is independent of the choice of isothermal coordinates; thus, for local considerations, we can suppose that φ has values in $\mathbb{C}$. We now examine the fibres of φ.

Lemma 2.1. *Suppose that $\varphi : U \to \mathbb{C}$ is a C^2 submersive harmonic morphism from an open subset of $\mathbb{R}^3_1$. Then the connected components of the fibres of φ are timelike geodesics, and so are segments of straight lines.*

This follows from the immediate generalization to semi-Riemannian manifolds of the theorem of Baird and Eells [2] that a submersive harmonic morphism with values in a surface has minimal fibres.

In order to proceed, we shall suppose that $\varphi : U \to \mathbb{C}$ is a C^2 harmonic morphism from an open subset of $\mathbb{R}^3_1$ which satisfies the following conditions (cf. [3]):

$$\begin{cases} \text{(a)} & \varphi \text{ is submersive on } U \text{ (and so non-degenerate),} \\ \text{(b)} & \text{each fibre is connected,} \\ \text{(c)} & \text{no fibre is part of a line which passes through the origin.} \end{cases} \tag{2.1}$$

Note that, given any point p where φ is submersive, by shifting the origin if necessary, we can always choose a neighbourhood U of p such that these assumptions hold.

Set $V = \varphi(U)$; note that V is open. Let ℓ be a fibre of φ, i.e. $\ell = \varphi^{-1}(z)$ for some $z \in V$. Then ℓ is a timelike line. Write $\varphi = \varphi_1 + \mathrm{i}\varphi_2$. For each $p \in U$, orient $\mathcal{H}_p$ so that $\mathrm{d}\varphi_p|_{\mathcal{H}_p}$ is orientation preserving, equivalently $\{\operatorname{grad}\varphi_1, \operatorname{grad}\varphi_2\}$ is an oriented basis; then orient ℓ by choosing its unit positive tangent vector $\boldsymbol{\gamma}$ such that $\{\operatorname{grad}\varphi_1, \operatorname{grad}\varphi_2, \boldsymbol{\gamma}\}$ is an oriented basis. We can now proceed as for the Riemannian case, defining the fibre position vector to be the unique $\boldsymbol{c} \in \mathbb{R}^3$ satisfying $\langle \boldsymbol{c}, \boldsymbol{\gamma}\rangle_1 = 0$ and with endpoint on ℓ; then $\boldsymbol{c}$ is necessarily spacelike. Noting that $\mathcal{H}_p$ is spacelike, let $J^{\mathcal{H}}$ denote rotation through $+\pi/2$ on $\mathcal{H}_p$ and define the complex vector $\boldsymbol{\xi} = \boldsymbol{\xi}(z)$ by

$$\boldsymbol{\xi} = (\boldsymbol{c} + \mathrm{i}J^{\mathcal{H}}\boldsymbol{c})/|\boldsymbol{c}|_1^2 \tag{2.2}$$

where $|\boldsymbol{c}|_1^2 := \langle \boldsymbol{c}, \boldsymbol{c}\rangle_1$. On extending the inner product $\langle\ ,\ \rangle_1$ on $\mathbb{R}^3_1$ by complex-bilinearity to vectors in $\mathbb{C}^3$, the equation of ℓ can be written as a single 'complex' equation:

$$\langle \boldsymbol{\xi}(z)\,,\, \boldsymbol{x}\rangle_1 = 1, \quad \text{explicitly,} \quad -\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 = 1\,; \tag{2.3}$$

note that this is equivalent to the pair of real equations: $\langle \mathrm{Re}\,\boldsymbol{\xi}\,, \boldsymbol{x}\rangle_1 = 1$, $\langle \mathrm{Im}\,\boldsymbol{\xi}\,, \boldsymbol{x}\rangle_1 = 0$. From (2.2) we see that the complex vector $\boldsymbol{\xi}$ is *null* in the sense that

$$\langle \boldsymbol{\xi}, \boldsymbol{\xi}\rangle_1 = 0\,, \quad \text{equivalently,} \quad |\mathrm{Re}\,\boldsymbol{\xi}|_1^2 = |\mathrm{Im}\,\boldsymbol{\xi}|_1^2 \text{ and } \langle \mathrm{Re}\,\boldsymbol{\xi}, \mathrm{Im}\,\boldsymbol{\xi}\rangle_1 = 0\,. \tag{2.4}$$

Also the *Hermitian square norm* $|\boldsymbol{\xi}|_1^2 := \langle \boldsymbol{\xi}, \overline{\boldsymbol{\xi}}\rangle_1 = |\mathrm{Re}\,\boldsymbol{\xi}|_1^2 + |\mathrm{Im}\,\boldsymbol{\xi}|_1^2$ satisfies $|\boldsymbol{\xi}|_1^2 = 2/|\boldsymbol{c}|_1^2$, so that we have a one-to-one correspondence between vectors $\boldsymbol{\xi} \in \mathbb{C}^3$ which satisfy (2.4) and have positive Hermitian square norm:

$$|\boldsymbol{\xi}|_1^2 > 0 \tag{2.5}$$

and non-zero spacelike vectors $\boldsymbol{c} \in \mathbb{R}^3_1$; the inverse is given by

$$\boldsymbol{c} = 2\,\mathrm{Re}\,\boldsymbol{\xi}/|\boldsymbol{\xi}|_1^2\,, \quad \text{so that} \quad J^{\mathcal{H}}\boldsymbol{c} = 2\,\mathrm{Im}\,\boldsymbol{\xi}/|\boldsymbol{\xi}|_1^2\,.$$

Now, as z varies, so does the fibre $\ell = \varphi^{-1}(z)$, so that $z \mapsto \boldsymbol{\xi}(z)$ defines a mapping on $V = \varphi(U)$. Then, just as in [4, Lemma 1.3.3], $\boldsymbol{\xi} : V \to \mathbb{C}^3$ is holomorphic, leading to the following result.

Proposition 2.2. *Any C^2 harmonic morphism $\varphi : U \to \mathbb{C}$ from an open subset of $\mathbb{R}^3_1$ which satisfies conditions* (2.1) *is a solution $z = \varphi(\boldsymbol{x})$ to the equation* (2.3) *for some holomorphic map $\boldsymbol{\xi} : V \to \mathbb{C}^3$ from an open subset of $\mathbb{C}$ which satisfies* (2.4) *and* (2.5). □

Holomorphic mappings $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) : V \to \mathbb{C}^3$ satisfying (2.4) with $\xi_2 - \mathrm{i}\xi_3$ nowhere zero are all of the form

$$\boldsymbol{\xi} = \frac{1}{2h}\big(2g, 1 + g^2, \mathrm{i}(1 - g^2)\big), \tag{2.6}$$

where $g, h : V \to \mathbb{C}$ are holomorphic functions, with h nowhere zero, given by $g = \xi_1\big/(\xi_2 - \mathrm{i}\xi_3)$ and $h = 1\big/(\xi_2 - \mathrm{i}\xi_3)$. Then the representation (2.3) takes the form

$$-2g(z)\,x_1 + (1 + g(z)^2)\,x_2 + \mathrm{i}(1 - g(z)^2)\,x_3 = 2h(z)\,. \tag{2.7}$$

A simple calculation gives $|\boldsymbol{\xi}|_1^2 = \big(1 - |g|^2\big)^2\big/(4|h|^2)$, hence $|\boldsymbol{\xi}|_1^2 = 0$ if and only if $|g| = 1$. Now, by using equation (2.7) rather than (2.3), we can allow h to be zero; on recalling that conditions (2.1) are always satisfied locally, we obtain the following result.

Proposition 2.3. *Any C^2 submersive harmonic morphism $\varphi : U \to \mathbb{C}$ from an open subset of $\mathbb{R}^3_1$ is locally a solution $z = \varphi(\boldsymbol{x})$ to* (2.7) *for some holomorphic maps $g, h : V \to \mathbb{C}$ defined on an open subset of $\mathbb{C}$ with $|g(z)| - 1$ nowhere zero, possibly after a change of coordinates $(x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3)$.* □

Remark 2.4. (i) The change of coordinates is only necessary to avoid having $\xi_2 - \mathrm{i}\xi_3 = 0$ which would correspond to a pole of g. This case can be included if we allow g and h to be meromorphic, as in [4, Chapter 1].

(ii) The theorem shows that any C^2 *submersive* harmonic morphism defined on an open subset of $\mathbb{R}^3_1$ with values in a Riemann surface is, in fact, real analytic. This is false for degenerate harmonic morphisms, see below.

We can interpret g and h as in the Riemannian case: Let $\times$ denote the cross product in $\mathbb{R}^3_1$ given by

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = \big((a_2b_3 - a_3b_2),\, -(a_3b_1 - a_1b_3),\, -(a_1b_2 - a_2b_1)\big).$$

Then a positively oriented unit vector along the line (2.7) is given by

$$\gamma(z) = \frac{\mathrm{Re}\,\boldsymbol{\xi} \times \mathrm{Im}\,\boldsymbol{\xi}}{|\mathrm{Re}\,\boldsymbol{\xi} \times \mathrm{Im}\,\boldsymbol{\xi}|} = \frac{1}{1 - |g|^2}\left(1 + |g|^2, 2g\right), \tag{2.8}$$

so that $g(z)$ represents the direction of the fibre over z. More precisely, let $H^2 = \{\boldsymbol{x} \in \mathbb{R}^3_1 : -{x_1}^2 + {x_2}^2 + {x_3}^3 = -1\}$ denote the *hyperbola of two sheets* in $\mathbb{R}^3_1$ and let $\sigma : H^2 \to \mathbb{C} \cup \{\infty\} \setminus \{|z| = 1\}$ be stereographic projection from $(-1, 0, 0)$ given by

$$\sigma(x_1, x_2, x_3) = (x_2 + \mathrm{i}x_3)/(1 + x_1) = (x_1 - 1)/(x_2 - \mathrm{i}x_3)\,. \tag{2.9}$$

Then, as in the Riemannian case [4, Chapter 1], $g(z) = \sigma(\boldsymbol{\gamma}(z))$, and $h(z)$ represents $\boldsymbol{c}(z)$ in the chart given by σ, that is, $h(z) = \mathrm{d}\sigma_{\boldsymbol{\gamma}(z)}(\boldsymbol{c}(z))$.

Note that H^2 has two components $H^2_\pm = \{(x_1, x_2, x_3) \in H^2 : \pm x_1 > 0\}$ corresponding under stereographic projection to the two components of $\mathbb{C} \cup \{\infty\} \setminus \{|z| = 1\}$. If $|g(z)| < 1$, then $\boldsymbol{\gamma}(z) \in H^2_+$ is future-pointing, and if $|g(z)| > 1$, then $\boldsymbol{\gamma}(z) \in H^2_-$ is past-pointing.

We now obtain a converse to Proposition 2.2 as a consequence of a general construction of complex-valued harmonic morphisms due in the $\mathbb{R}^3$ case to Jacobi [12]. It is a semi-Riemannian version of [4, Theorem 9.2.1]. Let (M, g) be an arbitrary Riemannian or semi-Riemannian manifold; denote the corresponding inner product on TM (or its complex-bilinear extension to $T^cM = TM \otimes \mathbb{C}$) by $\langle\ ,\ \rangle_M$, and the Laplace–Beltrami operator by Δ^M.

Proposition 2.5. *Let A be an open subset of $M \times \mathbb{C}$ and let $G : A \to \mathbb{C}$, $(x, z) \mapsto G(x, z)$ be a C^2 mapping which is* (i) *a harmonic morphism in its first argument, i.e., for each fixed z, $x \mapsto G_z(x) := G(x, z)$ is a harmonic morphism $\big((x, z) \in A\big)$;* (ii) *holomorphic in its second argument z. Let $\varphi : U \to \mathbb{C}$ be a C^2 solution to the equation $G(x, \varphi(x)) = \mathrm{const.}$ on an open subset U of M, and suppose that* $\operatorname{grad} G_z(x, \varphi(x))$ *is non-zero on a dense subset of U. Then φ is a harmonic morphism.*

Proof: The hypothesis that G is a harmonic morphism in its first argument means that

$$\text{(a)}\quad \Delta^M G_z = 0\,, \quad \text{(b)}\quad \langle \operatorname{grad} G_z\,, \operatorname{grad} G_z\rangle_M = 0 \quad \big((x, z) \in A\big)\,. \tag{2.10}$$

To show that φ is a harmonic morphism we must show that

$$\text{(a)}\quad \Delta^M \varphi = 0\,, \quad \text{(b)}\quad \langle \operatorname{grad} \varphi\,, \operatorname{grad} \varphi\rangle_M = 0\,. \tag{2.11}$$

We do this by applying the chain rule, as follows. Let $p \in U$ be a point where $\operatorname{grad} G_z$ is non-zero. Let $(x^1, \ldots, x^m)$ be coordinates centred on p which are normal in the sense that the Christoffel symbols vanish at p. Then, on a neighbourhood of p we have $G\big(x^1, \ldots, x^m, \varphi(x^1, \ldots, x^m)\big) = \mathrm{const.}$ Differentiating this with respect to x^α $(\alpha \in \{1, \ldots, m\})$ gives

$$\frac{\partial G}{\partial z}\frac{\partial \varphi}{\partial x^\alpha} + \frac{\partial G}{\partial x^\alpha} = 0\,, \tag{2.12}$$

hence,

$$\left(\frac{\partial G}{\partial z}\right)^2 \langle \operatorname{grad} \varphi\,, \operatorname{grad} \varphi\rangle_M = \langle \operatorname{grad} G_z\,, \operatorname{grad} G_z\rangle_M\,.$$

From (2.12) and our assumption on $\operatorname{grad} G_z$ it follows that $\partial G/\partial z$ is non-zero, hence (2.11b) follows from (2.10b).

Next, we differentiate (2.12) with respect to x^β $(\beta \in \{1, \ldots, m\})$ to give

$$\frac{\partial G}{\partial z}\frac{\partial^2 \varphi}{\partial x^\alpha \partial x^\beta} + \frac{\partial^2 G}{\partial z^2}\frac{\partial \varphi}{\partial x^\alpha}\frac{\partial \varphi}{\partial x^\beta} + \frac{\partial^2 G}{\partial z \partial x^\beta}\frac{\partial \varphi}{\partial x^\alpha} + \frac{\partial^2 G}{\partial x^\alpha \partial x^\beta} = 0\,.$$

Since the coordinates are normal at p, on multiplying by $g^{\alpha\beta}$ and summing, we obtain at p,

$$\frac{\partial G}{\partial z}\Delta^M \varphi + \frac{\partial^2 G}{\partial z^2}\langle \operatorname{grad} \varphi, \operatorname{grad} \varphi\rangle_M + g^{\alpha\beta}\frac{\partial^2 G}{\partial z \partial x^\beta}\frac{\partial \varphi}{\partial x^\alpha} + \Delta^M G_z = 0\,. \tag{2.13}$$

From (2.10b) we have $g^{\alpha\beta}\dfrac{\partial G}{\partial x^\alpha}\dfrac{\partial G}{\partial x^\beta}=0$. Differentiating with respect to z (and using $g^{\beta\alpha}=g^{\alpha\beta}$) gives $g^{\alpha\beta}\dfrac{\partial^2 G}{\partial z\partial x^\beta}\dfrac{\partial G}{\partial x^\alpha}=0$. Hence, from (2.12), the third term of (2.13) vanishes; from (2.10b), so does the second, hence (2.13) reads

$$\frac{\partial G}{\partial z}\Delta^M\varphi+\Delta^M G_z=0\,,$$

and (2.11b) follows. □

We apply this to the case of interest: $M=\mathbb{R}^3_1$.

Theorem 2.6. *Let $\boldsymbol{\xi}:V\to\mathbb{C}^3$, $\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3)$ be a holomorphic map from an open subset of $\mathbb{C}$ or a Riemann surface which satisfies (2.4). Then any C^2 solution $\varphi:U\to V$, $z=\varphi(\boldsymbol{x})$ to (2.3) on an open subset U of $\mathbb{R}^3_1$ is a harmonic morphism of rank at least one everywhere. It is degenerate at the points of the fibres $\varphi^{-1}(z)$ for which $|\boldsymbol{\xi}(z)|_1^2=0$.*

Conversely, every submersive *C^2 harmonic morphism from an open subset of $\mathbb{R}^3_1$ to a Riemann surface is given this way locally, after shifting the origin if necessary.*

Proof: Set

$$G(\boldsymbol{x},z)=\langle\boldsymbol{\xi}(z)\,,\,\boldsymbol{x}\rangle_1\,. \tag{2.14}$$

Then $\operatorname{grad}G_z=\boldsymbol{\xi}(z)$, but this is non-zero at any point $z=\varphi(\boldsymbol{x})$ by (2.3). It follows from Proposition 2.5 that φ is a harmonic morphism; from (2.12) we see that $\mathrm{d}\varphi\neq 0$ at all points of U, so that φ has rank at least one everywhere.

Let $z\in V$. Suppose that $|\boldsymbol{\xi}(z)|_1^2\neq 0$. Then, $\boldsymbol{\xi}(z)\neq\mathbf{0}$ so the fibre $\varphi^{-1}(z)$ is non-empty; from (2.4) we see that $\operatorname{Re}\boldsymbol{\xi}(z)$ and $\operatorname{Im}\boldsymbol{\xi}(z)$ are spacelike, orthogonal and have non-zero norm, and φ is submersive at all points on the fibre.

Suppose instead that $|\boldsymbol{\xi}(z)|_1^2=0$. Then from (2.4), $\operatorname{Re}\boldsymbol{\xi}(z)$ and $\operatorname{Im}\boldsymbol{\xi}(z)$ are lightlike and orthogonal and so must be linearly dependent. Hence, from (2.3), the fibre $\varphi^{-1}(z)$ is non-empty if and only if $\operatorname{Re}\boldsymbol{\xi}(z)\neq\mathbf{0}$ but $\operatorname{Im}\boldsymbol{\xi}(z)=\mathbf{0}$, in which case it is the degenerate plane $<\operatorname{Re}\boldsymbol{\xi}(z),\boldsymbol{x}>_1=1$, all of whose points are degenerate points of φ.

The converse follows from Proposition 2.2. □

Remark 2.7. Given a holomorphic $\boldsymbol{\xi}:V\to\mathbb{C}^3$ which satisfies (2.4), as z varies, the lines (2.3) form a *congruence*, i.e., a two-parameter family of lines, which may or may not be a foliation. The proof, equation (2.12) and the implicit function theorem shows that there is a local C^2 solution $z=\varphi(\boldsymbol{x})$ to (2.3) though a point (p,z_0) if and only if $\partial G/\partial z\equiv\langle\boldsymbol{\xi}'(z),\boldsymbol{x}\rangle_1$ is non-zero at that point. Indeed, at such a point, the lines (2.3) form a foliation. If, on the other hand, $\partial G/\partial z=0$ at (p,z_0), then the lines (2.3) meet to first order; we call such a point an *envelope point* of the congruence.

We can give a converse to Proposition 2.3, dropping the condition $|g(z)|\neq 1$ as follows.

Corollary 2.8. *Let $g,h:V\to\mathbb{C}\cup\{\infty\}$ be holomorphic maps from an open subset of $\mathbb{C}$ (or of a Riemann surface). Then any C^2 solution $\varphi:U\to V$, $z=\varphi(x_1,x_2,x_3)$ to (2.7) is a harmonic morphism with rank at least one everywhere. Further,*

(i) *If $|g(z)|\neq 1$, then the fibre $\varphi^{-1}(z)$ is non-empty and φ is regular at all of its points.*

(ii) *If $|g(z)|=1$ and $h(z)/g(z)$ is real, then $\varphi^{-1}(z)$ is non-empty and φ is degenerate at all of its points.*

(iii) *If $|g(z)|=1$ and $h(z)/g(z)$ is not real, then $\varphi^{-1}(z)$ is empty.*

Proof: This follows from Theorem 2.6, noting that, when $|g(z)| = 1$, we have $\operatorname{Im} \boldsymbol{\xi}(z) = 0$ if and only if $\operatorname{Im}(h(z)/g(z)) = 0$. Indeed, when $|g(z)| = 1$, writing $g(z) = \mathrm{e}^{\mathrm{i}\theta(z)}$ with $\theta(z) \in \mathbb{R}$, the real and imaginary parts of (2.7) read

$$\left.\begin{array}{rcl} \cos\theta\,(-x_1 + \cos\theta\, x_2 + \sin\theta\, x_3) & = & \operatorname{Re} h \\ \sin\theta\,(-x_1 + \cos\theta\, x_2 + \sin\theta\, x_3) & = & \operatorname{Im} h \end{array}\right\} ;$$

this system has a solution if and only if $h(z) = s(z)\, e^{\mathrm{i}\theta(z)}$ for some $s(z) \in \mathbb{R}$, in which case $\varphi^{-1}(z)$ is the degenerate plane

$$-x_1 + \cos\theta(z)\, x_2 + \sin\theta(z)\, x_3 = s(z)\,. \tag{2.15}$$

□

We shall see in Corollary 4.6 that all C^2 submersive harmonic morphisms which are degenerate everywhere satisfy (2.15).

In the following examples we write $q = x_2 + \mathrm{i}x_3$.

Example 2.9. (Orthogonal projection) Define $g, h : \mathbb{C} \to \mathbb{C}$ by $g(z) = 0,\ h(z) = z/2$. Then (2.7) becomes: $q = z$. This defines the congruence of lines parallel to the x_1-axis. These lines are the fibres of the globally defined harmonic morphism $\varphi : \mathbb{R}^3_1 \to \mathbb{C}$ given by $\varphi(x_1, x_2, x_3) = x_2 + \mathrm{i}x_3$.

Example 2.10. (Radial projection) Define $g, h : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ by $g(z) = z,\ h(z) = 0$. Then (2.7) becomes

$$z^2 \overline{q} - 2z\, x_1 + q = 0\,. \tag{2.16}$$

This has solutions

$$z_{\pm} = \big(x_1 \pm \sqrt{{x_1}^2 - |q|^2}\,\big)/\overline{q}\,. \tag{2.17}$$

Note that $|z_+||z_-| = 1$. Let $C = \{(x_1, x_2, x_3) : {x_1}^2 = |q|^2\}$ denote the light cone and $U = \{(x_1, x_2, x_3) : {x_1}^2 > |q|^2\}$ its interior. Then (2.17) defines smooth solutions $z_{\pm} : U \setminus \{(x_1, 0, 0) : x_1 \in \mathbb{R}\} \to \mathbb{C}$; on setting $z_+(x_1, 0, 0) = 0$ and $z_-(x_1, 0, 0) = \infty$ these extend to smooth solutions $z_+ : U \to D^2,\ z_- : U \to \mathbb{C} \cup \{\infty\} \setminus \overline{D^2}$, where D^2 is the open unit disc. If we now put $\varphi_{\pm} = \sigma^{-1} \circ z_{\pm}$, where σ is stereographic projection (2.9), then we obtain smooth submersive harmonic morphisms $\varphi_{\pm} : U \to H^2$ defined by

$$\varphi_{\pm} = \mp \frac{1}{\sqrt{{x_1}^2 - {x_2}^2 - {x_3}^2}}\,(x_1, x_2, x_3)\,.$$

Geometrically, $\varphi_{\pm}$ is $\mp$-*radial projection* centred on the origin. Its fibres are the half-lines of U from the origin.

If, on the other hand, we restrict $z_{\pm}$ to the exterior $\overline{U}^c = \{(x_1, x_2, x_3) : {x_1}^2 < |q|^2\}$ of the light cone, then $|z_+| = |z_-| = 1$ and we obtain everywhere-degenerate harmonic morphisms $z_{\pm} : \overline{U}^c \to S^1 \subset \mathbb{C}$. The fibres of these harmonic morphisms are degenerate planes tangent to the light cone C; each point $\boldsymbol{x}$ of $\overline{U}^c$ lies on two such planes, as $\boldsymbol{x}$ approaches the light cone both of these planes tend to the tangent plane.

Example 2.11. (Disc example) Define $g, h : \mathbb{C} \to \mathbb{C}$ by $g(z) = z$, $h(z) = \mathrm{i}z$. Then (2.7) becomes

$$z^2 \overline{q} - 2z(\mathrm{i} + x_1) + q = 0\,. \tag{2.18}$$

This has solutions

$$z_{\pm} = \big(\mathrm{i} + x_1 \pm \sqrt{(\mathrm{i} + x_1)^2 - |q|^2}\,\big)/\overline{q}\,.$$

Noting that $(\mathrm{i} + x_1)^2 - |q|^2 = -1 - |\boldsymbol{x}|_1^2 + 2\mathrm{i}x_1$ never lies on the non-negative real axis, write

$$(\mathrm{i} + x_1)^2 - |q|^2 = r\mathrm{e}^{\mathrm{i}\theta} \quad (r > 0,\ 0 < \theta < 2\pi)\,;$$

then on taking $\sqrt{(\mathrm{i}+x_1)^2-|q|^2}=\sqrt{r}e^{\mathrm{i}\theta/2}$, we see that the maps $z_\pm$ are smooth on $\mathbb{R}^3_1\setminus\{(x_1,0,0)\}$. Setting $z_-(x_1,0,0)=0$, $z_+(x_1,0,0)=\infty$ extends these to smooth harmonic morphisms $z_-:\mathbb{R}^3_1\to D^2$ and $z_+:\mathbb{R}^3_1\to\mathbb{C}\cup\{\infty\}\setminus\overline{D^2}$. Note that $z_+(x_1,q)=1/z_-(x_1,\overline{q})$, $\big((x_1,q)\in\mathbb{R}^3_1\big)$. Equation (2.18) is invariant under rotations $z\mapsto e^{\mathrm{i}\theta}z$, $q\mapsto e^{\mathrm{i}\theta}q$, so that it defines a congruence of lines which is rotationally symmetric about the x_1-axis. Hence, to describe this congruence, it suffices to determine the directions of the lines through the points $(0,u,0)$ for $u>0$. At such a point,

$$z_\pm=\big(\mathrm{i}\pm\sqrt{-1-u^2}\big)/u=\mathrm{i}\big(1\pm\sqrt{1+u^2}\big)/u\,.$$

Comparing with (2.9), we see that the direction $\boldsymbol{\gamma}$ of the fibre at z is given by $\boldsymbol{\gamma}(z)=\big(\mp\sqrt{1+u^2},0,-u\big)$; this direction is perpendicular to the radius from $(0,0,0)$ to $(0,u,0)$ and inclined at an angle $\arctan\big(u/\sqrt{1+u^2}\big)$ (and pointing 'clockwise') to the negative (resp. positive) x_1-axis. As u increases from 0 to ∞, this angle increases from 0 to $\pi/4$. We thus obtain surjective submersive harmonic morphisms $z_-:\mathbb{R}^3_1\to D^2$ and $z_+:\mathbb{R}^3_1\to\mathbb{C}\cup\{\infty\}\setminus\overline{D^2}$. Composing with σ^{-1} gives surjective submersive harmonic morphisms $\varphi_-:\mathbb{R}^3_1\to H^2_+$ and $\varphi_+:\mathbb{R}^3_1\to H^2_-$.

Note that we may introduce a real parameter $t\neq 0$ and set $h(z)=\mathrm{i}tz$ (with $g(z)=z$ unchanged). This gives the same example scaled by a factor of t; as $t\to 0$, this scaled disc example tends to radial projection (Example 2.10).

Corollary 2.12. *There is a globally defined surjective submersive harmonic morphism from Minkowski 3-space* $\mathbb{M}^3=\mathbb{R}^3_1$ *to the unit disc.*

Indeed, both the disc example and orthogonal projection (Example 2.9) define harmonic morphisms *globally* on Minkowski 3-space. This is in contrast to the Riemannian case, where we established a Bernstein-type theorem [3] (see also [4, Theorem 6.7.3]) that orthogonal projection is the only globally defined harmonic morphism from $\mathbb{R}^3$ to a surface, up to postcomposition with weakly conformal maps. Globally defined harmonic morphisms from higher-dimensional Minkowski spaces can be obtained by precomposing such harmonic morphisms with orthogonal projections $\mathbb{R}^m_1\to\mathbb{R}^3_1$ for any $m>3$.

3 Harmonic morphisms from Minkowski 3-space to a Lorentz surface

We recall some facts about hyperbolic numbers. Let $\mathbb{D}=\{(x_1,x_2)\in\mathbb{R}^2\}$ equipped with the usual coordinatewise addition, but with multiplication given by

$$(x_1,x_2)\,(y_1,y_2)=(x_1y_1+x_2y_2\,,\,x_1y_2+x_2y_1)\,.$$

We call the commutative ring $\mathbb{D}$ the set of *hyperbolic* or *double numbers*. Write $\mathrm{j}=(0,1)$; then we have $(x_1,x_2)=x_1+x_2\mathrm{j}$ with $\mathrm{j}^2=1$. Note that, unlike the complex numbers, $\mathbb{D}$ has zero divisors, namely the numbers $a(1\pm\mathrm{j})$ $(a\in\mathbb{R})$. Multiplication by j defines an involution I^D on D called the *characteristic involution*, explicitly, $I^D(x_1,x_2)=(x_2,x_1)$.

For $z=x_1+x_2\mathrm{j}$, $(x_1,x_2\in\mathbb{R})$, we write $x_1=\operatorname{Re}z$, $x_2=\operatorname{Im}z$ and $\overline{z}=x_1-x_2\mathrm{j}$. We shall often identify $z\in\mathbb{D}$ with the point (x_1,x_2) in standard coordinates in Minkowski 2-space $\mathbb{M}^2=\mathbb{R}^2_1$, then the standard Minkowski square norm $|z|_1^2=\langle z,z\rangle_1=-{x_1}^2+{x_2}^2$ is given by $|z|_1^2=-z\overline{z}$.

From the chain rule, we obtain

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)\quad\text{and}\quad\frac{\partial}{\partial\overline{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right),$$

so that, in standard coordinates (x_1,x_2), the Laplacian on $\mathbb{M}^2$ is given by

$$\Delta^{\mathbb{M}^2}=-\frac{\partial^2}{\partial {x_1}^2}+\frac{\partial^2}{\partial {x_2}^2}=-4\frac{\partial^2}{\partial\overline{z}\partial z}=-4\frac{\partial^2}{\partial z\partial\overline{z}}\,.$$

By analogy with the complex numbers, we say that a C^2 map $\varphi : U \to \mathbb{D}$, $w = \varphi(z)$, from an open subset of $\mathbb{D}$ is *H-holomorphic* (resp., *H-antiholomorphic*) if we have

$$\frac{\partial w}{\partial \bar{z}} = 0 \qquad \left(\text{resp., } \frac{\partial w}{\partial z} = 0\right) ;$$

equivalently, on writing $z = x_1 + x_2\mathrm{j}$, $w = u_1 + u_2\mathrm{j}$, the map φ satisfies the *H-Cauchy-Riemann equations*:

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2} \text{ and } \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1} \qquad \left(\text{resp., } \frac{\partial u_1}{\partial x_1} = -\frac{\partial u_2}{\partial x_2} \text{ and } \frac{\partial u_1}{\partial x_1} = -\frac{\partial u_2}{\partial x_2}\right).$$

These conditions are equivalent to demanding that the differential of φ intertwine the characteristic involutions, viz., $\mathrm{d}\varphi \circ I^D = I^D \circ \mathrm{d}\varphi$ (resp., $\mathrm{d}\varphi \circ I^D = -I^D \circ \mathrm{d}\varphi$).

By a *Lorentz* surface, we mean a smooth surface equipped with a conformal equivalence class of Lorentzian metrics — here two metrics g, g' on N^2 are said to be *conformally equivalent* if $g' = \mu g$ for some (smooth) function $\mu : N^2 \to \mathbb{R} \setminus \{0\}$. Any Lorentz surface is locally conformally equivalent to 2-dimensional Minkowski space $\mathbb{M}^2$, see, for example, [4]. Let $\varphi : U \to N_1^2$ be a C^2 mapping from an open subset U of $\mathbb{R}_1^3$ to a Lorentz surface. For local considerations, we can assume that φ has values in $\mathbb{M}^2$. Then, on identifying $\mathbb{M}^2$ with the space $\mathbb{D}$ of hyperbolic numbers as above and writing $\varphi = \varphi_1 + \varphi_2\mathrm{j}$, the map φ is a harmonic morphism if and only if it satisfies equations (1.2) with $m = 3$, where now φ has values in $\mathbb{D}$.

From now on, suppose that $\varphi : U \to \mathbb{M}^2 = \mathbb{D}$ is a non-constant harmonic morphism defined on an open subset U of $\mathbb{R}_1^3$. As in the last section, by a generalization of [2], its fibres are straight lines, more precisely,

Lemma 3.1. *Let $p \in U$ be a point where φ is submersive. Then the connected component of the fibre of φ through p is a spacelike geodesic.*

To proceed, we make the assumptions (2.1) of the previous section.

Write $V = \varphi(U)$ and let ℓ be a fibre of $\varphi : U \to \mathbb{D}$, i.e. $\ell = \varphi^{-1}(z)$ for some $z \in V$. Then, in contrast to the last section, ℓ is a *spacelike* line. Now the directions of spacelike lines are parametrized by the *pseudosphere* $S_1^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -{x_1}^2 + {x_2}^2 + {x_3}^2 = 1\}$. Let ℓ have direction $\boldsymbol{\gamma} \in S_1^2 \subset \mathbb{R}_1^3$. We proceed by analogy with the last section, replacing the rotation on the horizontal space by a characteristic involution.

Let $\boldsymbol{c} \in \mathbb{R}^3$ be the unique vector which satisfies $\langle \boldsymbol{c}, \boldsymbol{\gamma} \rangle_1 = 0$ and has endpoint on ℓ; note that $\boldsymbol{c}$ can be timelike, null or spacelike. Write $\varphi = \varphi_1 + \varphi_2\mathrm{j}$. For each $\boldsymbol{x} \in U$, orient $\mathcal{H}_{\boldsymbol{x}}$ so that $\mathrm{d}\varphi_{\boldsymbol{x}}|_{\mathcal{H}_{\boldsymbol{x}}}$ is orientation preserving, equivalently, $\{\operatorname{grad}\varphi_1, \operatorname{grad}\varphi_2\}$ is an oriented basis; then orient ℓ by choosing its unit positive tangent vector $\boldsymbol{\gamma}$ such that $\{\operatorname{grad}\varphi_1, \operatorname{grad}\varphi_2, \boldsymbol{\gamma}\}$ is an oriented basis. Let $I^{\mathcal{H}}$ denote the characteristic involution in the 2-plane $\mathcal{H}_{\boldsymbol{x}}$ obtained by lifting I^D from $\mathbb{D}$, equivalently $I^{\mathcal{H}}$ interchanges $\operatorname{grad}\varphi_1$ and $\operatorname{grad}\varphi_2$. If $\boldsymbol{c}$ is non-null (spacelike or timelike), then $|\boldsymbol{c}|_1^2 \equiv \langle \boldsymbol{c}, \boldsymbol{c} \rangle_1$ is non-zero and we may define a 'hyperbolic' vector $\boldsymbol{\xi} = \boldsymbol{\xi}(z) \in \mathbb{D}^3$ by

$$\boldsymbol{\xi} = (\boldsymbol{c} + \mathrm{j}\, I^{\mathcal{H}}\boldsymbol{c})/|\boldsymbol{c}|_1^2 . \tag{3.1}$$

Then, in a way analogous to that in the last section, the equation of ℓ can be written as a single 'hyperbolic' equation:

$$\langle \boldsymbol{\xi}(z), \boldsymbol{x} \rangle_1 = 1 ; \tag{3.2}$$

this is identical to (2.3) except that the inner product $\langle\ ,\ \rangle_1$ on $\mathbb{R}_1^3$ is extended by *hyperbolic* bilinearity to $\mathbb{D}^3 = \mathbb{R}_1^3 \otimes \mathbb{D}$. In the case when $\boldsymbol{c}$ is null, this equation defines a (degenerate) plane which contains the line ℓ; we shall discuss this case below. Again, $\boldsymbol{\xi}$ is *null* in the sense that it satisfies $\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle_1 = 0$, explicitly (note the difference of sign to that in (2.4)),

$$|\mathrm{Re}\, \boldsymbol{\xi}(z)|_1^2 = -|\mathrm{Im}\, \boldsymbol{\xi}(z)|_1^2 \quad \text{and} \quad \langle \mathrm{Re}\, \boldsymbol{\xi}(z), \mathrm{Im}\, \boldsymbol{\xi}(z) \rangle_1 = 0\,. \tag{3.3}$$

The *hyperbolic square norm* $|\boldsymbol{\xi}|_1^2 := \langle \boldsymbol{\xi}, \overline{\boldsymbol{\xi}} \rangle_1 = |\mathrm{Re}\, \boldsymbol{\xi}(z)|_1^2 - |\mathrm{Im}\, \boldsymbol{\xi}(z)|_1^2$ satisfies $|\boldsymbol{\xi}|_1^2 = 2/|\boldsymbol{c}|_1^2$ where $|\boldsymbol{c}|_1^2 = \langle \boldsymbol{c}, \boldsymbol{c} \rangle_1$, so that (3.1) gives a one-to-one correspondence between $\boldsymbol{\xi} \in \mathbb{D}^3$ which satisfy $\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle_1 = 0$ and have $|\boldsymbol{\xi}|_1^2 \neq 0$ and vectors $\boldsymbol{c} \in \mathbb{R}_1^3$ which have $|\boldsymbol{c}|_1^2 \neq 0$; the inverse is given by

$$\boldsymbol{c} = 2\,\mathrm{Re}\, \boldsymbol{\xi}/|\boldsymbol{\xi}|_1^2\,, \quad \text{so that} \quad I^{\mathcal{H}} \boldsymbol{c} = 2\,\mathrm{Im}\, \boldsymbol{\xi}/|\boldsymbol{\xi}|_1^2\,.$$

As in the previous section, if $\varphi : U \to \mathbb{D}$ is a harmonic morphism satisfying assumptions (2.1), then $\boldsymbol{\xi} : V = \varphi(U) \to \mathbb{D}^3$ is H-holomorphic. Conversely, there is a version of Proposition 2.5 where $\mathbb{C}$ is replaced by $\mathbb{D}$, but now we must impose the stronger condition that $|\mathrm{grad}\, G|_1^2$ is non-zero to ensure that $\partial G/\partial z$ is not a zero divisor; applying this as before we obtain the following version of Theorem 2.6.

Theorem 3.2. *Let $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) : V \to \mathbb{D}^3$ be an H-holomorphic map from an open subset of $\mathbb{D}$ (or of a Lorentz surface) which is null: $\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle_1 = 0$ and has non-zero hyperbolic square norm $|\boldsymbol{\xi}|_1^2$ on a dense open subset of V. Then any C^2 solution $\varphi : U \to \mathbb{M}^2 = \mathbb{D}$, $z = \varphi(\boldsymbol{x})$ on an open subset of $\mathbb{R}_1^3$ to equation* (3.2) *is a harmonic morphism.*

Conversely, every C^2 submersive *harmonic morphism from an open subset of $\mathbb{R}_1^3$ to a Lorentz surface is given this way locally, after shifting the origin if necessary.* □

H-holomorphic functions $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) : V \to \mathbb{D}^3$ satisfying $\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle_1 = 0$ with $\xi_1 - \xi_2 \mathrm{j}$ not zero and not a zero divisor are all given by

$$\boldsymbol{\xi} = \frac{1}{2h(z)} \big(-(1 + g(z)^2)\,,\, \mathrm{j}(1 - g(z)^2)\,,\, -2g(z) \big)\,, \tag{3.4}$$

where $g, h : V \to \mathbb{D}$ ($h \neq 0$) are H-holomorphic functions; explicitly,

$$g = \xi_3/(\xi_1 - \xi_2 \mathrm{j}) = (\xi_1 + \xi_2 \mathrm{j})/\xi_3\,, \quad h = -1/(\xi_1 - \xi_2 \mathrm{j})\,.$$

Then the representation (3.2) takes the form

$$(1 + g(z)^2)\, x_1 + \mathrm{j}\,(1 - g(z)^2)\, x_2 - 2g(z)\, x_3 = 2h(z)\,. \tag{3.5}$$

From (3.4) we have $|\boldsymbol{\xi}|_1^2 = \big(1 - |g|^2\big)^2/(4|h|^2)$; we deduce the following from Theorem 3.2.

Corollary 3.3. *Let $g, h : V \to \mathbb{D}$ be H-holomorphic functions from an open subset of $\mathbb{D}$ (or of a Lorentz surface) with $|g(z)| \not\equiv 1$. Then any C^2 solution $\varphi : U \to V$, $z = \varphi(x_1, x_2, x_3)$, to* (3.5) *is a harmonic morphism which is not degenerate everywhere.*

Conversely, any C^2 submersive harmonic morphism φ is given locally this way, possibly after a change of coordinates. □

We can interpret g and h in a way analogous to previous cases. Indeed, let $\mathcal{K}^1 = \{(x_1, x_2, x_3) \in S_1^2 : x_3 = -1\}$ and $\mathcal{H}^1 = \{z \in \mathbb{D} : |z|^2 = -1\}$. Then we can identify $S_1^2 \setminus \mathcal{K}^1$ with $\mathbb{D} \setminus \mathcal{H}^1$ by stereographic projection $\sigma_H : (x_1, x_2, x_3) = (x_1 + x_2 \mathrm{j})/(1 + x_3)$. Then $g(z) = \sigma_H(\boldsymbol{\gamma}(z))$ and $h(z) = (\mathrm{d}\sigma_H)_{\boldsymbol{\gamma}(z)}(\boldsymbol{c}(z))$.

Example 3.4. (Orthogonal projection) Define $g, h : \mathbb{D} \to \mathbb{D}$ by $g(z) = 0$, $h(z) = z/2$. Then (3.5) becomes: $x_1 + x_2 \mathrm{j} = z$. This defines the congruence of lines parallel to the x_3-axis. These lines are the fibres of the globally defined harmonic morphism $\varphi : \mathbb{M}^3 = \mathbb{R}_1^3 \to \mathbb{M}^2 = \mathbb{D}$ given by $\varphi(x_1, x_2, x_3) = x_1 + x_2 \mathrm{j}$.

Example 3.5. (Radial projection) Define $g, h : \mathbb{D} \to \mathbb{D}$ by $g(z) = z,\ h(z) = 0$. Then (3.5) becomes:

$$z^2(x_1 - x_2\mathrm{j}) - 2z\,x_3 + (x_1 + x_2\mathrm{j}) = 0\,. \tag{3.6}$$

This can be solved on $\mathbb{R}^3_1 \setminus \{x_1 = \pm x_2\}$ to give

$$z = \frac{x_3 + \varepsilon\sqrt{-x_1^{\,2} + x_2^{\,2} + x_3^{\,2}}}{x_1 - x_2\mathrm{j}} = \frac{\big(x_3 + \varepsilon\sqrt{-x_1^{\,2} + x_2^{\,2} + x_3^{\,2}}\,\big)\big(x_1 + x_2\mathrm{j}\big)}{x_1^{\,2} - x_2^{\,2}}\,;$$

here we set $\varepsilon = \pm 1, \pm\mathrm{j}$ to get all possible square roots in $\mathbb{D}$. Note that on the *exterior* $\overline{U}^c = \{(x_1, x_2, x_3) \in \mathbb{D} : -x_1^{\,2} + x_2^{\,2} + x_3^{\,2} > 0\}$ of the light cone C, taking $\varepsilon = \pm 1$ gives two smooth harmonic morphisms $z_\pm : \overline{U}^c \setminus \{x_1 = \pm x_2\} \to \mathbb{M}^2$, which can be interpreted as compositions $z_\pm = \sigma_H \circ \varphi_\pm$, where $\varphi_\pm$ is the restriction to $\overline{U}^c \setminus \{x_1 = \pm x_2\}$ of radial projection (or its negative) $\overline{U}^c \to S^2_1$:

$$\boldsymbol{x} = (x_1, x_2, x_2) \mapsto \mp\frac{\boldsymbol{x}}{\sqrt{|\boldsymbol{x}|_1^2}} = \mp\frac{1}{\sqrt{-x_1^{\,2} + x_2^{\,2} + x_3^{\,2}}}\,(x_1, x_2, x_3)\,.$$

When $\boldsymbol{x} \in C$, (3.5) has repeated solutions z and the fibre through $\boldsymbol{x}$ is the (degenerate) tangent plane to C at that point. Note that both $\mathbb{M}^2$ and S^2_1 have conformal compactification given by a quadric Q^2_1 in $\mathbb{R}P^3$, see [4, Example 14.1.22]; as $\boldsymbol{x}$ approaches a point on C, $\varphi_\pm(\boldsymbol{x})$ tends to a point at infinity of S^2_1 in Q^2_1, and the harmonic morphism can be regarded as having values in Q^2_1.

When $\boldsymbol{x}$ lies inside the light cone there is no value of $z \in \mathbb{M}^2$ satisfying (3.5) (contrast with Example 2.10).

Alternatively, we can take $\varepsilon = \pm\mathrm{j}$ to get the other two values of the square root, in which case

$$z_\pm = \frac{x_3 \pm \big(\sqrt{-x_1^{\,2} + x_2^{\,2} + x_3^{\,2}}\big)\mathrm{j}}{x_1 - x_2\mathrm{j}} = \frac{x_1 + x_2\mathrm{j}}{x_3 \mp \big(\sqrt{-x_1^{\,2} + x_2^{\,2} + x_3^{\,2}}\big)\mathrm{j}}\,.$$

Then $|z_\pm|_1^2 = -1$ and $z_\pm$ is an everywhere-degenerate harmonic morphism $\overline{U}^c \setminus \{x_1 \pm x_2\} \to \mathbb{M}^2$ with values on the hyperbola $\mathcal{H}^1$. The fibres of these harmonic morphisms are the degenerate tangent planes to the light cone C. As $\boldsymbol{x}$ tends to a point in the set $\{x_1 = \pm x_2\}$, $z_\pm$ tends to the point at infinity on the hyperbola and we can regard $z_\pm$ as extending to an everywhere-degenerate harmonic morphism from $\overline{U}^c$ to the compactification Q^2_1 of $\mathbb{M}^2$.

Example 3.6. (Disc example) Define $g, h : \mathbb{D} \to \mathbb{D}$ by $g(z) = z$, $h(z) = z\mathrm{j}$. Then (3.5) becomes

$$z^2(x_1 - x_2\mathrm{j}) - 2z(x_3 + \mathrm{j}) + x_1 + x_2\mathrm{j} = 0\,. \tag{3.7}$$

This can be solved on $\mathbb{R}^3_1 \setminus \{x_1 = \pm x_2\}$ to give

$$z_\varepsilon = \frac{x_3 + \mathrm{j} + \varepsilon\sqrt{-x_1^{\,2} + x_2^{\,2} + x_3^{\,2} + 1 + 2x_3\mathrm{j}}}{x_1 - x_2\mathrm{j}} \qquad (\varepsilon = \pm 1, \pm\mathrm{j})\,.$$

The square root is smooth on the region W where $\eta_1 = -x_1^{\,2} + x_2^{\,2} + x_3^{\,2} + 1 + 2x_3$ and $\eta_2 = -x_1^{\,2} + x_2^{\,2} + x_3^{\,2} + 1 - 2x_3$ are both positive, this is given by $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3_1 : \big(1 - |x_3|\big)^2 - x_1^{\,2} + x_2^{\,2} > 0\}$. Then on $W \setminus \{x_1 = \pm x_2\}$ we can compute the square root to give

$$z_\varepsilon = \frac{x_3 + \mathrm{j} + \varepsilon\big\{\frac{1}{2}(\sqrt{\eta_1} + \sqrt{\eta_2}) + \frac{1}{2}(\sqrt{\eta_1} - \sqrt{\eta_2})\mathrm{j}\big\}}{x_1 - x_2\mathrm{j}} \qquad (\varepsilon = \pm 1, \pm\mathrm{j})\,.$$

In order to describe these harmonic morphisms geometrically, first take $\varepsilon = 1$. Then at a point $(x_1, x_2, x_3) = (u, 0, 0)$, with $|u| < 1$ so that it lies in W, we have $z_1 = (\mathrm{j} + \sqrt{1 - u^2})/u$ and so γ consists of multiples of the vectors $(\sqrt{1 - u^2}, 1, 0)$; it is easily seen that the fibres of z_1 are tangent to the hyperbola: $x_1^{\,2} - x_2^{\,2} = 1$, $x_3 = 0$. As x_3 increases from 0, the lines start tilting.

With $\varepsilon = \mathrm{j}$, we find that, at $(x_1, x_2, 0)$,

$$z_{\mathrm{j}} = \frac{\mathrm{j} + \left(\sqrt{1 - x_1^2 + x_2^2}\right)\mathrm{j}}{x_1 - x_2\mathrm{j}}$$

and $\boldsymbol{\gamma}$ consists of multiples of the vectors $(x_2, x_1, -\sqrt{1 - r^2})$ if $x_1^2 > x_2^2$, and $(x_2, x_1, -1)$ if $x_1^2 < x_2^2$, where $r^2 = -x_1^2 + x_2^2$. Thus, at any point $P(x_1, x_2, 0)$, the fibre is perpendicular in a Lorentzian sense to the radius OP; as P travels along the radius from O, it starts vertically down and then swivels until it is horizontal *either* when it hits the hyperbola: $x_1^2 - x_2^2 = 1$, $x_3 = 0$ (i.e. if $x_1^2 > x_2^2$), *or*, if it avoids the hyperbola (i.e. if $x_1^2 < x_2^2$), at infinity. It is thus a hyperbolic analogue of the disc example that occurs in the Riemannian case [4, Example 1.5.3]. Note that since (3.7) is invariant under the change of coordinates $(x_1, x_2, x_3, z) \mapsto (x_1, -x_2, x_3, 1/z)$, the cases $\varepsilon = -1, -\mathrm{j}$ are equivalent to the above cases.

Note that, as in Example 2.11 we may introduce a real parameter $t \neq 0$ and set $h = tz\mathrm{j}$ (with $g = z$ unchanged); this gives the same example scaled by a factor of t. Again, as $t \to 0$, this scaled disc example tends to radial projection (Example 3.5).

4 Degenerate harmonic morphisms on Minkowski spaces

By definition (see the Introduction), a C^1 horizontally weakly conformal map is degenerate at a point x if and only if the kernel of $\mathrm{d}\varphi_x$ is degenerate. It follows [4, Remark 14.5.5] that an everywhere-degenerate harmonic morphism φ from a Lorentzian manifold M_1^m to an arbitrary semi-Riemannian manifold N necessarily has rank one everywhere; further, by [4, Proposition 14.5.8], it factors locally into the composition of an everywhere-degenerate harmonic morphism from M_1^m to $\mathbb{R}$ and an immersion of $\mathbb{R}$ into N. Hence, to determine all such φ, it suffices to take $N = \mathbb{R}$. In the case that M_1^m is an open subset U of m-dimensional Minkowski space $\mathbb{M}^m = \mathbb{R}_1^m$, an everywhere-degenerate harmonic morphism is just a null *real-valued* solution of the wave equation, i.e. a solution $\varphi : U \to \mathbb{R}$ of the system (1.2).

To solve this problem, we need the following version of Proposition 2.5; note that it is empty if M is Riemannian. As the proof uses the same calculations, we omit it.

Proposition 4.1. *Let M be an arbitrary semi-Riemannian manifold. Let A be an open subset of $M \times \mathbb{R}$ and let $G : A \to \mathbb{R}$, $(x, z) \mapsto G(x, z)$, be a C^2 mapping which is an everywhere-degenerate harmonic morphism in its first argument, i.e, writing $G_z(x) = G(x, z)$,*

$$\text{(a)} \quad \Delta^M G_z = 0\,, \quad \text{(b)} \quad \langle \operatorname{grad} G_z\,, \operatorname{grad} G_z \rangle_M = 0 \quad \big((x, z) \in A\big)\,. \tag{4.1}$$

Let $\varphi : U \to \mathbb{C}$ be a C^2 solution to equation $G(x, \varphi(x)) = \text{const.}$ on an open subset U of M and suppose that $\operatorname{grad} G_z(x, \varphi(x))$ is non-zero on a dense subset of U. Then φ is an everywhere-degenerate harmonic morphism, i.e., it satisfies the system

$$\text{(a)} \quad \Delta^M \varphi = 0\,, \quad \text{(b)} \quad \langle \operatorname{grad} \varphi\,, \operatorname{grad} \varphi \rangle_M = 0\,. \tag{4.2}$$

□

In the Lorentzian case this gives

Lemma 4.2. *Let $\varphi(x_1, x_2, \ldots, x_m)$ satisfy*

$$\tau\big(\varphi(x_1, x_2, \ldots, x_m),\, x_2,\, \ldots, x_m\big) = x_1\,. \tag{4.3}$$

Then φ satisfies the system

$$\text{(a)} \quad \Box \varphi = 0\,, \quad \text{(b)} \quad \langle \operatorname{grad} \varphi\,, \operatorname{grad} \varphi \rangle_1 = 0 \tag{4.4}$$

if and only if, for each fixed x_1, τ satisfies the system

$$\text{(a)}\quad \Delta^{\mathbb{R}^{m-1}}\tau = 0\,, \quad \text{(b)}\quad \langle \operatorname{grad}\tau\,,\, \operatorname{grad}\tau\rangle_{\mathbb{R}^{m-1}} = 1\,; \tag{4.5}$$

that is, φ is a null solution to the wave equation if and only if, on each slice $x_1 = \text{const.}$, τ is a harmonic function with $|\operatorname{grad}\tau|^2 = 1$.

Proof: Set $G(\varphi, x_1, x_2, \dots, x_m) = \tau(\varphi, x_2, \dots, x_m) - x_1$. Then

$$\left(\frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \dots, \frac{\partial G}{\partial x_m}\right) = \left(-1, \frac{\partial \tau}{\partial x_2}, \dots, \frac{\partial \tau}{\partial x_m}\right)$$

so that

$$\begin{aligned}\langle \operatorname{grad} G\,,\, \operatorname{grad} G\rangle_1 &= \langle \operatorname{grad}\tau\,,\, \operatorname{grad}\tau\rangle_{\mathbb{R}^{m-1}} - 1 \quad \text{and}\\ \Box G \equiv \Delta^{\mathbb{M}^m} G &= \Delta^{\mathbb{R}^{m-1}}\tau\,.\end{aligned}$$

The result follows. □

Solutions of the system (4.5) are easy to find, as follows.

Lemma 4.3. *Any C^2 solution $\varphi : U \to \mathbb{R}$ on an open subset of $\mathbb{R}^{m-1}$ to the system* (4.5) *is affine, i.e.,*

$$\tau(x_2, \dots, x_m) = \ell_1 + \sum_{i=2}^{m} \ell_i x_i \tag{4.6}$$

for some constants $\ell_1, \ell_2, \dots, \ell_m$ with $\sum_{i=2}^m {\ell_i}^2 = 1$.

Proof: Since τ is harmonic, it is smooth. Set $T = \operatorname{grad}\tau : U \to \mathbb{R}^m$. Then T is harmonic and has image in the unit sphere. By the maximum principle, T is constant. Indeed, choose any point $p \in U$ and set $\ell = T(p)$. Then the function $\boldsymbol{x} \mapsto \langle T(\boldsymbol{x}), \ell\rangle$ is harmonic and has a maximum at p and so is constant. Integrating yields (4.6). □

We deduce the following result.

Theorem 4.4. (Collins [6]) *Let $\varphi : U \to \mathbb{R}$ be a null C^2 solution to the wave equation, i.e. a solution to* (4.4)*, on an open set of $\mathbb{M}^m$. Suppose that $\partial\varphi/\partial x_1 \neq 0$. Then, locally, $z = \varphi(x_1, \dots, x_m)$ satisfies*

$$\ell_1(z) + \sum_{i=2}^{m} \ell_i(z) x_i = x_1 \tag{4.7}$$

for some C^2 functions $\ell_1, \ell_2, \dots, \ell_m : V \to \mathbb{R}$ defined on an open subset of $\mathbb{R}$ with $\sum_{i=2}^m {\ell_i}^2 = 1$.

Conversely, any C^2 solution to (4.7) *is a null solution to the wave equation.*

Proof: By the implicit function theorem we can solve $\varphi(x_1, x_2, \dots, x_m) = z$ to give

$$x_1 = \tau(z, x_2, \dots, x_m). \tag{4.8}$$

Then, by Lemma 4.2, on each slice $x_1 = \text{const.}$, τ satisfies (4.5). By Lemma 4.3, $\tau|_{x_1=\text{const.}}$ is affine, thus,

$$\tau(z, x_2, \dots, x_m) = \ell_1(z) + \sum_{i=2}^{m} \ell_i(z) x_i$$

with $\sum_{i=2}^m {\ell_i}^2 = 1$. Then (4.8) yields (4.7). □

Corollary 4.5. *The level sets of a C^2 null solution to the wave equation are degenerate hyperplanes.* □

Corollary 4.6. *Any C^2 harmonic morphism $\varphi : U \to \mathbb{R}$, $z = \varphi(x_1, x_2, x_3)$ from an open subset of $\mathbb{M}^3 = \mathbb{R}^3_1$ which is submersive and degenerate everywhere is locally the solution to an equation*

$$-x_1 + \cos\theta(z)\, x_2 + \sin\theta(z)\, x_3 = r(z)\,, \tag{4.9}$$

for some C^2 functions $\theta, r : V \to \mathbb{R}$ defined on an open subset of $\mathbb{R}$.

Conversely, any C^2 solution to this equation on an open subset of $\mathbb{R}^3_1$ is a harmonic morphism which is degenerate everywhere. □

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