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Twistorial maps between (para)quaternionic projective spaces

by

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To Professor S. Ianuş on the occasion of his 70th Birthday

Abstract

By extending an integrability result previously proved for almost quaternionic manifolds, [M1], a characterization of manifolds with an integrable paraquaternionic structure as locally paraquaternionic projective manifolds is given. An alternative proof of a characterization of twistorial maps between quaternionic projective spaces,[IMOP], is developed and extended to the paraquaternionic case. The extension of these results to maps between locally (para)quaternionic Grassmannian manifolds is discussed and partially proved, in line with [M1] and its extension [M2] to quaternionic tensor product structures.

Key Words: (para)quaternionic projective spaces - (para)quaternionic manifolds - (para)quaternionic integrability -(para)quaternionic maps
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1 Introduction

In this paper we deal with G-structures related to the algebra $\mathbb{K} = \mathbb{H}, \widetilde{\mathbb{H}}$ of quaternions, paraquaternions respectively and with classes of natural maps for such structures, as introduced in [IMOP] from a twistorial point of view. We denote $GL(m, \mathbb{K})$ the group of invertible right $\mathbb{K}$-linear endomorphisms of the m-dimensional numerical space $\mathbb{K}^m$.

An *almost quaternionic structure* on a manifold M^{4m} is a reduction of its frame bundle to the group $G = GL(m, \mathbb{H}) \cdot GL(1, \mathbb{H})$ which can be identified with the group of $\mathbb{R}$-automorphisms $\eta = T(\xi)$ of $\mathbb{H}^n$ of the form: $\eta = A\xi q$, $A \in GL(m, \mathbb{H}), q \in GL(1, \mathbb{H})$. It is known that the *integrability condition* of such G-structure, i.e. the condition that the corresponding reduction of the frame bundle is given by the cocycle determined by a coordinate atlas, is *very restrictive*: in [M1], [K] it was proved that it holds if and only if the manifold is *locally quaternionic projective*, i.e. the differentiable structure is induced by an atlas of quaternionic coordinates which change by quaternionic linear fractional transformations, as it happens for non homogeneous quaternionic projective coordinate systems of the *quaternionic projective space* $\mathbb{H}P^m$. The proof in [M1] bases on the fact that the integrability condition for the more

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restricted group $G = GL(m, \mathbb{H})$ characterizes the locally quaternionic affine manifolds. In [M2] it was proved also that, more generally, an *almost tensor product quaternionic structure* of *type* (p, k) on a manifold M^{4pk}, that is a G-structure where $G = GL(p, \mathbb{H}) \cdot GL(k, \mathbb{H})$, is integrable if and only if the manifold is *locally quaternionic Grassmannian.* A generalization of the first result was given in [IMOP] dealing with quaternionic maps, where as an application of the given characterization of such maps as twistorial maps it was proved that *quaternionic maps between open sets of quaternionic projective spaces* $\mathbb{H}P^n, \mathbb{H}P^m$ *are induced by* $\mathbb{H}$*-linear maps from* $\mathbb{H}^{n+1}$ *to* $\mathbb{H}^{m+1}$.

The primary aim of present paper was to present an alternative, direct and more elementary, proof of the stated characterization of quaternionic maps between quaternionic projective spaces, following the line of [M1],[M2]. By performing such task, whose basic step essentially consisted in a careful inspection of the proof of main results in [M1] and [M2], we realized first that analogous results about integrability are still valid for almost paraquaternionic manifolds and, also, for manifolds with an almost paraquaternionic tensor product structure: i.e. *the paraquaternionic G-structures where* $G = GL(m, \widetilde{\mathbb{H}}) \cdot GL(1, \widetilde{\mathbb{H}})$ *and* $G = GL(p, \widetilde{\mathbb{H}}) \cdot GL(k, \widetilde{\mathbb{H}})$ *are integrable if and only if the manifold is locally paraquaternionic projective and, respectively, locally paraquaternionic Grassmannian.* (For an account on almost paraquaternionic manifolds and on tensor product structures let see the bibliographical references). Then we were able to give a proof of the said characterization of quaternionic maps which carries over to paraquaternionic case: *paraquaternionic maps between open sets of paraquaternionic projective spaces* $\widetilde{\mathbb{H}}P^n, \widetilde{\mathbb{H}}P^m$ *are induced by* $\widetilde{\mathbb{H}}$*-linear maps from* $\widetilde{\mathbb{H}}^{n+1}$ *to* $\widetilde{\mathbb{H}}^{m+1}$.

Finally, by going over again the above results, and referring in particular to those in [M2], we are naturally induced to *conjecture that the extension to analogous results holds for maps between open sets of (para)quaternionic Grassmannian manifolds,* $\varphi : U \subset G_p(\mathbb{K}^{p+k}) \to G_p(\mathbb{K}^{p+k'})$, $\mathbb{K} = \mathbb{H}$ *or* $\widetilde{\mathbb{H}}$, $k \leq k'$.

2 Preliminaries

$\mathbb{K}$ denotes an associative, non commutative, algebra over the real field $\mathbb{R}$ coinciding with the algebra $\mathbb{H}$ of *quaternions* or the algebra $\widetilde{\mathbb{H}}$ of *paraquaternions.* An element of $\mathbb{K}$ is a *(para)quaternion*[1] of the form $q = q_0 + iq_1 + jq_2 + kq_3$ where $q_t \in \mathbb{R}$, $t = 0, 1, 2, 3$, and the imaginary units $i_1 = i, i_2 = j, i_3 = k$ verify the relations

$$i^2 = -1, j^2 = k^2 = -\epsilon\,;\; ij = -ji = k\,,\; jk = -kj = \epsilon i\,,\; ki = -ik = j$$

being $\epsilon = 1$ or $\epsilon = -1$ if $\mathbb{K} = \mathbb{H}$ or $\mathbb{K} = \widetilde{\mathbb{H}}$ respectively. Also notation $i_0 = 1$ will be used.

The *conjugate* of q is $\overline{q} = q_0 - iq_1 - jq_2 - kq_3$; $\mathrm{Re}(q) = q_0$ is the *real part* and $\mathrm{Im}(q) = iq_1 + jq_2 + kq_3$ is the *imaginary part* of q.

The *square norm* of q is

$$\|q\|^2 = q\overline{q} \equiv (q_0)^2 + (q_1)^2 + \epsilon((q_2)^2 + (q_3)^2)\,.$$

We recall that

$$\mathrm{Re}(p+q) = \mathrm{Re}(p) + \mathrm{Re}(q) \quad , \quad \mathrm{Re}(pq) = \mathrm{Re}(qp)$$

$$\overline{p+q} = \overline{p} + \overline{q} \quad , \quad \overline{pq} = \overline{q}\,\overline{p} \qquad (p, q \in \mathbb{K})$$

For $q = q_0 + iq_1 + jq_2 + kq_3 \in \mathbb{K}$ let define the *associated quaternions*

$$q' = -iqi \quad , \quad q'' = -\epsilon jqj \quad , \quad q''' = -\epsilon kqk$$

[1] In the following, by writing say *(para)quaternion* we intend that the statement is to be made respectively for a quaternion or a paraquaternion, that is for $\mathbb{H}$ or $\widetilde{\mathbb{H}}$.

(cfr. [Bo]). Hence

$$\left\{\begin{array}{lcl} q & = & q_0+iq_1+jq_2+kq_3 \\ q' & = & q_0+iq_1-jq_2-kq_3 \\ q'' & = & q_0-iq_1+jq_2-kq_3 \\ q''' & = & q_0-iq_1-jq_2+kq_3 \end{array}\right. , \quad \left\{\begin{array}{lcl} q_0 & = & \frac{1}{4}(q+q'+q''+q''') \\ q_1 & = & -\frac{i}{4}(q+q'-q''-q''') \\ q_2 & = & -\frac{\epsilon j}{4}(q-q'+q''-q''') \\ q_3 & = & -\frac{\epsilon k}{4}(q-q'-q''+q''') \end{array}\right. \tag{1}$$

Let $\xi=\xi_0+i\xi_1+j\xi_2+k\xi_3$ be a (para)quaternionic variable or function whose differential is

$$d\xi = d\xi_0+id\xi_1+id\xi_1+jd\xi_2+kd\xi_3$$

Then

$$\left\{\begin{array}{lcl} d\xi_0 & = & \frac{1}{4}(d\xi+d\xi'+d\xi''+d\xi''') \\ d\xi_1 & = & -\frac{i}{4}(d\xi+d\xi'-d\xi''-d\xi''') \\ d\xi_2 & = & -\frac{\epsilon j}{4}(d\xi-d\xi'+d\xi''-d\xi''') \\ d\xi_3 & = & -\frac{\epsilon k}{4}(d\xi-d\xi'-d\xi''+d\xi''') \end{array}\right. \tag{2}$$

If $F=F(\xi)$ is a (para)quaternionic function of ξ, whose real components are $F_i=F_i(\xi)\equiv F_i(\xi_0,\xi_1,\xi_2,\xi_3), i=0,1,2,3$, i.e. $F(\xi)=F_0+iF_1+jF_2+kF_3$, then the differential of F,

$$dF = dF_0+idF_1+jdF_2+kdF_3\,, \tag{3}$$

where

$$dF_i = \frac{\partial F_i}{\partial \xi_0}d\xi_0+\frac{\partial F_i}{\partial \xi_1}d\xi_1+\frac{\partial F_i}{\partial \xi_2}d\xi_2+\frac{\partial F_i}{\partial \xi_3}d\xi_3 \qquad (i=0,1,2,3)\,, \tag{4}$$

could be written at any point as a linear function of the (para)quaternionic differentials $d\xi, d\xi', d\xi'', d\xi'''$

$$dF = \frac{\partial F}{\partial \xi}d\xi+\frac{\partial F}{\partial \xi'}d\xi'+\frac{\partial F}{\partial \xi''}d\xi''+\frac{\partial F}{\partial \xi'''}d\xi''' \tag{5}$$

and the unicity of such expression gives also the definition of *(para)quaternionic derivatives* $\frac{\partial F}{\partial \xi}, \frac{\partial F}{\partial \xi'}, \frac{\partial F}{\partial \xi''}, \frac{\partial F}{\partial \xi'''}$.

These definitions and results can be carried over to a function $\eta=F(\xi^1,\dots,\xi^n)$ of n (para)quaternionic variables $\xi^1,\dots,\xi^n$. In particular, one has the (para)quaternionic expression of the differential

$$dF = \sum_{\alpha=1}^{n}\left(\frac{\partial F}{\partial \xi^\alpha}d\xi^\alpha+\frac{\partial F}{\partial \xi^{\alpha'}}d\xi^{\alpha'}+\frac{\partial F}{\partial \xi^{\alpha''}}d\xi^{\alpha''}+\frac{\partial F}{\partial \xi^{\alpha'''}}d\xi^{\alpha'''}\right) \tag{6}$$

(In the following we will omit the summation symbol by adopting Einstein convention).

A basic result, which goes back to C. Ehresmann (see [Bo]), states that a quaternionic function of n quaternionic variables which is differentiable on the right (resp. on the left) in quaternionic sense has to be linear. For completeness and in view of extensions to the paraquaternionic case we state the following proposition. In the quaternionic case the proof, which is elementary, was indicated to us by G.B. Rizza several years ago, before [M1].

Proposition 2.1. *Let $F(\xi^1,\dots,\xi^n)$ be a (para)quaternionic function of the n (para)quaternionic variables $\xi^1,\dots,\xi^n$. Let assume that F is C^k, $k\geq 2$, and that it is differentiable on the right in (para)quaternionic sense, that is*

$$dF = A_\beta d\xi^\beta \qquad (\beta=1,\dots,n) \tag{7}$$

with $A_\beta=A_\beta(\xi^1,\dots,\xi^n)$ a (para)quaternionic function, of class C^{k-1}. Then F is (right-)linear, i.e. of the form $F=a_\beta\xi^\beta+b$ with $a_\beta=A_\beta, b$ (para)quaternionic constants $(\beta=1,\dots,n)$.

Proof. (7) is equivalent to the (set of) identities

$$\frac{\partial F}{\partial \xi^{\beta_a}} = A_\beta i_a \qquad (a = 0,1,2,3; \beta = 1,\dots,n)$$

and also to the identities

$$\frac{\partial F}{\partial \xi^{\beta_0}} = A_\beta \quad , \quad \frac{\partial F}{\partial \xi^{\beta_a}} = \frac{\partial F}{\partial \xi^{\beta_0}} i_a \quad (a = 0,1,2,3; \beta = 1,\dots,n). \tag{8}$$

Hence, by further differentiation and using equality of mixed partials,

$$\frac{\partial^2 F}{\partial \xi^{\gamma_b} \partial \xi^{\beta_a}} = \frac{\partial^2 F}{\partial \xi^{\gamma_b} \partial \xi^{\beta_0}} i_a = \frac{\partial^2 F}{\partial \xi^{\beta_0} \partial \xi^{\gamma_b}} i_a = \frac{\partial^2 F}{\partial \xi^{\beta_0} \partial \xi^{\gamma_0}} i_b i_a \tag{9}$$

and by applying again the equality of mixed partials, it results

$$\frac{\partial^2 F}{\partial \xi^{\beta_0} \partial \xi^{\gamma_0}} (i_b i_a - i_a i_b) = 0 \quad (a, b = 0,1,2,3; \beta, \gamma = 1,\dots,n)\,.$$

For $a \neq b, a, b \in \{1,2,3\}$, $i_b i_a - i_a i_b = -2 i_a i_b \neq 0$, then

$$\frac{\partial^2 F}{\partial \xi^{\beta_0} \partial \xi^{\gamma_0}} = 0 \qquad (\beta, \gamma = 1,\dots,n)$$

and by (9) all second derivatives of F vanish. Hence $F = A_\beta \xi^\beta + b$. □

3 The (para)quaternionic projective space $\mathbb{K}P^n$

Let $\mathbb{K}^{n+1}$ be the *(n+1)-dimensional (para)quaternionic numerical space*, formed by vectors $\mathbf{x} = (x^0, x^1, \dots, x^n)$, $(x^i \in \mathbb{K})$. A vector $\mathbf{x}$ is called *isotropic* if $\|\mathbf{x}\|^2 := \sum_i \|x^i\|^2 = 0$; for $\mathbb{K} = \mathbb{H}$ there are no non zero isotropic vectors. Denote $\mathbb{K}_0 = \{q \in \mathbb{K} | \|q\| \neq 0\}$ the multiplicative group of invertible (para)quaternions. Let $\mathbb{K}_0^{n+1}$ be the open set of nonisotropic vectors of $\mathbb{K}^{n+1}$ and $\sim$ be the equivalence relation $\mathbf{x} \sim \mathbf{y}$ if and only if $\mathbf{y} = \mathbf{x}\lambda$ for some $\lambda \in \mathbb{K}_0$.

The (*right*) n-dimensional *(para)quaternionic projective space* $\mathbb{K}P^n$, the space of (nonsingular) (para)quaternionic lines of $\mathbb{K}^{n+1}$, can be defined as the quotient space

$$\mathbb{K}P^n = \mathbb{K}_0^{n+1} \diagdown \mathbb{K}_0\,.$$

(see [B], [GMV], [L]). We denote $[\mathbf{x}]$ the point of $\mathbb{K}P^n$ which is determined by the vector $\mathbf{x} \in \mathbb{K}_0^{n+1}$, i.e. $[\mathbf{x}]$ is the equivalence class of $\mathbf{x}$ in $\mathbb{K}_0^{n+1}$.

$\mathbb{K}P^n$ is endowed by a natural (para)quaternionic structure that is an integrable almost (para)quaternionic structure given by a canonical atlas of (paraquaternionic) coordinates.

Let first recall the following definitions (for this point of view in quaternionic case, see [Bo])

Definition 3.1. *An almost (para)quaternionic structure on a manifold M^{4n} is defined by:*
1) a locally trivial bundle $p : \mathcal{K}(M) \to M$ with fiber type the (para)quaternionic algebra $\mathbb{K}$
2) a field $\mathcal{R}$ of effective representations $\mathcal{R}_x$ of $p^{-1}(x) = \mathbb{K}_x(M)$ in T_xM, $x \in M$.

Let

$$\mathbf{x} \bullet \mathbf{y} := \sum_{1,\dots,n+1} x^i \overline{y^i} \qquad , \qquad \mathbf{x}, \mathbf{y} \in \mathbb{K}^{n+1}$$

be the *standard (para)quaternionic hermitian product* in $\mathbb{K}^{n+1}$ and

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{K}} := \mathrm{Re}(\mathbf{x} \bullet \mathbf{y}) = \sum_{1,\dots,n+1} (x_0^i y_0^i + x_1^i y_1^i + \epsilon x_2^i y_2^i + \epsilon x_3^i y_3^i)$$

be the corresponding standard scalar product in $\mathbb{R}^{4(n+1)}_{\mathbb{K}} \equiv \mathbb{K}^{n+1}$ where $\mathbb{R}^{4(n+1)}_{\mathbb{K}}$ is $\mathbb{R}^{4(n+1)}$ endowed by the standard positively defined euclidean product if $\mathbb{K} = \mathbb{H}$ and $\mathbb{R}^{4(n+1)}_{2(n+1)}$, that is $\mathbb{R}^{4(n+1)}$ endowed by the standard pseudoeuclidean scalar product $\sum_{1,\dots,n+1}(x^i_0 y^i_0 + x^i_1 y^i_1 + \epsilon x^i_2 y^i_2 + \epsilon x^i_3 y^i_3)$ of *neutral signature* $(2n+2, 2n+2)$, if $\mathbb{K} = \widetilde{\mathbb{H}}$.

Denote $S^{4n+3}_{\mathbb{K}}$ the unit (pseudo)sphere of the (para)hermitian space $\mathbb{R}^{4(n+1)}_{\mathbb{K}}$ formed by the non singular vectors $\mathbf{x}$ of unitary norm, defined by the equation

$$\mathbf{x} \bullet \mathbf{x} = 1 .$$

$S^{4n+3}_{\mathbb{H}} \subset \mathbb{R}^{4(n+1)}$ is the unit sphere S^{4n+3} of the euclidean space $\mathbb{R}^{4(n+1)}$ and $S^{4n+3}_{\widetilde{\mathbb{H}}} \subset \mathbb{R}^{4(n+1)}$ is the unit pseudosphere S^{4n+3}_{2n+1} of the euclidean space $\mathbb{R}^{4(n+1)}_{2n+2}$.

We denote by $S_{\mathbb{K}}$ the unit 3-dimensional (pseudo)sphere of $\mathbb{R}^4_{\mathbb{K}}$, that is $S_{\mathbb{H}} \equiv S^3 \subset \mathbb{R}^4$, $S_{\widetilde{\mathbb{H}}} \equiv S^1_2 \subset \mathbb{R}^4_2$ respectively. $S_{\mathbb{K}}$ is formed by the unitary (para)quaternions $\lambda \in \mathbb{K}$ such that $\lambda\overline{\lambda} = 1$.

$S_{\mathbb{K}} \subset \mathbb{K}_0$ acts on the right on $\mathbb{K}^{n+1}_0$ and the n-dimensional (para)quaternionic projective space can be considered as the quotient

$$\mathbb{K}P^n = S^{4n+3}_{\mathbb{K}}/S_{\mathbb{K}} .$$

that is we can write

$$\mathbb{K}P^n = \{[\mathbf{x}] \,|\, \mathbf{x} \in \mathbb{K}^{n+1}_0, \mathbf{x} \bullet \mathbf{x} = 1, \mathbf{x} \sim \mathbf{x}\lambda, \forall \lambda \in S_{\mathbb{K}}\} .$$

Then the fibre space of tangent vectors $T(\mathbb{K}P^n)$ can be realized as the quotient

$$\begin{aligned} T(\mathbb{K}P^n) = \{ \quad & [\mathbf{x}, \mathbf{v}] | \mathbf{x} \in \mathbb{K}^{n+1}_0, \mathbf{x} \bullet \mathbf{x} = 1, \mathbf{v} \in \mathbb{K}^{n+1}, \mathbf{x} \bullet \mathbf{v} = 0, \\ & (\mathbf{x}, \mathbf{v}) \sim (\mathbf{x}\lambda, \mathbf{v}\lambda) \, \forall \lambda \in S_{\mathbb{K}}\}, \end{aligned}$$

The fibre space $\mathcal{K}(\mathbb{K}P^n)$ can be defined as the quotient of the set of pairs $(\mathbf{x}, q)$, where $\mathbf{x} \in S^{4n+3}_{\mathbb{K}}$ and $q \in \mathbb{K}$, by the equivalence relation:

two pairs $(\mathbf{x}, q)$ and $(\mathbf{x}', q')$ are equivalent if and only if it exists a unitary $\lambda \in S_{\mathbb{K}}$ such that

$$\mathbf{x}' = \mathbf{x}\lambda \qquad \text{and} \qquad q' = \lambda^{-1} q \lambda$$

i.e.

$$\mathcal{K}(\mathbb{K}P^n) = \{[\mathbf{x}, q] | \mathbf{x} \in \mathbb{K}^{n+1}_0, \mathbf{x} \bullet \mathbf{x} = 1,\, q \in \mathbb{K}, (\mathbf{x}, q) \sim (\mathbf{x}\lambda, \lambda^{-1} q \lambda) \, \forall \lambda \in S_{\mathbb{K}}\}$$

The representation of $\mathcal{K}(\mathbb{K}P^n)$ on $T(\mathbb{K}P^n)$ is defined by

$$\mathcal{R}_{[\mathbf{x}]}[\mathbf{x}, q] : [\mathbf{x}, \mathbf{v}] \to [\mathbf{x}, \mathbf{v}q] \,, \forall [\mathbf{x}, \mathbf{v}] \in T(\mathbb{K}P^n)$$

(One verifies that $\mathbf{x} \bullet (\mathbf{v}q) = 0$, $[\mathbf{x}, \mathbf{v}q] \sim [\mathbf{x}\lambda, \mathbf{v}\lambda\lambda^{-1}q\lambda] = [\mathbf{x}\lambda, \mathbf{v}q\lambda]$.)

For any basis $\{\mathbf{f}_0, \mathbf{f}_1, \dots, \mathbf{f}_n\}$ of $\mathbb{K}^{n+1}$ and corresponding (para)quaternionic coordinates $(y^0, y^1, \dots, y^n)$ a system of (non homogeneous) coordinates $(\xi^1, \dots, \xi^n)$ is defined on the open set where $y_0 \neq 0$ by assuming $\xi^\alpha = y^\alpha (y^0)^{-1}$, $\alpha = 1, \dots, n$. If $\{\mathbf{f}_0{}', \mathbf{f}_1{}', \dots, \mathbf{f}_n{}'\}$ is another basis of $\mathbb{K}^{n+1}$ and $\xi^{\alpha'} = y^\alpha (y^{0'})^{-1}$ the induced coordinates the change of homogeneous coordinates is a linear transformation

$$y^{\alpha'} = b^{\alpha'}_\beta y^\beta$$

(by using Einstein convention for sum) and the corresponding change of non homogeneos coordinates is a (para)quaternionic linear fractional transformation

$$\xi^{\alpha'} = (b^{\alpha'}_\beta \xi^\beta + b^{\alpha'}_0)(b^{0'}_\gamma \xi^\gamma + b^{0'}_0)^{-1} \qquad (\alpha', \beta, \gamma = 1, \dots, n) \tag{10}$$

By differentiating one obtains

$$d\xi^{\alpha'} = A^{\alpha'}_\gamma d\xi^\gamma Q \qquad (\alpha, \gamma = 1, \dots, n)$$

where

$$A^{\alpha'}_{\gamma} = -\xi^{\alpha'} b^{0'}_{\gamma} + b^{\alpha'}_{\gamma} \quad , \quad Q = (b^{0'}_{\beta}\xi^{\beta} + b^{0'}_{0})^{-1} .$$

The coordinate systems $(\xi^1, \ldots, \xi^n)$ form the *standard coordinate atlas of* $\mathbb{K}P^n$.

The standard atlas induces a $GL(n,\mathbb{K}) \cdot GL(1,\mathbb{K})$-structure on the manifold $\mathbb{K}P^n$, which thus is an almost (para)quaternionic manifold, and moreover such structure is integrable.

Let M^{4n} be a 4n-dimensional differentiable manifold and assume that it is endowed by a *(para)quaternionic coordinate atlas*, consisting of local systems of (para)quaternionic coordinates $(\xi^{\alpha}) \equiv (\xi^1, \ldots, \xi^n)$ on open sets U of a covering of M^{4n}, such that in the non empty intersection $U \cap U'$ of two charts the corresponding coordinate systems are related by linear fractional (para)quaternionic transformations of type (10); then we say that M^{4n} is *locally (para)quaternionic projective.* A locally (para)quaternionic projective manifold M^{4n} is automatically endowed by an integrable (smooth) almost (para)quaternionic structure.

As an extension of the main result in [M1], proved for $\mathbb{H}$, we can state that the converse is always true for both algebras $\mathbb{K}$.

Proposition 3.2. *An almost (para)quaternionic structure, of class $C^t, t \geq 3$, is integrable if an only if it is induced by a locally (para)quaternionic structure.*

Proof. For $\mathbb{K} = \mathbb{H}$ it was proved in [M1] and by carefully following the proof given there we checked, by a long but rather trivial work, that the same steps can be carried over to the paraquaternionic case, $\mathbb{K} = \widetilde{\mathbb{H}}$. (We will return on that proof when considering proposition 4.1). □

In fact, we mention that by checking the analogous proof of the results in [M2] it is possible to state the following generalization of 3.2.

Recall that the group $G = GL(p,\mathbb{K}) \cdot GL(k,\mathbb{K})$ is isomorphic to the group of invertible endomorphisms $Y = T(X)$ of $\mathbb{K}^{pk}, Y = (y^{\alpha}_i), X = (x^{\beta}_j) \in \mathbb{K}^{pk}$, of the form $Y = AXQ$, where $A = (A^{\alpha}_{\beta}) \in GL(p,\mathbb{K}), Q = (Q^i_j) \in GL(k,\mathbb{K}), \quad (\alpha, \beta = 1, \ldots, p; i, j = 1, \ldots, k)$. Note also that a change of Pontrjagin coordinates $\Gamma = (\gamma^{\alpha}_i), \Xi = (\xi^{\beta}_j)$, $(\alpha, \beta = 1, \ldots, p;\ i, j = 1, \ldots, k)$ in $G_p(\mathbb{K}^{p+k})$ is given by a linear fractional (para)quaternionic transformation

$$\Gamma = (B\Xi + C)(D\Xi + E)^{-1}$$

where the (para)quaternionic matrix

$$\begin{pmatrix} B & C \\ D & E \end{pmatrix} = \begin{pmatrix} (b^{\alpha}_{\beta}) & (c^{\alpha}_j) \\ (d^{\rho}_j) & (e^r_s)) \end{pmatrix}$$

is invertible. Moreover, at any point the transformation between the differentials $d\Gamma, d\Xi$ belongs to $G = GL(p,\mathbb{K}) \cdot GL(k,\mathbb{K})$, see [M2] for $\mathbb{K} = \mathbb{H}$.

Proposition 3.3. *On a real differentiable manifold M^{4pk}, of class $C^t, t \geq 3$, an almost (para)quaternionic tensor product structure of type (p,k), is integrable if and only if it is locally (para)quaternionic Grassmannian, of type $G_p(\mathbb{K}^{p+k})$, for $\mathbb{K} = \mathbb{H}, \widetilde{\mathbb{H}}$ respectively.*

4 (Para)quaternionic maps between (para)quaternionic manifolds

Let M^{4n}, N^{4m} be two almost (para)quaternionic manifolds, whose fields of representations are $\mathcal{R}^M_x, \mathcal{R}^N_y, x \in M, y \in N$ respectively. A map $\varphi : M \to N$ is a *is a (para)quaternionic map* if there exists a map $\widetilde{\varphi} : \mathcal{K}(M) \to \mathcal{K}(N)$ such that for any vector $\mathbf{v} \in T_xM, x \in M$ one has

$$d\varphi \circ \mathcal{R}^M_x(q_x) = (\mathcal{R}^N_{\varphi(x)} \circ \widetilde{\varphi})(q_x) \circ d\varphi \quad , \quad \forall q_x \in \mathcal{K}_x\, ,\, x \in M\, .$$

(see [IMOP]).

Example: *Linear projective maps between (para)quaternionic projective spaces.* Let $A : \mathbb{K}^{n+1} \to \mathbb{K}^{m+1}$ be a (para)quaternionic linear map. Let $\hat{U} = \mathbb{K}_0^{n+1} - \{\mathrm{Ker}A\}$ and $U \subset \mathbb{K}P^n$ be the corresponding open set through the canonical projection $\mathbb{K}_0^{n+1} \to \mathbb{K}P^n$. Then the map $\varphi : \mathbb{K}P^n \supseteq U \to \mathbb{K}P^m$ is (para)quaternionic. To see it let $(x^i), i = 0, 1, \ldots, n, (y^j), j = 0, 1, \ldots, m,$ be (para)quaternionic cartesian coordinate systems on $\mathbb{K}^{n+1}, \mathbb{K}^{m+1}$ respectively and

$$y^j = a_i^j x^i \qquad (j = 0, 1, \ldots, m)$$

the equations of A, with (para)quaternionic matrix (a_j^i). In terms of non homogeneous (para)quaternionic projective coordinates $(\xi^\alpha = x^\alpha (x^0)^{-1}),\ \alpha = 0, 1, \ldots n, (\eta^r = y^r (y^0)^{-1}), r = 1, \ldots, m$ on $\mathbb{K}P^m, \mathbb{K}P^n$ respectively, the (local) equations of the differential of the map φ have the form:

$$d\eta^r = B_\alpha^r d\xi^\alpha Q \qquad (r = 1, \ldots, m) \tag{11}$$

where

$$B_\alpha^r = a_\alpha^r - \eta^r a_\alpha^0 \quad , \quad Q = (a_\beta^0 \xi^\beta + a_0^0)^{-1} \qquad (r = 1, \ldots, m; \alpha = 1, \ldots, n)\,.$$

Hence φ is a (para)quaternionic map.

In fact, linear projective maps are the only (para)quaternionic maps between (para)quaternionic projective spaces. For the quaternionic case, $\mathbb{K} = \mathbb{H}$, the result was proved in [IMOP]. Here we will give a more elementary proof of it, which applies also to the paraquaternionic case.

Proposition 4.1. *(See [IMOP]). Let $M^{4n} \subset \mathbb{K}P^n$ be a connected open subset of $\mathbb{K}P^n$. A (para)-quaternionic C^3-map $\varphi : M^{4n} \to \mathbb{K}P^m$ is a linear projective map, i.e. it is induced by a linear map $A : \mathbb{K}^{n+1} \to \mathbb{K}^{m+1}$ through the canonical projections $\mathbb{K}_0^{n+1} \to \mathbb{K}P^n$, $\mathbb{K}_0^{m+1} \to \mathbb{K}P^m$.*

Proof. For simplicity, we will proceed by referring to the quaternionic case and step by step, without special mention, we will take into account that the same procedure formally works for the paraquaternionic case, if one substitute $\widetilde{\mathbb{H}}$ to $\mathbb{H}$.

Let $(x^i), i = 0, 1, \ldots, n, (y^j), j = 0, 1, \ldots, m,$ be quaternionic cartesian coordinate systems on $\mathbb{H}^{n+1}, \mathbb{H}^{m+1}$ respectively. In terms of non homogeneous quaternionic projective coordinates $(\xi^\alpha = x^\alpha (x^0)^{-1}), \alpha = 1, \ldots n, (\eta^r = y^r (y^0)^{-1}), r = 1, \ldots, m,$ on $\mathbb{H}P^n, \mathbb{H}P^m$ respectively, the equations of the differential of a quaternionic map φ have the form:

$$d\eta^r = \sum_{\alpha=1}^{n} B_\alpha^r d\xi^\alpha Q \qquad (r = 1, \ldots, m) \tag{12}$$

where Q, B_α^a are quaternionic smooth functions of the quaternionic variables $\xi^1, \ldots, \xi^n$.

To prove the proposition amount to deduce that the functions η^r have to be of the form

$$\eta^r = (a_\alpha^r \xi^\alpha + a_0^r)(a_\beta^0 \xi^\beta + a_0^0)^{-1} \qquad (r = 1, \ldots, m)$$

where the coefficients $a_\beta^i, a_\beta^0, a_0^0$ are constant (we use the Einstein convention for summation). This in turn is equivalent to be

$$f B_\alpha^r = a_\alpha^r - \eta^r a_\alpha^0 \quad , \quad f Q^{-1} = a_\beta^0 \xi^\beta + a_0^0$$

for some constants $a_\alpha^r \in \mathbb{H}$ and a real function $f = f(\xi^1, \ldots, \xi^n)$.

In [M1], as recalled concerning proposition (4.1), it was proved that for n=m and $Q, B = (B_\alpha^\gamma)$ invertible, that is in the case of a diffeomorphism, the functions $\eta^{\alpha'}$ are in fact of the form

$$\eta^\rho = (a_\beta^\rho \xi^\beta + a_0^\rho)(a_\alpha^0 \xi^\alpha + a_0^0)^{-1} \qquad (\rho = 1, \ldots, n)$$

To prove the result in full generality it has to be proved essentially only the following *key lemma.*

Lemma 4.2. *A smooth quaternionic function* $\eta = F(\xi^1, \dots, \xi^n)$ *of the quaternionic variables* $\xi^1, \dots, \xi^n$ *whose differential can be written in the quaternionic form*

$$d\eta = B_\alpha d\xi^\alpha Q \tag{13}$$

where Q, B_α *are quaternionic smooth functions of* $\xi^1, \dots, \xi^n$ *is a linear fractional function, i.e. it has the form*

$$\eta = (a_\beta \xi^\beta + b)(c_\alpha \xi^\alpha + d)^{-1}$$

where the coefficients a_β, c_α, b, d *are constant.*

Proof of Lemma 4.2: We use induction on n. In fact the result is true for n=1, basing on the quoted result in [M1], that is: if the smooth quaternionic function $\eta = f(\xi)$ of the quaternionic variable ξ verify $d\eta = Bd\xi Q$ then it is of the form

$$\eta = (a\xi + b)(c\xi + d)^{-1}$$

for constants $a, b, c, d \in \mathbb{H}$. One can easily reduce to show that the result is true on an open set of points where $Q \neq 0$.

Now we make the following **two remarks**:

- 1) for a given quaternionic function $\eta = F(\xi^1, \dots, \xi^n)$ whose differential verifies equation (13) the function $\eta' = F'(\xi^1, \dots, \xi^{n-1})$ obtained by fixing the value of the variable $\xi^n = \xi^n_0$, i.e.

$$F'(\xi^1, \dots, \xi^{n-1}) = F(\xi^1, \dots, \xi^{n-1}, \xi^n_0),$$

verifies

$$d\eta' = \sum_{\alpha=1}^{n-1} B'_\alpha d\xi^\alpha Q' \tag{14}$$

where $B'_\alpha = B_\alpha(\xi^1, \dots, \xi^{n-1}, \xi^n_0), Q' = Q(\xi^1, \dots, \xi^{n-1}, \xi^n_0)$.

- 2) if we have a quaternionic linear fractional function

$$\eta = (a_\beta \xi^\beta + b)(c_\alpha \xi^\alpha + d)^{-1} \tag{15}$$

and we fix the value of one variable, say $\xi^n = \xi^n_0$, we still obtain a quaternionic linear fractional function in the remaining variable.

Basing on these remarks, the induction is performed as follows.

Let consider a quaternionic function $\eta = F(\xi^1, \xi^2)$ verifying

$$d\eta = (B_1 d\xi^1 + B_2 d\xi^2)Q\,. \tag{16}$$

We can assume that B_1, B_2 are both non zero (otherwise η depends only on one variable or it is constant and we are sent back to case n=1). By considering η as a function of the variable ξ^1 alone it results that it has the form

$$\eta = (a_1\xi^1 + b_1)(c_1\xi^1 + d_1)^{-1} \tag{17}$$

where a_1, b_1, c_1, d_1 are functions of ξ^2.

Analogously, by considering F as a function of the variable ξ^2 alone it results that it has the form

$$\eta = (a_2\xi^2 + b_2)(c_2\xi^2 + d_2)^{-1} \tag{18}$$

where a_2, b_2, c_2, d_2 are functions of ξ^1. Then one has to compare the two expressions and, possibly, get the desired result.

By resuming we have made the

first step: one has

$$a_1\xi^1 + b_1 = \eta(c_1\xi^1 + d^1) \qquad , \qquad a_2\xi^2 + b_2 = \eta(c_2\xi^2 + d^2) \tag{19}$$

where

$$a_1 = a_1(\xi^2)\,,\; b_1 = b_1(\xi^2)\,,\; c_1 = c_1(\xi^2)\,,\; d_1 = d_1(\xi^2)$$

and

$$a_2 = a_2(\xi^1)\,,\; b_2 = b_2(\xi^1)\,,\; c_2 = c_2(\xi^1)\,,\; d_2 = d_2(\xi^1)$$

Remark 4.3. *If we could prove, for example, that a_1, c_1 are constant and that b_2, d_2 have respectively the form $b_2 = a\xi^1 + b, d_2 = c\xi^2 + d$ where a, b, c, d are constants then the conclusion will follow.*

But we have to note also the following.

Remark 4.4. *The functions B_α, Q in the expression (16) of the differential $d\eta$ are determined up to a smooth real function f: i.e., for ${B_\alpha}' = B_\alpha f$ $(\alpha = 1, 2)$ and $Q' = Qf^{-1}$ one still has*

$$d\eta = (B_1' d\xi^1 + B_2' d\xi^2)Q'\,.$$

second step: Let's show that *in (16) one can assume*

$$Q = (c_2\xi^2 + d_2)^{-1}$$

that is

$$d\eta = (B_1 d\xi^1 + B_2 d\xi^2)(c_2\xi^2 + d_2)^{-1}\,. \tag{20}$$

In fact, let start from (16). By differentiating the second of identities (19) one has

$$\begin{aligned} da_2\xi^2 + a_2 d\xi^2 + db_2 = \quad & (B_1 d\xi^1 + B_2 d\xi^2)Q(c_2\xi^2 + d_2) \\ & +\eta(dc_2\xi^2 + c_2 d\xi^2 + dd_2) \end{aligned} \quad . \tag{21}$$

By equating separately the terms containing $d\xi^1$ and the terms containing $d\xi^2$ one has respectively the identities

$$da_2\xi^2 + db_2 = B_1 d\xi^1 Q(c_2\xi^2 + d_2) + \eta(dc_2\xi^2 + dd_2) \tag{22}$$

and

$$a_2 d\xi^2 = B_2 d\xi^2 Q(c_2\xi^2 + d_2) + \eta c_2 d\xi^2$$

that is

$$(a_2 - \eta c_2)d\xi^2 = B_2 d\xi^2 Q(c_2\xi^2 + d_2)\,. \tag{23}$$

The last identity (23) has the form

$$U d\xi^2 = V d\xi^2 W$$

for some $U = U(\xi^1, \xi^2), V = V(\xi^1, \xi^2), W = W(\xi^1, \xi^2)$ and if $U \neq 0$, as in our case otherwise $\eta = a_2 c_2^{-1}$ would be a function of ξ^1 alone, it implies that W *is a real function.* (In fact: for $d\xi^2 = 1, i, j, k$ respectively we have

$$U = VW\,,\; Ui = ViW\,,\; Uj = VjW\,,\; Uk = VkW\,,$$

and hence

$$VWi = ViW\,,\; VWj = VjW\,,\; VWk = VkW.$$

If $V \neq 0$ it follows that the values of W belong to the center $\mathbb{R}$ of $\mathbb{H}$.) In our case, from (23) it results that $f = Q(c_2\xi^2 + d_2)$ is a real function and hence for $Q' = Qf^{-1}$ one has $Q' = (c_2\xi^2 + d_2)^{-1}$, as to be proved.

third step: *Under the made assumption on* Q, the differentiation of the first identity (21) gives

$$da_2\xi^2 + a_2 d\xi^2 + db_2 = (B_1 d\xi^1 + B_2 d\xi^2) + \eta(dc_2\xi^2 + c_2 d\xi^2 + dd_2)\,. \tag{24}$$

and this last identity is equivalent to the identities

$$B_2 = a_2 - \eta c_2 \tag{25}$$

$$da_2\xi^2 + db_2 = B_1 d\xi^1 + \eta(dc_2\xi^2 + dd_2)\,. \tag{26}$$

fourth step: (26) implies that a_2 *and* c_2 *are constant* and b_2, d_2 *are linear functions of* ξ^1.

Proof of the fourth step: Since the quaternionic differentials $d\xi, d\xi', d\xi'', d\xi'''$ are independent, (26) is equivalent to the identities

$$\frac{\partial a_2}{\partial \xi^1} d\xi^1 \xi^2 + \frac{\partial b_2}{\partial \xi^1} d\xi^1 = B_1 d\xi^1 + \eta(\frac{\partial c_2}{\partial \xi^1} d\xi^1 \xi^2 + \frac{\partial d_2}{\partial \xi^1} d\xi^1)$$

and

$$\begin{aligned} &(\tfrac{\partial a_2}{\partial \xi^{1'}} d\xi^{1'} + \tfrac{\partial a_2}{\partial \xi^{1''}} d\xi^{1''} + \tfrac{\partial a_2}{\partial \xi^{1'''}} d\xi^{1'''})\xi^2 + (\tfrac{\partial b_2}{\partial \xi^{1'}} d\xi^{1'} + \tfrac{\partial b_2}{\partial \xi^{1''}} d\xi^{1''} + \tfrac{\partial b_2}{\partial \xi^{1'''}} d\xi^{1'''} = \\ &\eta\big((\tfrac{\partial c_2}{\partial \xi^{1'}} d\xi^{1'} + \tfrac{\partial c_2}{\partial \xi^{1''}} d\xi^{1''} + \tfrac{\partial c_2}{\partial \xi^{1'''}} d\xi^{1'''})\xi^2 + (\tfrac{\partial d_2}{\partial \xi^{1'}} d\xi^{1'} + \tfrac{\partial d_2}{\partial \xi^{1''}} d\xi^{1''} + \tfrac{\partial d_2}{\partial \xi^{1'''}} d\xi^{1'''})\big) \end{aligned} \tag{27}$$

that is, respectively

$$(\frac{\partial a_2}{\partial \xi^1} - \eta \frac{\partial c_2}{\partial \xi^1}) d\xi^1 \xi^2 = (B_1 + \eta \frac{\partial d_2}{\partial \xi^1} - \frac{\partial b_2}{\partial \xi^1}) d\xi^1 \tag{28}$$

and

$$\begin{aligned} &(\tfrac{\partial a_2}{\partial \xi^{1'}} d\xi^{1'} \xi^2 + \tfrac{\partial b_2}{\partial \xi^{1'}}) d\xi^{1'} = \eta(\tfrac{\partial c_2}{\partial \xi^{1'}} d\xi^{1'} \xi^2 + \tfrac{\partial d_2}{\partial \xi^{1'}} d\xi^{1'})\,, \\ &(\tfrac{\partial a_2}{\partial \xi^{1''}} d\xi^{1''} \xi^2 + \tfrac{\partial b_2}{\partial \xi^{1''}}) d\xi^{1''} = \eta(\tfrac{\partial c_2}{\partial \xi^{1''}} d\xi^{1''} \xi^2 + \tfrac{\partial d_2}{\partial \xi^{1''}} d\xi^{1''})\,, \\ &(\tfrac{\partial a_2}{\partial \xi^{1'''}} d\xi^{1'''} \xi^2 + \tfrac{\partial b_2}{\partial \xi^{1'''}}) d\xi^{1''} = \eta(\tfrac{\partial c_2}{\partial \xi^{1'''}} d\xi^{1'''} \xi^2 + \tfrac{\partial d_2}{\partial \xi^{1'''}} d\xi^{1'''})\,. \end{aligned} \tag{29}$$

Now, (28) has the form $Rd\xi^1\xi^2 = Sd\xi^1$. By doing like in step 2 and, moreover, by taking into account the independence of ξ^2, it follows

$$R \equiv \frac{\partial a_2}{\partial \xi^1} - \eta \frac{\partial c_2}{\partial \xi^1} = 0 \tag{30}$$

and successively, since we assumed η depending effectively on both ξ^1, ξ^2, one has

$$\frac{\partial a_2}{\partial \xi^1} = \frac{\partial c_2}{\partial \xi^1} = 0 \tag{31}$$

and also

$$B_1 + \eta \frac{\partial d_2}{\partial \xi^1} - \frac{\partial b_2}{\partial \xi^1} = 0 \tag{32}$$

From first of (29), for ex., one has

$$(\frac{\partial a_2}{\partial \xi^{1'}} - \eta \frac{\partial c_2}{\partial \xi^{1'}}) d\xi^{1'} \xi^2 = (\eta \frac{\partial d_2}{\partial \xi^{1'}} - \frac{\partial b_2}{\partial \xi^{1'}}) d\xi^{1'}$$

and by the same argument as above it follows

$$\frac{\partial a_2}{\partial \xi^{1'}} - \eta \frac{\partial c_2}{\partial \xi^{1'}} = 0 \quad , \quad \eta \frac{\partial d_2}{\partial \xi^{1'}} - \frac{\partial b_2}{\partial \xi^{1'}} = 0$$

and successively

$$\frac{\partial a_2}{\partial \xi^{1'}} = \frac{\partial c_2}{\partial \xi^{1'}} = 0 \quad , \quad \frac{\partial d_2}{\partial \xi^{1'}} = \frac{\partial b_2}{\partial \xi^{1'}} = 0 \tag{33}$$

and also, by repeating the procedure on second and third identity of (29) respectively, one has

$$\frac{\partial a_2}{\partial \xi^{1''}} = \frac{\partial c_2}{\partial \xi^{1''}} = 0 \quad , \quad \frac{\partial d_2}{\partial \xi^{1''}} = \frac{\partial b_2}{\partial \xi^{1''}} = 0 \tag{34}$$

$$\frac{\partial a_2}{\partial \xi^{1''}} = \frac{\partial c_2}{\partial \xi^{1''}} = 0 \quad , \quad \frac{\partial d_2}{\partial \xi^{1''}} = \frac{\partial b_2}{\partial \xi^{1''}} = 0 \tag{35}$$

final step: By summarizing, we proved that

$$\frac{\partial a_2}{\partial \xi^{1}} = \frac{\partial a_2}{\partial \xi^{1'}} = \frac{\partial a_2}{\partial \xi^{1''}} = \frac{\partial a_2}{\partial \xi^{1'''}} = 0 \, , \, \frac{\partial c_2}{\partial \xi^{1}} = \frac{\partial c_2}{\partial \xi^{1'}} = \frac{\partial c_2}{\partial \xi^{1''}} = \frac{\partial c_2}{\partial \xi^{1'''}} = 0$$

hence a_2, c_2 *are constant*, and that

$$\frac{\partial b_2}{\partial \xi^{1'}} = \frac{\partial b_2}{\partial \xi^{1''}} = \frac{\partial b_2}{\partial \xi^{1'''}} = 0 \quad , \quad \frac{\partial d_2}{\partial \xi^{1'}} = \frac{\partial c_2}{\partial \xi^{1''}} = \frac{\partial d_2}{\partial \xi^{1'''}} = 0$$

hence b_2, d_2 *are (right) linear functions of* ξ^1, of the form

$$b_2 = b\xi^1 + p \qquad , \qquad d_2 = d\xi^1 + q \qquad p, q \in \mathbb{H}$$

The conclusion follows by recalling remark (4.3).

The same proof can be easily performed to deal with the case of a quaternionic function $\eta(\xi^1, \ldots, \xi^n)$, for any n. $\square$

5 Generalizations in the same line

- (I) - By using the result of [M2] it should be possible to carry over the same proof of proposition 4.1 above for maps between locally grassmannian (para)quaternionic manifolds, and prove the following proposition.

Proposition 5.1. (**Conjecture**) *Let* $M^{4pk} \subset G_p(\mathbb{K}^{p+k})$ *be a connected open subset of* $G_p(\mathbb{K}^{p+k})$. *A tensor product map* $\varphi : M^{4pk} \to G_p(\mathbb{K}^{p+k'})$, $k \leq k'$, *is a linear projective map, i.e. it is induced by a linear map* $A : \mathbb{K}^{p+k} \to \mathbb{K}^{p+k'}$.

Note that a quaternionic tensor product structure belongs to an interesting class of G-structures for which a good twistor theory can be considered, [AG1].

- (II) - Th. Hangan proved a characterization of real or complex manifolds admitting an integrable (p,q) tensorial product structure, under the hypothesis that $p \neq 1$ and $q \neq 1$, as locally grassmannian manifolds. One could wonder if the above generalization (I) holds true also in real and complex case. Of course, in the real and complex case Lemma (4.2) cannot be proved and the hypothesis $p \neq 1$ and $q \neq 1$ is necessary.

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References

[AG1] D.V. Alekseevsky - M. M. Graev *Twistors and G-structures* Russian Acad. Sci. Izv. Math., **40** (1993) No. 1, 1-31

[AG2] D.V. Alekseevsky - M. M. Graev *Grassmann and hyper-Kähler structures on some spaces of sections of holomorphic bundles* Manifolds and Geometry (Pisa, 1993), 1-19, Symposia Mathematica, vol. XXXVI, 1995, Cambridge Univ. Press, Cambridge, 1996

[AM1] D.V. Alekseevsky - S. Marchiafava *A report on quaternionic-like structures on a manifold* Proceedings of the International Workshop on Differential Geometry and its Applications (Bucharest, 1993), Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., **55** (1993), no. 3-4, 9-34

[AM2] D.V. Alekseevsky - S. Marchiafava *Quaternionic structures on a manifold and subordinated structures*, Ann. Mat. Pura Appl., **171** (1996), 205-273

[B] N. Blazic *paraquaternionic projective space and pseudo-riemannian geometry*, Publ. Math. (Beograd) **60** (1996), 101-107

[Bo] E. Bonan *Sur les G-structures de type quaternionien*, Ca. Topol. Géom. Diff., **9**, 4 (1967) 389-463

[BMR] N. Bokan - P. Matzeu - Z. Razic *Geometric quantities of manifolds with Grassmann structure*, Nagoya Math. J., **180** (2005), 45-76

[GMV] E. García-Río - Y. Matsushita R. Vázquez-Lorenzo *Paraquaternionic Kähler manifolds*, Rocky Mountain J. of Math., **31**, No 1 (2001), 237-260

[Ha] T. Hangan *Tensor product tangent bundles*, Archiv. der Math., **19** (4) (1968), 436-448

[IMOP] S. Ianus - S. Marchiafava - L. Ornea - R. Pantilie *Twistorial maps between quaternionic manifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), (to appear), arXiv:0801.4587.

[K] R. S. Kulkarni, *On the principle of uniformization*, J. Diff. Geom., **13** (1978), 109-138

[L] M.-A. Lawn, *Pseudo-Riemannian surfaces of arbitrary signature in* $\mathbb{S}^{2,2}$, preprint, january 2009, 1-26

[M1] S. Marchiafava *Sulle varieta' a struttura quaternionale generalizzata*, Rend. Mat. Roma (6) **3**, (1970), 529-545

[M2] S. Marchiafava *Varieta' localmente grassmanniane quaternionali*, Rend. Accad. Naz. Lincei, (8) **57**, no. 1-2 Ferie (1974), 80-89

[Sa] S. Salamon *Differential geometry of quaternionic manifolds*, Ann. Scient. Ec. Norm. Sup., 4ème série, **19** (1986), 31-55

[V] S. Vukmirović *Para-quaternionic reduction*, ArXiv: math.DG/0304424v1, 2003

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