Harmonic morphisms from Minkowski space and hyperbolic numbers
by
Paul Baird and John C. Wood*

To Professor S. Ianuș on the occasion of his 70th Birthday

Abstract

We show that all harmonic morphisms from 3-dimensional Minkowski space with values in a surface have a Weierstrass representation involving the complex numbers or the hyperbolic numbers depending on the signature of the codomain. We deduce that there is a non-trivial globally defined submersive harmonic morphism from Minkowski 3-space to a surface, in contrast to the Riemannian case. We show that a degenerate harmonic morphism on a Minkowski space is precisely a null real-valued solution to the wave equation, and we find all such.

Key Words: harmonic morphism, harmonic map, wave equation, hyperbolic number

2000 Mathematics Subject Classification: Primary 58E20, Secondary 53C43.

1 Introduction

A $C^2$ map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds is called a harmonic morphism if, for every harmonic function $f : V \to \mathbb{R}$ from an open subset $V$ of $N$ with $\varphi^{-1}(V)$ non-empty, the composition $f \circ \varphi : \varphi^{-1}(V) \to \mathbb{R}$ is harmonic. It is a fundamental result of Fuglede and Ishihara [7, 10], that $\varphi$ is a harmonic morphism if and only if it is both a harmonic map and horizontally weakly conformal. If we allow the metrics $g$ and $h$ to be indefinite, the situation becomes more subtle due to the three possible types of tangent vector that can occur: spacelike, timelike or null. However, provided sufficient care is taken over the definitions, the same characterization applies [8, 4]. In this more general setting, we say that a $C^1$-map $\varphi : (M, g) \to (N, h)$ between semi-Riemannian manifolds is horizontally (weakly) conformal or semiconformal at $x \in M$ with square dilation $\Lambda(x)$ if

$$g(d\varphi^*_x(U), d\varphi^*_x(V)) = \Lambda(x) h(U, V) \quad (U, V \in T_{\varphi(x)}N)$$

for some $\Lambda(x) \in \mathbb{R}$, where $d\varphi^*_x : T_{\varphi(x)}N \to T_xM$ denotes the adjoint of $d\varphi_x$. If $\varphi$ is horizontally weakly conformal at every point, then we shall simply say that $\varphi$ is horizontally weakly conformal. Note that, contrary to the Riemannian case, the function $\Lambda : M \to \mathbb{R}$ can take on nonpositive values. In fact, recall that a subspace $W$ of $T_xM$ is called degenerate if there exists a non-zero vector $v \in W$ such that $g(v, w) = 0$ for all $w \in W$, and null if $g(v, w) = 0$ for all $v, w \in W$; then we have three types of points, as follows (see [4] Proposition 14.5.4).

*JCW thanks the Gulbenkian foundation, CMAF and the Faculdade de Ciências, Universidade de Lisboa, for support and hospitality during part of the preparation of this work
Proposition 1.1. Let \( \varphi : (M, g) \to (N, h) \) be a \( C^1 \) horizontally weakly conformal map. Then, for each \( x \in M \), precisely one of the following holds:

(i) \( d\varphi_x = 0 \), thus \( d\varphi \) has rank 0 at \( x \);

(ii) \( \Lambda(x) \neq 0 \). Then \( \varphi \) is submersive at \( x \) and \( d\varphi_x \) maps the horizontal space \( H_x := (\ker d\varphi_x) ^\perp \) conformally onto \( T_{\varphi(x)} N \) with square conformality factor \( \Lambda(x) \), i.e., \( h(d\varphi_x(X), d\varphi_x(Y)) = \Lambda(x) g(X, Y) \) \( (X, Y \in H_x) \), we call \( x \) a regular point of \( \varphi \);

(iii) \( \Lambda(x) = 0 \) but \( d\varphi_x \neq 0 \). Then the vertical space \( V_x := \ker d\varphi_x \) is degenerate and \( H_x \subseteq V_x \); equivalently, \( H_x \) is null and non-zero. We say that \( x \) is a degenerate point of \( \varphi \), or that \( \varphi \) is degenerate at \( x \).

We call \( \varphi \) non-degenerate if it has no degenerate points, i.e., all points are of type (i) or (ii) above; this is always the case when the domain is Riemannian. Points that are not regular, i.e. points of type (i) or (iii), are called critical points.

Recall that a \( C^2 \) map \( \varphi : (M, g) \to (N, h) \) is harmonic if it satisfies the harmonicity equation \( \tau(\varphi) = 0 \) where \( \tau(\varphi) = \text{Tr} \nabla d\varphi \) is the tension field of \( \varphi \), see [H] Chapters 3 and 14 for an account adapted to our needs. When the domain is of Riemannian signature, the harmonicity equation is elliptic; in particular, for maps between Euclidean spaces, it is Laplace’s equation. On the other hand, when \( (M, g) \) is of Lorentzian signature, the harmonicity equation is hyperbolic. In particular, recall that \( m \)-dimensional Minkowski space \( M^m = \mathbb{R}^m \) is defined to be \( \mathbb{R}^m \) endowed with the metric of signature \( (1, m-1) \) given in standard coordinates \((x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \) by \( g = -dx_1^2 + dx_2^2 + \ldots + dx_m^2 \). Then a map \( \varphi : M^m \to \mathbb{R} \) or \( \mathbb{C} \) is harmonic if and only if it satisfies the wave equation \((1.2)\) below.

Harmonic morphisms to surfaces are particularly nice; from the definition it is clear that the composition of such a map with a conformal or weakly conformal map of surfaces is again a harmonic morphism. In particular, the concept of harmonic morphism depends only on the conformal class of the metric on the surface; hence, when it is of Riemannian signature and oriented, we can take it to be a Riemann surface. A map \( \varphi : M^m \to N^2 \) from Minkowski \( m \)-space to a Riemann surface is a harmonic morphism if and only if, in any local complex coordinate system on \( N^2 \), it satisfies

\[
\begin{cases}
(a) & \Box \varphi = \frac{\partial^2 \varphi}{\partial x_i^2} + \sum_{i=2}^m \frac{\partial^2 \varphi}{\partial x_i \partial x_i} = 0, \\
(b) & \langle \text{grad} \varphi, \text{grad} \varphi \rangle_1 = -\left( \frac{\partial \varphi}{\partial x_1} \right)^2 + \sum_{i=2}^m \left( \frac{\partial \varphi}{\partial x_i} \right)^2 = 0,
\end{cases}
\]

for \((x_1, \ldots, x_m) \in U\); the second equation being the condition of horizontal weak conformality. Here \( \langle , \rangle_1 \) denotes the standard Lorentzian inner product defined for \( v = (v_1, v_2, \ldots, v_m),\ w = (w_1, w_2, \ldots, w_m) \in \mathbb{R}^m \) by

\[
\langle v, w \rangle_1 = -v_1 w_1 + v_2 w_2 + \ldots + v_m w_m.
\]

Harmonic morphisms from domains of Euclidean 3-space into a Riemann surface have a particularly elegant description in terms of holomorphic data [3] which we called a Weierstrass representation as the data coincides with that well-known representation of minimal surface in \( \mathbb{R}^3 \). More precisely, the fibres of a harmonic morphism \( \varphi : U \to N^2 \) from a domain \( U \) of \( \mathbb{R}^3 \) with values in a Riemann surface form a foliation by line segments which determines a holomorphic curve in the mini-twistor space of all lines in \( \mathbb{R}^3 \), a complex space. Conversely, such a curve determines a foliation by line segments, and so a harmonic morphism, on some open subset of \( \mathbb{R}^3 \). A detailed account of this correspondence is given in [H] Chapter 1.

From the Weierstrass representation and some geometrical arguments, one can deduce a Bernstein Theorem that the only harmonic morphism defined globally on \( \mathbb{R}^3 \) with values in a surface is orthogonal projection onto a two-dimensional subspace, followed by a weakly conformal map [3].
2 Harmonic morphisms from Minkowski 3-space to a Riemann surface

We begin by characterizing those submersive (and so non-degenerate) harmonic morphisms defined on open subsets of Minkowski 3-space $M^3 = \mathbb{R}^3$ with values in a Riemann surface. All manifolds and tensors defined on them are assumed to be smooth ($C^\infty$).

Let $\varphi : U \to \mathbb{C}$ be a $C^2$ mapping from an open subset $U$ of $\mathbb{R}^3$ onto a 2-dimensional Riemannian manifold. Let $(u, v)$ be isothermal coordinates on a domain of $N^2$; then $u + iv$ gives a local complex coordinate with respect to which we write $\varphi(x_1, x_2, x_3) = \varphi_1(x_1, x_2, x_3) + i\varphi_2(x_1, x_2, x_3)$. Then, $\varphi$ is a harmonic morphism if and only if it satisfies the pair of equations (1.2) with $\ell = 1$; (2.3)

\[
\begin{align*}
\varphi \text{ is submersive on } U \text{ (and so non-degenerate)}, \\
\text{each fibre is connected}, \\
\text{no fibre is part of a line which passes through the origin}. 
\end{align*}
\]

Note that, given any point $p$ where $\varphi$ is submersive, by shifting the origin if necessary, we can always choose a neighbourhood $U$ of $p$ such that these assumptions hold.

Set $V = \varphi(U)$; note that $V$ is open. Let $\ell$ be a fibre of $\varphi$, i.e. $\ell = \varphi^{-1}(z)$ for some $z \in V$. Then $\ell$ is a timelike line. Write $\varphi = \varphi_1 + i\varphi_2$. For each $p \in U$, orient $\mathcal{H}_p$ so that $d\varphi_p|_{\mathcal{H}_p}$ is orientation preserving, equivalently $\{\text{grad} \varphi_1, \text{grad} \varphi_2\}$ is an oriented basis; then orient $\ell$ by choosing its unit positive tangent vector $\gamma$ such that $\{\text{grad} \varphi_1, \text{grad} \varphi_2, \gamma\}$ is an oriented basis. We can now proceed as for the Riemannian case, defining the fibre position vector to be the unique $c \in \mathbb{R}^3$ satisfying $(c, \gamma)_1 = 0$ and with endpoint on $\ell$; then $c$ is necessarily spacelike. Noting that $\mathcal{H}_p$ is spacelike, let $J^N$ denote rotation through $+\pi/2$ on $\mathcal{H}_p$ and define the complex vector $\xi = \xi(z)$ by

\[ \xi = (c + iJ^Nc)/|c|^2 \]  

where $|c|^2 := (c, c)_1$. On extending the inner product $(\ , \ )_1$ on $\mathbb{R}^3$ by complex-bilinearity to vectors in $\mathbb{C}^3$, the equation of $\ell$ can be written as a single ‘complex’ equation:

\[ (\xi(z), x)_1 = 1, \]  

explicitly,

\[ -\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 = 1; \]
note that this is equivalent to the pair of real equations: \(\langle \text{Re}\,\xi, x\rangle_1 = 1, \text{ Re}\,\xi, x\rangle_1 = 0\). From (2.2) we see that the complex vector \(\xi\) is null in the sense that

\[
\langle \xi, \xi \rangle_1 = 0, \quad \text{equivalently, } |\text{Re}\,\xi|^2 = |\text{Im}\,\xi|^2 \text{ and } (\text{Re}\,\xi, \text{Im}\,\xi)_1 = 0.
\]

(2.4)

Also the Hermitian square norm \(|\xi|^2 := \langle \xi, \overline{\xi} \rangle_1 = |\text{Re}\,\xi|^2 + |\text{Im}\,\xi|^2\) satisfies \(|\xi|^2 = 2/|c|^2\), so that we have a one-to-one correspondence between vectors \(\xi \in \mathbb{C}^3\) which satisfy (2.4) and have positive Hermitian square norm:

\[
|\xi|^2 > 0
\]

(2.5)

and non-zero spacelike vectors \(c \in \mathbb{R}^3_1\); the inverse is given by

\[
c = 2\text{Re}\,\xi/|\xi|^2, \quad \text{so that } J_H^n c = 2\text{Im}\,\xi/|\xi|^2.
\]

Now, as \(z\) varies, so does the fibre \(\ell = \varphi^{-1}(z)\), so that \(z \mapsto \xi(z)\) defines a mapping on \(V = \varphi(U)\). Then, just as in [1] Lemma 1.3.3, \(\xi : V \to \mathbb{C}^3\) is holomorphic, leading to the following result.

**Proposition 2.2.** Any \(C^2\) harmonic morphism \(\varphi : U \to \mathbb{C}\) from an open subset of \(\mathbb{R}^3_1\) which satisfies conditions (2.1) is a solution \(z = \varphi(x)\) to the equation (2.3) for some holomorphic map \(\xi : V \to \mathbb{C}^3\) from an open subset of \(\mathbb{C}\) which satisfies (2.4) and (2.5).

Holomorphic mappings \(\xi = (\xi_1, \xi_2, \xi_3) : V \to \mathbb{C}^3\) satisfying (2.4) with \(\xi_2 - i\xi_3\) nowhere zero are all of the form

\[
\xi = \frac{1}{2h}(2g, 1 + g^2, i(1 - g^2)),
\]

(6)

where \(g, h : V \to \mathbb{C}\) are holomorphic functions, with \(h\) nowhere zero, given by \(g = \xi_1/(\xi_2 - i\xi_3)\) and \(h = 1/(\xi_2 - i\xi_3)\). Then the representation (2.5) takes the form

\[
-2g(x) x_1 + (1 + g(x)^2) x_2 + i(1 - g(x)^2) x_3 = 2h(x).
\]

(2.7)

A simple calculation gives \(|\xi|^2 = (1 - |g|^2)^2/(4h^2)\), hence \(|\xi|^2 = 0\) if and only if \(|g| = 1\). Now, by using equation (2.7) rather than (2.3), we can allow \(h\) to be zero; on recalling that conditions (2.1) are always satisfied locally, we obtain the following result.

**Proposition 2.3.** Any \(C^2\) submersive harmonic morphism \(\varphi : U \to \mathbb{C}\) from an open subset of \(\mathbb{R}^3_1\) is locally a solution \(z = \varphi(x)\) to (2.7) for some holomorphic maps \(g, h : V \to \mathbb{C}\) defined on an open subset of \(\mathbb{C}\) with \(|g(z)| = 1\) nowhere zero, possibly after a change of coordinates \((x_1, x_2, x_3) \to (x_1 - x_2, -x_3)\).

**Remark 2.4.** (i) The change of coordinates is only necessary to avoid having \(\xi_2 - i\xi_3 = 0\) which would correspond to a pole of \(g\). This case can be included if we allow \(g\) and \(h\) to be meromorphic, as in [1] Chapter 1.

(ii) The theorem shows that any \(C^2\) submersive harmonic morphism defined on an open subset of \(\mathbb{R}^3_1\) with values in a Riemann surface is, in fact, real analytic. This is false for degenerate harmonic morphisms, see below.

We can interpret \(g\) and \(h\) as in the Riemannian case: Let \(\times\) denote the cross product in \(\mathbb{R}^3_1\) given by

\[
(a_1, a_2, a_3) \times (b_1, b_2, b_3) = ((a_2 b_3 - a_3 b_2), -(a_3 b_1 - a_1 b_3), -(a_1 b_2 - a_2 b_1)).
\]

Then a positively oriented unit vector along the line (2.7) is given by

\[
\gamma(z) = \frac{\text{Re}\,\xi \times \text{Im}\,\xi}{|\text{Re}\,\xi \times \text{Im}\,\xi|} = \frac{1}{1 - |g|^2} \left(1 + |g|^2, 2g\right).
\]

(2.8)
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so that \( g(z) \) represents the direction of the fibre over \( z \). More precisely, let \( H^2 = \{ x \in \mathbb{R}^3_1 : -x^2 + x^2 + x^3 = -1 \} \) denote the \textit{hyperbola of two sheets} in \( \mathbb{R}^3_1 \) and let \( \sigma : H^2 \to \mathbb{C} \cup \{ \infty \} \setminus \{ |z| = 1 \} \) be stereographic projection from \((-1,0,0)\) given by

\[
\sigma(x_1, x_2, x_3) = (x_2 + i x_3)/(1 + x_1) = (x_1 - 1)/(x_2 - i x_3).
\]

(2.9)

Then, as in the Riemannian case \cite[Chapter 1]{H} \( g \) given by \( R^\text{valued harmonic morphisms due in the stereographic projection to the two components of } p \mid \text{is future-pointing, and if } M, g \text{ corresponding inner product on } \text{Harmonic morphisms from Minkowski space } g \text{ so that } x \text{ and the Laplace–Beltrami operator by } \Delta^M \text{ From (2.12) and our assumption on grad } a \text{ Symbols vanish at } \Delta^M G\).

\[
\text{To show that } \phi \text{ is a harmonic morphism we must show that (a) } \Delta^M \phi = 0, \quad \text{ and (b) } \langle \text{grad } \phi, \text{ grad } \phi \rangle_M = 0 \quad ((x, z) \in A). \quad (2.10)
\]

We do this by applying the chain rule, as follows. Let \( p \in U \) be a point where \( \text{grad } G \) is non-zero. Let \( (x^1, \ldots, x^m) \) be coordinates centred on \( p \) which are normal in the sense that the Christoffel symbols vanish at \( p \). Then, on a neighbourhood of \( p \) we have \( G(x^1, \ldots, x^m, \phi(x^1, \ldots, x^m)) = \text{const.} \)

Differentiating this with respect to \( x^\alpha (\alpha \in \{ 1, \ldots, m \}) \) gives

\[
\frac{\partial G}{\partial z} \frac{\partial \phi}{\partial x^\alpha} + \frac{\partial G}{\partial x^\alpha} = 0,
\]

(2.12)

From (2.12) and our assumption on grad \( G \) it follows that \( \partial G/\partial z \) is non-zero, hence (2.11) follows from (2.10).

Next, we differentiate (2.12) with respect to \( x^\beta (\beta \in \{ 1, \ldots, m \}) \) to give

\[
\frac{\partial G}{\partial z} \frac{\partial^2 \phi}{\partial z \partial x^\beta} + \frac{\partial^2 G}{\partial z ^2} \frac{\partial \phi}{\partial x^\beta} + \frac{\partial^2 G}{\partial z \partial x^\beta} \frac{\partial \phi}{\partial x^\alpha} + \frac{\partial^2 G}{\partial x^\alpha \partial x^\beta} = 0.
\]

Since the coordinates are normal at \( p \), on multiplying by \( g^{\alpha \beta} \) and summing, we obtain at \( p \),

\[
\frac{\partial G}{\partial z} \Delta^M \phi + \frac{\partial^2 G}{\partial z ^2} \langle \text{grad } \phi, \text{ grad } \phi \rangle_M + g^{\alpha \beta} \frac{\partial^2 G}{\partial z \partial x^\beta} \frac{\partial \phi}{\partial x^\alpha} + \Delta^M G = 0.
\]

(2.13)
From (2.10b) we have $g^{\alpha\beta} \frac{\partial G}{\partial x^\alpha} \frac{\partial G}{\partial x^\beta} = 0$. Differentiating with respect to $z$ (and using $g^{\alpha\beta} = g^{\beta\alpha}$) gives $g^{\alpha\beta} \frac{\partial^2 G}{\partial x^\alpha \partial x^\beta} = 0$. Hence, from (2.12), the third term of (2.13) vanishes; from (2.10b), so does the second, hence (2.13) reads
\[
\frac{\partial G}{\partial z} \Delta^M \varphi + \Delta^M G_z = 0,
\]
and (2.11b) follows. \qed

We apply this to the case of interest: $M = \mathbb{R}_3^1$.

**Theorem 2.6.** Let $\xi : V \to \mathbb{C}^3$, $\xi = (\xi_1, \xi_2, \xi_3)$ be a holomorphic map from an open subset of $\mathbb{C}$ or a Riemann surface which satisfies (2.14). Then any $C^2$ solution $\varphi : U \to V$, $z = \varphi(x)$ to (2.3) on an open subset $U$ of $\mathbb{R}_3^1$ is a harmonic morphism of rank at least one everywhere. It is degenerate at the points of the fibres $\varphi^{-1}(z)$ for which $|\xi(z)|^2 = 0$.

Conversely, every submersive $C^2$ harmonic morphism from an open subset of $\mathbb{R}_3^1$ to a Riemann surface is given this way locally, after shifting the origin if necessary.

**Proof:** Set
\[
G(x, z) = (\xi(z), x)_1.
\] (2.14)
Then $G_z = \xi(z)$, but this is non-zero at any point $z = \varphi(x)$ by (2.3). It follows from Proposition 2.5 that $\varphi$ is a harmonic morphism; from (2.12) we see that $d\varphi \neq 0$ at all points of $U$, so that $\varphi$ has rank at least one everywhere.

Let $z \in V$. Suppose that $|\xi(z)|^2 \neq 0$. Then, $\xi(z) \neq 0$ so the fibre $\varphi^{-1}(z)$ is non-empty; from (2.3) we see that $\text{Re} \xi(z)$ and $\text{Im} \xi(z)$ are spacelike, orthogonal and have non-zero norm, and $\varphi$ is submersive at all points on the fibre.

Suppose instead that $|\xi(z)|^2 = 0$. Then from (2.3), $\text{Re} \xi(z)$ and $\text{Im} \xi(z)$ are lightlike and orthogonal and so must be linearly dependent. Hence, from (2.3), the fibre $\varphi^{-1}(z)$ is non-empty if and only if $\text{Re} \xi(z) \neq 0$ but $\text{Im} \xi(z) = 0$, in which case it is the degenerate plane $\text{Re} \xi(z), x_1 = 1$, all of whose points are degenerate points of $\varphi$.

The converse follows from Proposition 2.2. \qed

**Remark 2.7.** Given a holomorphic $\xi : V \to \mathbb{C}^3$ which satisfies (2.14), as $z$ varies, the lines (2.3) form a congruence, i.e., a two-parameter family of lines, which may or may not be a foliation. The proof, equation (2.12) and the implicit function theorem shows that there is a local $C^2$ solution $z = \xi(x)$ to (2.3) though a point $(p, z_0)$ if and only if $\partial G/\partial z \equiv (\xi'(z), x)_1$ is non-zero at that point. Indeed, at such a point, the lines (2.3) form a foliation. If, on the other hand, $\partial G/\partial z = 0$ at $(p, z_0)$, then the lines (2.3) meet to first order; we call such a point an envelope point of the congruence.

We can give a converse to Proposition 2.3 dropping the condition $|g(z)| \neq 1$ as follows.

**Corollary 2.8.** Let $g, h : V \to \mathbb{C} \cup \{\infty\}$ be holomorphic maps from an open subset of $\mathbb{C}$ (or of a Riemann surface). Then any $C^2$ solution $\varphi : U \to V$, $z = \varphi(x_1, x_2, x_3)$ to (2.7) is a harmonic morphism with rank at least one everywhere. Further,

(i) If $|g(z)| \neq 1$, then the fibre $\varphi^{-1}(z)$ is non-empty and $\varphi$ is regular at all of its points.

(ii) If $|g(z)| = 1$ and $h(z)/g(z)$ is real, then $\varphi^{-1}(z)$ is non-empty and $\varphi$ is degenerate at all of its points.

(iii) If $|g(z)| = 1$ and $h(z)/g(z)$ is not real, then $\varphi^{-1}(z)$ is empty.
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**Proof:** This follows from Theorem 2.9 noting that, when \(|g(z)| = 1\), we have \(\text{Im} \xi (z) = 0\) if and only if \(\text{Im} \left( h(z)/g(z) \right) = 0\). Indeed, when \(|g(z)| = 1\), writing \(g(z) = e^{i\theta(z)}\) with \(\theta(z) \in \mathbb{R}\), the real and imaginary parts of (2.7) read

\[
\begin{align*}
\cos \theta (-x_1 + \cos \theta x_2 + \sin \theta x_3) &= \text{Re} \, h \\
\sin \theta (-x_1 + \cos \theta x_2 + \sin \theta x_3) &= \text{Im} \, h
\end{align*}
\]

this system has a solution if and only if \(h(z) = s(z) \, e^{i\theta(z)}\) for some \(s(z) \in \mathbb{R}\), in which case \(\varphi^{-1}(z)\) is the degenerate plane

\[-x_1 + \cos \theta(z) x_2 + \sin \theta(z) x_3 = s(z)\, .
\] (2.15)

We shall see in Corollary 4.6 that all \(C^2\) submersive harmonic morphisms which are degenerate everywhere satisfy (2.15).

In the following examples we write \(q = x_2 + i x_3\).

**Example 2.9.** (Orthogonal projection) Define \(g, h : \mathbb{C} \to \mathbb{C}\) by \(g(z) = 0\), \(h(z) = z/2\). Then (2.7) becomes: \(q = z\). This defines the congruence of lines parallel to the \(x_1\)-axis. These lines are the fibres of the globally defined harmonic morphism \(\varphi : \mathbb{R}^2 \to \mathbb{C}\) given by \(\varphi(x_1, x_2, x_3) = x_2 + i x_3\).

**Example 2.10.** (Radial projection) Define \(g, h : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}\) by \(g(z) = z\), \(h(z) = 0\). Then (2.7) becomes

\[z^2 \bar{q} - 2 z x_1 + q = 0\, .
\] (2.16)

This has solutions

\[z_{\pm} = \left( x_1 \pm \sqrt{x_1^2 - |q|^2} \right)/\bar{q}\, .
\] (2.17)

Note that \(|z_+| = |z_-| = 1\). Let \(C = \{(x_1, x_2, x_3) : x_2^2 = |q|^2\}\) denote the light cone and \(U = \{(x_1, x_2, x_3) : x_2^2 > |q|^2\}\) its interior. Then (2.17) defines smooth solutions \(z_{\pm} : U \setminus \{(x_1, 0, 0) : x_1 \in \mathbb{R}\} \to \mathbb{C}\); on setting \(z_+(x_1, 0, 0) = 0\) and \(z_-(x_1, 0, 0) = \infty\) these extend to smooth solutions \(z_\pm : U \to D^2\), \(z_- : U \to \mathbb{C} \cup \{\infty\} \setminus \overline{D^2}\), where \(D^2\) is the open unit disc. If we now put \(\varphi_{\pm} = \sigma^{-1} \circ z_{\pm}\) where \(\sigma\) is stereographic projection (2.9), then we obtain smooth submersive harmonic morphisms \(\varphi_{\pm} : U \to H^2\) defined by

\(\varphi_{\pm} = \mp \frac{1}{\sqrt{x_1^2 - x_2^2 - x_3^2}} (x_1, x_2, x_3)\).

Geometrically, \(\varphi_{\pm}\) is \(\mp\)-radial projection centred on the origin. Its fibres are the half-lines of \(U\) from the origin.

If, on the other hand, we restrict \(z_{\pm}\) to the exterior \(U^c = \{(x_1, x_2, x_3) : x_1^2 < |q|^2\}\) of the light cone, then \(|z_+| = |z_-| = 1\) and we obtain everywhere-degenerate harmonic morphisms \(z_{\pm} : U^c \to S^1 \subset \mathbb{C}\). The fibres of these harmonic morphisms are degenerate planes tangent to the light cone \(C\); each point \(x\) of \(U^c\) lies on two such planes, as \(x\) approaches the light cone both of these planes tend to the tangent plane.

**Example 2.11.** (Disc example) Define \(g, h : \mathbb{C} \to \mathbb{C}\) by \(g(z) = z\), \(h(z) = iz\). Then (2.7) becomes

\[z^2 \bar{q} - 2z(i + x_1) + q = 0\, .
\] (2.18)

This has solutions

\[z_{\pm} = i + x_1 \pm \sqrt{(i + x_1)^2 - |q|^2}/\bar{q}\, .
\]

Noting that \((i + x_1)^2 - |q|^2 = -1 - |x_1|^2 + 2ix_1\) never lies on the non-negative real axis, write

\[(i + x_1)^2 - |q|^2 = r e^{i\theta} \quad (r > 0, \ 0 < \theta < 2\pi)\];
then on taking $\sqrt{(1 + x^2) - |q|^2} = \sqrt{v^{\theta/2}}$, we see that the maps $z_\pm$ are smooth on $\mathbb{R}^3 \setminus \{(x_1, 0, 0)\}$. Setting $z_\pm(x_1, 0, 0) = 0$, $z_\pm(x_1, 0, 0) = \infty$ extends these to smooth harmonic morphisms $z_\pm : \mathbb{R}^3 \to D^2$ and $z_\pm : \mathbb{R}^3 \to \mathbb{C} \cup \{\infty\} \setminus \partial D$. Note that $z_\pm(x_1, q) = 1/z_\pm(x_1, q)$, $(x_1, q) \in \mathbb{R}^3)$. Equation (2.18) is invariant under rotations $z \mapsto e^{i\theta}z$, $q \mapsto e^{i\theta}q$, so that it defines a congruence of lines which is rotationally symmetric about the $x_1$-axis. Hence, to describe this congruence, it suffices to determine the directions of the lines through the points $(0, u, 0)$ for $u > 0$. At such a point,

$$z_\pm = (i \pm \sqrt{1 - u^2})/u = i(1 \pm \sqrt{1 + u^2})/u.$$ 

Comparing with (2.17), we see that the direction $\gamma$ of the fibre at $z$ is given by $\gamma(z) = (±\sqrt{1 + u^2}, 0, -u)$; this direction is perpendicular to the radius from $(0, 0, 0)$ to $(0, u, 0)$ and inclined at an angle $\arctan(\sqrt{1 + u^2})$ (and pointing ‘clockwise’) to the negative (resp. positive) $x_1$-axis. As $u$ increases from 0 to $\infty$, this angle increases from 0 to $\pi/4$. We thus obtain surjective submersive harmonic morphisms $z_- : \mathbb{R}^3 \to D^2$ and $z_+ : \mathbb{R}^3 \to \mathbb{C} \cup \{\infty\} \setminus \partial D$. Composing with $\sigma^{-1}$ gives surjective submersive harmonic morphisms $\varphi_- : \mathbb{R}^3 \to H_2^+$ and $\varphi_+ : \mathbb{R}^3 \to H_2^-$.

Note that we may introduce a real parameter $t \neq 0$ and set $h(z) = itz$ (with $g(z) = z$ unchanged). This gives the same example scaled by a factor of $t$; as $t \to 0$, this scaled disc example tends to radial projection (Example 2.10).

**Corollary 2.12.** There is a globally defined surjective submersive harmonic morphism from Minkowski 3-space $\mathbb{M}^3 = \mathbb{R}^3$ to the unit disc.

Indeed, both the disc example and orthogonal projection (Example 2.9) define harmonic morphisms *globally* on Minkowski 3-space. This is in contrast to the Riemannian case, where we established a Bernstein-type theorem [3] (see also [4, Theorem 6.7.3]) that orthogonal projection is the only globally defined harmonic morphism from $\mathbb{R}^3$ to a surface, up to postcomposition with weakly conformal maps. Globally defined harmonic morphisms from higher-dimensional Minkowski spaces can be obtained by precomposing such harmonic morphisms with orthogonal projections $\mathbb{R}_{10}^n \to \mathbb{R}^3$ for any $m > 3$.

### 3 Harmonic morphisms from Minkowski 3-space to a Lorentz surface

We recall some facts about hyperbolic numbers. Let $\mathbb{D} = \{(x_1, x_2) \in \mathbb{R}^2\}$ equipped with the usual coordinatewise addition, but with multiplication given by

$$(x_1, x_2)(y_1, y_2) = (x_1y_1 + x_2y_2, x_1y_2 + x_2y_1).$$

We call the commutative ring $\mathbb{D}$ the set of *hyperbolic* or *double numbers*. Write $j = (0, 1)$; then we have $(x_1, x_2) = x_1 + x_2j$ with $j^2 = 1$. Note that, unlike the complex numbers, $\mathbb{D}$ has zero divisors, namely the numbers $a(1 + j)$ ($a \in \mathbb{R}$). Multiplication by $j$ defines an involution $I^D$ on $\mathbb{D}$ called the *characteristic involution*, explicitly, $I^D(x_1, x_2) = (x_2, x_1)$.

For $z = x_1 + x_2j$, $(x_1, x_2 \in \mathbb{R})$, we write $x_1 = \Re z$, $x_2 = \Im z$ and $\overline{z} = x_1 - x_2j$. We shall often identify $z \in \mathbb{D}$ with the point $(x_1, x_2)$ in standard coordinates in Minkowski 2-space $\mathbb{M}^2 = \mathbb{R}^4$, then the standard Minkowski square norm $|z|^2 = (z, \overline{z}) = -x_1^2 + x_2^2$ is given by $|z|^2 = -z\overline{z}$.

From the chain rule, we obtain

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right),$$

so that, in standard coordinates $(x_1, x_2)$, the Laplacian on $\mathbb{M}^2$ is given by

$$\Delta^{\mathbb{M}^2} = -\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = -4\frac{\partial^2}{\partial z \partial \overline{z}} = -4\frac{\partial^2}{\partial z \partial \overline{z}}.$$
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By analogy with the complex numbers, we say that a $C^2$ map $\varphi : U \rightarrow \mathbb{D}$, $w = \varphi(z)$, from an open subset of $\mathbb{D}$ is $H$-holomorphic (resp., $H$-anti holomorphic) if we have

$$\frac{\partial w}{\partial \overline{z}} = 0 \quad \left(\text{resp., } \frac{\partial w}{\partial z} = 0 \right);$$

equivalently, on writing $z = x_1 + x_2 j$, $w = u_1 + u_2 j$, the map $\varphi$ satisfies the $H$-Cauchy-Riemann equations:

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2} \quad \text{and} \quad \frac{\partial u_1}{\partial x_2} = -\frac{\partial u_2}{\partial x_1} \quad \left(\text{resp., } \frac{\partial u_1}{\partial x_1} = -\frac{\partial u_2}{\partial x_2} \text{ and } \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1} \right).$$

These conditions are equivalent to demanding that the differential of $\varphi$ intertwine the characteristic involutions, viz., $d\varphi \circ I^D = I^D \circ d\varphi$ (resp., $d\varphi \circ I^D = -I^D \circ d\varphi$).

By a Lorentz surface, we mean a smooth surface equipped with a conformal equivalence class of Lorentzian metrics — here two metrics $g, g'$ on $N^2$ are said to be conformally equivalent if $g' = \mu g$ for some (smooth) function $\mu : N^2 \rightarrow \mathbb{R} \setminus \{0\}$. Any Lorentz surface is locally conformally equivalent to 2-dimensional Minkowski space $M^2$, see, for example, [1]. Let $\varphi : U \rightarrow N^2$ be a $C^2$ mapping from an open subset $U$ of $\mathbb{R}^3_1$ to a Lorentz surface. For local considerations, we can assume that $\varphi$ has values in $M^2$. Then, on identifying $M^2$ with the space $D$ of hyperbolic numbers as above and writing $\varphi = \varphi_1 + \varphi_2 j$, the map $\varphi$ is a harmonic morphism if and only if it satisfies equations (1.2) with $m = 3$, where now $\varphi$ has values in $D$.

From now on, suppose that $\varphi : U \rightarrow M^2 = \mathbb{D}$ is a non-constant harmonic morphism defined on an open subset $U$ of $\mathbb{R}^3_1$. As in the last section, by a generalization of [2], its fibres are straight lines, more precisely,

**Lemma 3.1.** Let $p \in U$ be a point where $\varphi$ is submersive. Then the connected component of the fibre of $\varphi$ through $p$ is a spacelike geodesic.

To proceed, we make the assumptions (2.1) of the previous section.

Write $V = \varphi(U)$ and let $\ell$ be a fibre of $\varphi : U \rightarrow \mathbb{D}$, i.e. $\ell = \varphi^{-1}(z)$ for some $z \in V$. Then, in contrast to the last section, $\ell$ is a spacelike line. Now the directions of spacelike lines are parametrized by the pseudosphere $S^2_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -x_1^2 + x_2^2 + x_3^2 = 1\}$. Let $\ell$ have direction $\gamma \in S^2_1 \subset \mathbb{R}^3_1$. We proceed by analogy with the last section, replacing the rotation on the horizontal space by a characteristic involution.

Let $c \in \mathbb{R}^3$ be the unique vector which satisfies $\langle c, \gamma \rangle_1 = 0$ and has endpoint on $\ell$; note that $c$ can be timelike, null or spacelike. Write $\varphi = \varphi_1 + \varphi_2 j$. For each $x \in U$, orient $\mathcal{H}_x$ so that $d\varphi|_{\mathcal{H}_x}$ is orientation preserving, equivalently, $\{\text{grad } \varphi_1, \text{grad } \varphi_2\}$ is an oriented basis; then orient $\ell$ by choosing its unit positive tangent vector $\gamma$ such that $\{\text{grad } \varphi_1, \text{grad } \varphi_2, \gamma\}$ is an oriented basis. Let $I^H$ denote the characteristic involution in the 2-plane $\mathcal{H}_x$ obtained by lifting $I^D$ from $\mathbb{D}$, equivalently $I^H$ interchanges $\text{grad } \varphi_1$ and $\text{grad } \varphi_2$. If $c$ is non-null (spacelike or timelike), then $|c|^2 \equiv \langle c, c \rangle_1$ is non-zero and we may define a ‘hyperbolic’ vector $\xi = \xi(z) \in \mathbb{D}^3$ by

$$\xi = (c + j I^H c) / |c|^2. \quad (3.1)$$

Then, in a way analogous to that in the last section, the equation of $\ell$ can be written as a single ‘hyperbolic’ equation:

$$\langle \xi(z), x \rangle_1 = 1; \quad (3.2)$$

this is identical to (2.3) except that the inner product $\langle \cdot, \cdot \rangle_1$ on $\mathbb{R}^3_1$ is extended by hyperbolic bilinearity to $\mathbb{D}^3 = \mathbb{R}^3_1 \otimes \mathbb{D}$. In the case when $c$ is null, this equation defines a (degenerate) plane which contains...
The line $\ell$; we shall discuss this case below. Again, $\xi$ is null in the sense that it satisfies $\langle \xi, \xi \rangle_1 = 0$, explicitly (note the difference of sign to that in (2.4)),

$$|\text{Re} \xi(z)|^2 = -|\text{Im} \xi(z)|^2 \quad \text{and} \quad \langle \text{Re} \xi(z), \text{Im} \xi(z) \rangle_1 = 0. \quad (3.3)$$

The hyperbolic square norm $|\xi|^2 := \langle \xi, \xi \rangle_1 = |\text{Re} \xi(z)|^2 - |\text{Im} \xi(z)|^2$ satisfies $|\xi|^2 = 2/|c|^2$ where $|c|^2 = (c, c)_1$, so that (3.1) gives a one-to-one correspondence between $\xi \in \mathbb{D}^3$ which satisfy $\langle \xi, \xi \rangle_1 = 0$ and have $|\xi|^2 \neq 0$ and vectors $c \in \mathbb{R}^3$ which have $|c|^2 \neq 0$; the inverse is given by

$$c = 2 \text{Re} \xi/|\xi|^2, \quad \text{so that} \quad T^c c = 2 \text{Im} \xi/|\xi|^2.$$

As in the previous section, if $\varphi : U \to \mathbb{D}$ is a harmonic morphism satisfying assumptions (2.1), then $\xi : V = \varphi(U) \to \mathbb{D}^3$ is H-holomorphic. Conversely, there is a version of Proposition 2.3 where $\mathbb{C}$ is replaced by $\mathbb{D}$, but now we must impose the stronger condition that $|\text{grad} G|^2$ is non-zero to ensure that $\partial G/\partial z$ is not a zero divisor; applying this as before we obtain the following version of Theorem 2.6.

**Theorem 3.2.** Let $\xi = (\xi_1, \xi_2, \xi_3) : V \to \mathbb{D}^3$ be an H-holomorphic map from an open subset of $\mathbb{D}$ (or of a Lorentz surface) which is null: $\langle \xi, \xi \rangle_1 = 0$ and has non-zero hyperbolic square norm $|\xi|^2$ on a dense open subset of $V$. Then any $C^2$ solution $\varphi : U \to \mathbb{M} = \mathbb{D}, z = \varphi(x)$ on an open subset of $\mathbb{R}^3$ to equation (3.2) is a harmonic morphism.

Conversely, every $C^2$ submersive harmonic morphism from an open subset of $\mathbb{R}^3_1$ to a Lorentz surface is given this way locally, after shifting the origin if necessary.

H-holomorphic functions $\xi = (\xi_1, \xi_2, \xi_3) : V \to \mathbb{D}^3$ satisfying $\langle \xi, \xi \rangle_1 = 0$ with $\xi_1 - \xi_3$ not zero and not a zero divisor are all given by

$$\xi = \frac{1}{2h(z)} \left( -\left(1 + g(z)^2\right), j\left(1 - g(z)^2\right), -2g(z) \right), \quad (3.4)$$

where $g, h : V \to \mathbb{D}$ ($h \neq 0$) are H-holomorphic functions; explicitly,

$$g = \xi_3/(\xi_1 - \xi_3) = (\xi_1 + \xi_3)/\xi_3, \quad h = -1/(\xi_1 - \xi_3) .$$

Then the representation (3.2) takes the form

$$(1 + g(z)^2) x_1 + j(1 - g(z)^2) x_2 - 2g(z) x_3 = 2h(z) . \quad (3.5)$$

From (3.3) we have $|\xi|^2 = (1 - |g|^2)/4|h|^2$; we deduce the following from Theorem 3.2.

**Corollary 3.3.** Let $g, h : V \to \mathbb{D}$ be H-holomorphic functions from an open subset of $\mathbb{D}$ (or of a Lorentz surface) with $|g(z)| \neq 1$. Then any $C^2$ solution $\varphi : U \to \mathbb{M}, z = \varphi(x_1, x_2, x_3)$, to (3.5) is a harmonic morphism which is not degenerate everywhere.

Conversely, any $C^2$ submersive harmonic morphism $\varphi$ is given locally this way, possibly after a change of coordinates.

We can interpret $g$ and $h$ in a way analogous to previous cases. Indeed, let $\mathcal{K}^1 = \{(x_1, x_2, x_3) \in \mathbb{S}^2_1 : x_3 = -1\}$ and $\mathcal{H}^1 = \{z \in \mathbb{D} : |z|^2 = -1\}$. Then we can identify $\mathbb{S}^2_1 \setminus \mathcal{K}^1$ with $\mathbb{D} \setminus \mathcal{H}^1$ by stereographic projection $\sigma_H : (x_1, x_2, x_3) = (x_1 + x_2)/2$. Then $g(z) = \sigma_H(\gamma(z))$ and $h(z) = (d\sigma_H)_z(\gamma(z))$.

**Example 3.4.** (Orthogonal projection) Define $g, h : \mathbb{D} \to \mathbb{D}$ by $g(z) = 0, h(z) = z/2$. Then (3.3) becomes: $x_1 + x_2 j = z$. This defines the congruence of lines parallel to the $x_3$-axis. These lines are the fibres of the globally defined harmonic morphism $\varphi : \mathbb{M}^3 = \mathbb{R}^3_1 \to \mathbb{M}^2 = \mathbb{D}$ given by $\varphi(x_1, x_2, x_3) = x_1 + x_2 j$. 

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Example 3.5. (Radial projection) Define \( g, h : \mathbb{D} \rightarrow \mathbb{D} \) by \( g(z) = z \), \( h(z) = 0 \). Then (3.5) becomes:

\[
    z^2(x_1 - x_2j) - 2z x_3 + (x_1 + x_2j) = 0.
\] (3.6)

This can be solved on \( \mathbb{R}^3 \setminus \{x_1 = \pm x_2\} \) to give

\[
    z = \frac{x_3 + \varepsilon \sqrt{-x_1^2 + x_2^2 + x_3^2}}{x_1 - x_2j} = \frac{(x_3 + \varepsilon \sqrt{-x_1^2 + x_2^2 + x_3^2})(x_1 + x_2j)}{x_1^2 - x_2^2},
\]

where we set \( \varepsilon = \pm 1, \pm j \) to get all possible square roots in \( \mathbb{D} \). Note that on the exterior \( \overline{U}^c = \{(x_1, x_2, x_3) \in \mathbb{D} : -x_1^2 + x_2^2 + x_3^2 > 0\} \) of the light cone \( C \), taking \( \varepsilon = \pm 1 \) gives two smooth harmonic morphisms \( z_{ \pm : \overline{U}^c \setminus \{x_1 = \pm x_2\} \rightarrow \mathbb{M}^2 \) which can be interpreted as compositions \( \varphi_{ \pm } = \sigma_j \circ \varphi_{ \pm } \), where \( \varphi_{ \pm } \) is the restriction to \( \overline{U} \setminus \{x_1 = \pm x_2\} \) of radial projection (or its negative) \( \overline{U} \rightarrow S^2 \):

\[
    x = (x_1, x_2, x_3) \mapsto \frac{x}{\sqrt{x_1^2}} = \frac{1}{\sqrt{-x_1^2 + x_2^2 + x_3^2}} (x_1, x_2, x_3).
\]

When \( x \in C \), (3.5) has repeated solutions \( z \) and the fibre through \( x \) is the (degenerate) tangent plane to \( C \) at that point. Note that both \( \mathbb{M}^2 \) and \( S^2 \) have conformal compactification given by a quadric \( Q_1^2 \) in \( \mathbb{R}P^3 \), see [1] Example 14.1.22]; as \( x \) approaches a point on \( C \), \( \varphi_{ \pm } (x) \) tends to a point at infinity of \( S^2 \) in \( Q_1^2 \), and the harmonic morphism can be regarded as having values in \( Q_1^2 \).

When \( x \) lies inside the light cone there is no value of \( z \in \mathbb{M}^2 \) satisfying (3.5) (contrast with Example 2.10).

Alternatively, we can take \( \varepsilon = \pm j \) to get the other two values of the square root, in which case

\[
    z_{ \pm } = \frac{x_3 \pm (\sqrt{-x_1^2 + x_2^2 + x_3^2}) j}{x_1 - x_2} = \frac{x_3 + x_2j}{x_3 + (\sqrt{-x_1^2 + x_2^2 + x_3^2}) j}.
\]

Then \( |z_{ \pm }|^2 = -1 \) and \( z_{ \pm } \) is an everywhere-degenerate harmonic morphism \( \overline{U}^c \setminus \{x_1 = \pm x_2\} \rightarrow \mathbb{M}^2 \) with values on the hyperbola \( \mathcal{H}^1 \). The fibres of these harmonic morphisms are the degenerate tangent planes to the light cone \( C \). As \( x \) tends to a point in the set \( \{x_1 = \pm x_2\} \), \( z_{ \pm } \) tends to the point at infinity on the hyperbola and we can regard \( z_{ \pm } \) as extending to an everywhere-degenerate harmonic morphism from \( \overline{U}^c \) to the compactification \( Q_1^2 \) of \( M^2 \).

Example 3.6. (Disc example) Define \( g, h : \mathbb{D} \rightarrow \mathbb{D} \) by \( g(z) = z \), \( h(z) = zj \). Then (3.5) becomes

\[
    z^2(x_1 - x_2j) - 2z(x_3 + j) + x_1 + x_2j = 0.
\] (3.7)

This can be solved on \( \mathbb{R}^3 \setminus \{x_1 = \pm x_2\} \) to give

\[
    z = \frac{x_3 + j + \varepsilon \sqrt{-x_1^2 + x_2^2 + x_3^2 + 1 + 2x_3j}}{x_1 - x_2} \quad (\varepsilon = \pm 1, \pm j).
\]

The square root is smooth on the region \( W \) where \( \eta_1 = -x_1^2 + x_2^2 + x_3^2 + 1 + 2x_3 \) and \( \eta_2 = -x_1^2 + x_2^2 + x_3^2 + 1 - 2x_3 \) are both positive, this is given by \( W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (1 - |x_3|)^2 - x_1^2 + x_2^2 > 0\} \). Then on \( W \setminus \{x_1 = \pm x_2\} \) we can compute the square root to give

\[
    z = \frac{x_3 + j + 2\{\sqrt{\eta_1} + \sqrt{\eta_2}\}}{x_1 - x_2} \quad (\varepsilon = \pm 1, \pm j).
\]

In order to describe these harmonic morphisms geometrically, first take \( \varepsilon = 1 \). Then at a point \( (x_1, x_2, x_3) = (u, 0, 0) \), with \( |u| < 1 \) so that it lies in \( W \), we have \( z_1 = (j + \sqrt{1 - u^2})/u \) and so \( \gamma \) consists of multiples of the vectors \( (\sqrt{1 - u^2}, 1, 0) \); it is easily seen that the fibres of \( z_1 \) are tangent to the hyperbola: \( x_1^2 - x_2^2 = 1, x_3 = 0 \). As \( x_3 \) increases from 0, the lines start tilting.
With \( \varepsilon = j \), we find that, at \((x_1, x_2, 0)\),
\[
\tau = \frac{j + (\sqrt{1 - x_1^2} x_2)}{x_1 - x_2}
\]
and \( \gamma \) consists of multiples of the vectors \((x_2, x_1, -\sqrt{1 - r^2})\) if \(x_1^2 > x_2^2\), and \((x_2, x_1, -1)\) if \(x_1^2 < x_2^2\),
where \( r^2 = -x_1^2 + x_2^2 \). Thus, at any point \(P(x_1, x_2, 0)\), the fibre is perpendicular in a Lorentzian sense
to the radius \(OP\); as \(P\) travels along the radius from \(O\), it starts vertically down and then swivels until
it is horizontal \textit{either} when it hits the hyperbola: \(x_1^2 - x_2^2 = 1, x_3 = 0\) \(i.e.\) if \(x_1^2 > x_2^2\), \textit{or}, if it avoids
the hyperbola \(\text{\(i.e.\)}\) if \(x_1^2 < x_2^2\), at infinity. It is thus a hyperbolic analogue of the disc example that
occurs in the Riemannian case \cite[Example 1.5.3]{4}. Note that since \((3.7)\) is invariant under the change of
coordinates \((x_1, x_2, x_3) \rightarrow (x_1, -x_2, x_3, 1/z)\), the cases \(\varepsilon = -1, -j\) are equivalent to the above cases.

Note that, as in Example 3.5, we may introduce a real parameter \(t \neq 0\) and set \(h = tz\) \((\text{with}
g = z \text{unchanged})\); this gives the same example scaled by a factor of \(t\). Again, as \(t \to 0\), this scaled
disc example tends to radial projection \((\text{Example 3.5})\).

4 Degenerate harmonic morphisms on Minkowski spaces

By definition (see the Introduction), a \(C^1\) horizontally weakly conformal map is degenerate at a point
\(x\) if and only if the kernel of \(d\phi_\tau\) is degenerate. It follows \cite[Remark 14.5.5]{4} that an everywhere-degen-
erate harmonic morphism \(\phi\) from a Lorentzian manifold \(M^n\) to an arbitrary semi-Riemannian manifold
\(N\) necessarily has rank one everywhere; further, by \cite[Proposition 14.5.8]{4}, it factors locally into the
composition of an everywhere-degenerate harmonic morphism from \(M^n\) to \(\mathbb{R}\) and an immersion of \(\mathbb{R}\)
into \(N\). Hence, to determine all such \(\phi\), it suffices to take \(N = \mathbb{R}\). In the case that \(M^n\) is an open
subset \(U\) of an \(m\)-dimensional Minkowski space \(M^m = \mathbb{R}^m\), an everywhere-degenerate harmonic morphism
is just a null \textit{real-valued} solution of the wave equation, \textit{i.e.} a solution \(\varphi : U \rightarrow \mathbb{R}\) of the system \((4.5)\).

To solve this problem, we need the following version of Proposition 2.5; note that it is empty if \(M\)
is Riemannian. As the proof uses the same calculations, we omit it.

**Proposition 4.1.** Let \(M\) be an arbitrary semi-Riemannian manifold. Let \(A\) be an open subset of
\(M \times \mathbb{R}\) and let \(G : A \rightarrow \mathbb{R}, (x, z) \rightarrow G(x, z)\), be a \(C^2\) mapping which is an everywhere-degenerate harmonic morphism in its first argument, \textit{i.e.}, writing \(G_\tau(x) = G(x, z)\),
\[
\begin{array}{ll}
(4.1) & \Delta^M G_\tau = 0, \quad \langle \text{grad } G_\tau, \text{grad } G_\tau \rangle_M = 0 \quad (\text{if } (x, z) \in A). \\
\end{array}
\]
Let \(\varphi : U \rightarrow \mathbb{C}\) be a \(C^2\) solution to equation \(G(x, \varphi(x)) = \text{const.}\) on an open subset \(U\) of \(M\) and
suppose that \(\text{grad } G_\tau(x, \varphi(x)) \neq 0\) on a dense subset of \(U\). Then \(\varphi\) is an everywhere-degenerate harmonic morphism, \textit{i.e.}, it satisfies the system
\[
\begin{array}{ll}
(4.2) & \Delta^M \varphi = 0, \quad \langle \text{grad } \varphi, \text{grad } \varphi \rangle_M = 0. \\
\end{array}
\]
\[]

In the Lorentzian case this gives

**Lemma 4.2.** Let \(\varphi(x_1, x_2, \ldots, x_m)\) satisfy
\[
\tau(\varphi(x_1, x_2, \ldots, x_m), x_2, \ldots, x_m) = x_1. \quad (4.3)
\]
Then \(\varphi\) satisfies the system
\[
\begin{array}{ll}
(4.4) & \Box \varphi = 0, \quad \langle \text{grad } \varphi, \text{grad } \varphi \rangle_1 = 0. \\
\end{array}
\]
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if and only if, for each fixed \( x_1 \), \( \tau \) satisfies the system

\[
\begin{align*}
(a) & \quad \Delta^{m-1} \tau = 0, \\
(b) & \quad \langle \text{grad} \tau, \text{grad} \tau \rangle_{\mathbb{R}^{m-1}} = 1;
\end{align*}
\]

that is, \( \varphi \) is a null solution to the wave equation if and only if, on each slice \( x_1 = \text{const.} \), \( \tau \) is a harmonic function with \( |\text{grad} \tau|^2 = 1 \).

**Proof:** Set \( G(\varphi, x_1, x_2, \ldots, x_m) = \tau(\varphi, x_2, \ldots, x_m) - x_1 \). Then

\[
\left( \frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \ldots, \frac{\partial G}{\partial x_m} \right) = \left( -1, \frac{\partial \tau}{\partial x_2}, \ldots, \frac{\partial \tau}{\partial x_m} \right)
\]

so that

\[
\langle \text{grad} G, \text{grad} G \rangle_{\mathbb{R}^{m-1}} = \langle \text{grad} \tau, \text{grad} \tau \rangle_{\mathbb{R}^{m-1}} - 1 \quad \text{and} \quad \Box G = \Delta^{m-1} G = \Delta^{m-1} \tau.
\]

The result follows.

Solutions of the system (4.5) are easy to find, as follows.

**Lemma 4.3.** Any \( C^2 \) solution \( \varphi : U \to \mathbb{R} \) on an open subset of \( \mathbb{R}^{m-1} \) to the system (4.5) is affine, i.e.,

\[
\tau(x_2, \ldots, x_m) = \ell_1 + \sum_{i=2}^{m} \ell_i x_i
\]

for some constants \( \ell_1, \ell_2, \ldots, \ell_m \) with \( \sum_{i=2}^{m} \ell_i^2 = 1 \).

**Proof:** Since \( \tau \) is harmonic, it is smooth. Set \( T = \text{grad} \tau : U \to \mathbb{R}^m \). Then \( T \) is harmonic and has image in the unit sphere. By the maximum principle, \( T \) is constant. Indeed, choose any point \( p \in U \) and set \( \ell = T(p) \). Then the function \( x \mapsto \langle T(x), \ell \rangle \) is harmonic and has a maximum at \( p \) and so is constant. Integrating yields (4.6).

We deduce the following result.

**Theorem 4.4.** (Collins [6]) Let \( \varphi : U \to \mathbb{R} \) be a null \( C^2 \) solution to the wave equation, i.e. a solution to (4.1), on an open set of \( \mathbb{M}^m \). Suppose that \( \partial \varphi / \partial x_1 \neq 0 \). Then, locally, \( z = \varphi(x_1, \ldots, x_m) \) satisfies

\[
\ell_1(z) + \sum_{i=2}^{m} \ell_i(z) x_i = x_1
\]

for some \( C^2 \) functions \( \ell_1, \ell_2, \ldots, \ell_m : V \to \mathbb{R} \) defined on an open subset of \( \mathbb{R} \) with \( \sum_{i=2}^{m} \ell_i^2 = 1 \).

Conversely, any \( C^2 \) solution to (4.7) is a null solution to the wave equation.

**Proof:** By the implicit function theorem we can solve \( \varphi(x_1, x_2, \ldots, x_m) = z \) to give

\[
x_1 = \tau(z, x_2, \ldots, x_m).
\]

Then, by Lemma 4.2 on each slice \( x_1 = \text{const.} \), \( \tau \) satisfies (4.5). By Lemma 4.3 \( \tau|_{x_1=\text{const.}} \) is affine, thus,

\[
\tau(z, x_2, \ldots, x_m) = \ell_1(z) + \sum_{i=2}^{m} \ell_i(x) x_i
\]

with \( \sum_{i=2}^{m} \ell_i^2 = 1 \). Then (4.8) yields (4.7).
Corollary 4.5. The level sets of a $C^2$ null solution to the wave equation are degenerate hyperplanes.

Corollary 4.6. Any $C^2$ harmonic morphism $\varphi : U \to \mathbb{R}$, $z = \varphi(x_1, x_2, x_3)$ from an open subset of $M^3 = \mathbb{R}^3$ which is submersive and degenerate everywhere is locally the solution to an equation

$$-x_1 + \cos \theta(z) x_2 + \sin \theta(z) x_3 = r(z),$$

for some $C^2$ functions $\theta, r : V \to \mathbb{R}$ defined on an open subset of $\mathbb{R}^3$.

Conversely, any $C^2$ solution to this equation on an open subset of $\mathbb{R}^3$ is a harmonic morphism which is degenerate everywhere.

References


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Harmonic morphisms from Minkowski space

Received: 23.01.2009.

Département de Mathématiques,
Université de Bretagne Occidentale,
6 Avenue Le Gorgeu,
29285 Brest, France
E-mail: Paul.Baird@univ-brest.fr

Department of Pure Mathematics,
University of Leeds
Leeds LS2 9JT, Great Britain
E-mail: j.c.wood@leeds.ac.uk