

Generalized globally framed f -space-forms

by

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To Professor S. Ianuș on the occasion of his 70th Birthday

Abstract

Globally framed f -manifolds are studied from the point of view of the curvature. Generalized globally framed f -space-forms are introduced and the interrelation with generalized Sasakian and generalized complex space-forms is pointed out. Suitable differential equations allow to discuss the constancy of the φ -sectional curvatures. Further results are stated when the underlying structure is a \mathcal{K} -structure or an $f.pk$ -structure of Kenmotsu type.

Key Words: f -structure, space-form, generalized space form.

2000 Mathematics Subject Classification: Primary 53C15, Secondary 53C25.

Introduction

The study of the curvature and the classification of Riemannian manifolds under suitable curvature restrictions are classical problems. Analogous topics are treated in the context of almost Hermitian or contact Geometry. In particular, in [21], Tricerri and Vanhecke classified almost Hermitian manifolds M^{2n} which are generalized complex space-forms, in dimension $2n \geq 6$, whereas, in [20], Olszak stated the classification of such spaces for $n = 2$. In the context of almost contact metric manifolds, Alegre, Blair and Carriazo introduced the concept of generalized Sasakian-space-form, proving partial classification results. Even if the problem of the classification of these spaces is quite far from being solved, several efforts on the subject are due to Bueken and Vanhecke ([6]) and to Kim ([17]). In this paper, we extend the notion of generalized Sasakian-space-form to the class of metric $f.pk$ -manifolds $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \dots, s\}$. We call them generalized $f.pk$ -space-forms, requiring that the curvature involves a set of smooth functions F_1, F_2, F_{ij} , with $F_{ij} = F_{ji}$ for $i, j \in \{1, \dots, s\}$. General properties and a characterization of generalized $f.pk$ -space-forms are obtained. In particular, one gets the pointwise constancy of the φ -sectional curvatures $c = F_1 + 3F_2$. Several applications of the second Bianchi identity allow to relate the 1-forms dF_1, dF_2, dF_{ij} to the covariant derivatives $\nabla\varphi, \nabla\eta^i, \nabla$ denoting the Levi-Civita connection. We discuss the constancy of the functions F_1, F_2 along the distribution $\mathcal{D} = \text{Im}\varphi$, which in this case is CR -integrable. This allows to clarify the interrelation between generalized $f.pk$ -space-forms and generalized complex space-forms. The last two sections deal with the spaces whose underlying $f.pk$ -structure is of a particular type, namely it is either a \mathcal{K} -structure ([9]) or an $f.pk$ -structure of Kenmotsu type ([11]).

All manifolds are assumed to be connected. For the curvature of a Riemannian manifold we adopt the definitions $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ and $R(X, Y, Z, W) = g(X, R(Z, W, Y))$. We also use the Einstein convention, omitting the sum symbol for repeated indexes, if there is no doubt.

Acknowledgments. The author wish to thank Professor Stere Ianus for his long, precious collaboration and for his great friendship.

1 Preliminaries

In the class of f -structures introduced in 1963 by Yano [22], particularly interesting are the so-called f -structures with complemented frames, also called globally framed f -structures or f -structures with parallelizable kernel (briefly $f.pk$ -structures) ([3, 14, 10]). An $f.pk$ -manifold is a manifold M^{2n+s} on which is defined an f -structure, that is a $(1, 1)$ -tensor field φ satisfying $\varphi^3 + \varphi = 0$, of rank $2n$, such that the subbundle $\ker \varphi$ is parallelizable. Then, there exists a global frame $\{\xi_i\}$, $i \in \{1, \dots, s\}$, for the subbundle $\ker \varphi$, with dual 1-forms η^i , satisfying $\varphi^2 = -I + \eta^i \otimes \xi_i$, $\eta^i(\xi_j) = \delta_j^i$, from which $\varphi\xi_i = 0$, $\eta^i \circ \varphi = 0$ follow. An $f.pk$ -structure on a manifold M^{2n+s} is said to be *normal* if the tensor field $N = [\varphi, \varphi] + 2d\eta^i \otimes \xi_i$ vanishes, $[\varphi, \varphi]$ denoting the Nijenhuis torsion of φ . It is known that one can consider a Riemannian metric g on M^{2n+s} associated with an $f.pk$ -structure (φ, ξ_i, η^i) , such that $g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^s \eta^i(X)\eta^i(Y)$, for any $X, Y \in \Gamma(TM^{2n+s})$, and the structure $(\varphi, \xi_i, \eta^i, g)$ is then called a *metric $f.pk$ -structure*. Therefore, TM^{2n+s} splits as complementary orthogonal sum of its subbundles $\text{Im } \varphi$ and $\ker \varphi$. We denote their respective differentiable distributions by \mathcal{D} and \mathcal{D}^\perp . Let Φ denote the 2-form on M^{2n+s} defined by $\Phi(X, Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(TM^{2n+s})$.

Several subclasses have been studied from different points of view ([3, 4, 7, 10, 9, 11, 12]), also dropping the normality condition and, in this case, the term *almost* precedes the name of the considered structures or manifolds. As in ([3]), a metric $f.pk$ -structure is said a \mathcal{K} -structure if it is normal and the fundamental 2-form Φ is closed; a manifold with a \mathcal{K} -structure is called a \mathcal{K} -manifold. In particular, if $d\eta^i = \Phi$, for all $i \in \{1, \dots, s\}$, the \mathcal{K} -structure is said an \mathcal{S} -structure and M^{2n+s} an \mathcal{S} -manifold. Finally, if $d\eta^i = 0$ for all $i \in \{1, \dots, s\}$, then the \mathcal{K} -structure is called a \mathcal{C} -structure and M^{2n+s} is said a \mathcal{C} -manifold. Obviously, if $s = 1$, a \mathcal{K} -manifold M^{2n+1} is a quasi Sasakian manifold, a \mathcal{C} -manifold is a cosymplectic manifold and an \mathcal{S} -manifold is a Sasakian manifold.

We recall that the Levi-Civita connection ∇ of a metric $f.pk$ -manifold satisfies the following formula ([3, 10]):

$$\begin{aligned}
 2g((\nabla_X \varphi)Y, Z) = & 3d\Phi(X, \varphi Y, \varphi Z) - 3d\Phi(X, Y, Z) \\
 & + g(N(Y, Z), \varphi X) + N_j^{(2)}(Y, Z)\eta^j(X) \\
 & + 2d\eta^j(\varphi Y, X)\eta^j(Z) - 2d\eta^j(\varphi Z, X)\eta^j(Y),
 \end{aligned}
 \tag{1}$$

where $N_j^{(2)}$ is given by $N_j^{(2)}(X, Y) = 2d\eta^j(\varphi X, Y) - 2d\eta^j(\varphi Y, X)$.

Furthermore, for \mathcal{S} -manifolds we have $\nabla_X \xi_j = -\varphi X$, $j = 1, \dots, s$, ([3]). Putting $\bar{\xi} = \sum_{j=1}^s \xi_j$, $\bar{\eta} = \sum_{j=1}^s \eta^j$ is its dual form with respect to g and

$$(\nabla_X \varphi)Y = g(\varphi X, \varphi Y)\bar{\xi} + \bar{\eta}(Y)\varphi^2(X).
 \tag{2}$$

We remark that (2) together with $\mathcal{L}_{\xi_i} g = 0$ and $\mathcal{L}_{\xi_i} \eta^j = 0$, $i, j \in \{1, \dots, s\}$, characterizes the \mathcal{S} -manifolds among the metric $f.pk$ -manifolds.

A metric $f.pk$ -manifold $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ has pointwise constant (p.c.) φ -sectional curvature if at any $p \in M^{2n+s}$, $c(p) = R_p(X, \varphi X, X, \varphi X)$ does not depend on the φ -section spanned by $\{X, \varphi X\}$, for any unit $X \in \mathcal{D}_p$. Several results involving the pointwise constancy of the φ -sectional curvatures of an almost contact metric manifold (i.e. for $s = 1$) are recently obtained in [1, 2, 6, 17]. We refer to [5] for a systematic exposition of the classical curvature results on contact metric manifolds. We recall some known results.

Proposition 1. *A Sasaki manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ has p.c. φ -sectional curvature $c \in \mathfrak{F}(M^{2n+1})$ if and only if its curvature tensor field verifies*

$$\begin{aligned}
 R(X, Y, Z) = & \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\} \\
 & + \frac{c-1}{4}\{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z \\
 & + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
 & + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}
 \end{aligned} \tag{3}$$

for any $X, Y, Z \in \Gamma(TM^{2n+1})$.

A Sasaki manifold M^{2n+1} with constant φ -sectional curvature $c \in \mathbb{R}$ is called a *Sasakian-space-form* and denoted by $M^{2n+1}(c)$. It is well known that, if $n \geq 2$, a Sasaki manifold M^{2n+1} with p.c. φ -sectional curvature c is a Sasakian-space-form.

Definition 1. ([1, 6]) An almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is a *generalized Sasakian-space-form*, denoted by $M^{2n+1}(f_1, f_2, f_3)$, if it admits three smooth functions f_1, f_2, f_3 such that its curvature tensor field verifies, for any $X, Y, Z \in \Gamma(TM^{2n+1})$,

$$\begin{aligned}
 R(X, Y, Z) = & f_1\{g(Y, Z)X - g(X, Z)Y\} \\
 & + f_2\{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\} \\
 & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
 & + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}
 \end{aligned} \tag{4}$$

Remark 1. Any generalized Sasakian-space-form has p.c. φ -sectional curvature $c = f_1 + 3f_2$. Obviously, a Sasaki manifold of p.c. φ -sectional curvature c satisfies (4) with $f_1 = (c + 3)/4$, and $f_2 = f_3 = (c - 1)/4$. A cosymplectic manifold with p.c. φ -sectional curvature c satisfies (4) with $f_1 = f_2 = f_3 = c/4$.

Proposition 2. ([19, 3]) *An S -manifold M^{2n+s} has p.c. φ -sectional curvature $c \in \mathcal{F}(M^{2n+s})$ if and only if its curvature tensor field verifies*

$$\begin{aligned}
 R(X, Y, Z) = & \frac{c+3s}{4}\{g(\varphi X, \varphi Z)\varphi^2 Y - g(\varphi Y, \varphi Z)\varphi^2 X\} \\
 & + \frac{c-s}{4}\{g(Z, \varphi Y)\varphi X - g(Z, \varphi X)\varphi Y + 2g(X, \varphi Y)\varphi Z\} \\
 & + \bar{\eta}(X)\bar{\eta}(Z)\varphi^2 Y - \bar{\eta}(Y)\bar{\eta}(Z)\varphi^2 X \\
 & + g(\varphi Y, \varphi Z)\bar{\eta}(X)\bar{\xi} - g(\varphi X, \varphi Z)\bar{\eta}(Y)\bar{\xi},
 \end{aligned} \tag{5}$$

for any $X, Y, Z \in \Gamma(TM^{2n+s})$.

An S -manifold M^{2n+s} with constant φ -sectional curvature $c \in \mathbb{R}$ is called an *S -space-form* and denoted by $M^{2n+s}(c)$. Moreover, it is also well known that if $n \geq 2$ then an S -manifold with p.c. φ -sectional curvature c is an S -space-form. We remark that for $s = 1$ (5) reduces to (3).

2 Generalized $f.pk$ -space-forms

Proposition 3. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ be a metric $f.pk$ -manifold. Then we have*

- a) $g(\varphi(\nabla_X \varphi)Y, Z) + g((\nabla_X \varphi)\varphi Y, Z) = 0$, for any $Y, Z \in \mathcal{D}$
- b) $g(\varphi(N(Y, Z)) + N(\varphi Y, Z), X) = 0$, for any $X, Y \in \mathcal{D}$,
- c) $2g((\nabla_X \varphi)Y, Z) = 3d\Phi(X, \varphi Y, \varphi Z) - 3d\Phi(X, Y, Z) + g(N(Y, Z), \varphi X)$, for any $X, Y, Z \in \mathcal{D}$.

Proof: Covariantly differentiating $\varphi^2 = -I + \eta^k \otimes \xi_k$ with respect to any $X \in \Gamma(TM^{2n+s})$, we get $\varphi \circ (\nabla_X \varphi) + (\nabla_X \varphi) \circ \varphi = (\nabla_X \eta^k) \otimes \xi_k + \eta^k \otimes \nabla_X \xi_k$ and the first relation follows. We obtain b) since, for $Y, Z \in \Gamma(TM^{2n+s})$, using the definition of N , we have

$$\begin{aligned} \varphi(N(Y, Z)) = & -N(\varphi Y, Z) - \eta^k([Y, \varphi Z]\xi_k + \eta^k(Y)[\xi_k, \varphi Z] \\ & - \varphi(Z)(\eta^k(Y))\xi_k - \eta^k(Y)\varphi([\xi_k, Z]) + 2d\eta^k(\varphi Y, Z)\xi_k. \end{aligned}$$

Finally, c) follows from (1). □

Let \mathcal{F} denote any set of smooth functions F_{ij} on M^{2n+s} such that $F_{ij} = F_{ji}$ for any $i, j \in \{1, \dots, s\}$.

Definition 2. A *generalized f.pk-space-form*, denoted by $M^{2n+s}(F_1, F_2, \mathcal{F})$, is a metric *f.pk*-manifold $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ which admits smooth functions F_1, F_2, \mathcal{F} such that its curvature tensor field verifies

$$\begin{aligned} R(X, Y, Z) = & F_1 \{g(\varphi X, \varphi Z)\varphi^2 Y - g(\varphi Y, \varphi Z)\varphi^2 X\} \\ & + F_2 \{g(Z, \varphi Y)\varphi X - g(Z, \varphi X)\varphi Y + 2g(X, \varphi Y)\varphi Z\} \\ & + \sum_{i,j=1}^s F_{ij} \{\eta^i(X)\eta^j(Z)\varphi^2 Y - \eta^i(Y)\eta^j(Z)\varphi^2 X \\ & + g(\varphi Y, \varphi Z)\eta^i(X)\xi_j - g(\varphi X, \varphi Z)\eta^i(Y)\xi_j\}, \end{aligned} \tag{6}$$

For $s = 1$, we obtain a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ with $f_1 = F_1, f_2 = F_2$ and $f_3 = F_1 - F_{11}$. In particular, if the given structure is either Sasakian, or Kenmotsu, or possibly cosymplectic, then (6) holds with $F_{11} = 1, F_1 = (c+3)/4, F_2 = (c-1)/4$ and $f_3 = F_1 - F_{11} = (c-1)/4 = f_2$ in the first case, $F_{11} = -1, F_1 = (c-3)/4, F_2 = (c+1)/4$ and $f_3 = F_1 - F_{11} = (c+1)/4 = f_2$ in the second case, and $F_{11} = 0, F_1 = c/4, F_2 = c/4$ and $f_3 = c/4$ in the last case.

From (6), by direct computations, we get the following links between the curvatures and the functions in \mathcal{F} .

Proposition 4. Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a generalized *f.pk*-space-form. We have:

- 1) the φ -sectional curvature is p.c. $c = F_1 + 3F_2$,
- 2) $R(X, \xi_h, X, \xi_k) = F_{hk}$, for any unit $X \in \mathcal{D}$ and any $h, k \in \{1, \dots, s\}$,
- 3) for any unit $X \in \mathcal{D}$, for the sectional curvature we get $K(\xi_k, X) = F_{kk}$ and $K(\bar{\xi}, X) = \sum_{h,k=1}^s F_{hk}$.

Proposition 5. Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a generalized *f.pk*-space-form. For any $k \in \{1, \dots, s\}$ we have:

- 1) $R(X, \xi_k, Z) = -\sum_{j=1}^s F_{kj} \{g(\varphi Z, \varphi X)\xi_j + \eta^j(Z)\varphi^2(X)\}$, for any X, Z ,
- 2) $R(X, Z, \xi_k) = \sum_{j=1}^s F_{kj} \{\eta^j(X)\varphi^2(Z) - \eta^j(Z)\varphi^2(X)\}$, for any X, Z ,
- 3) $R(X, \xi_k, Z) = -g(Z, X) \sum_{j=1}^s F_{kj}\xi_j, X, Z \in \mathcal{D}$,
- 4) $R(X, Z, \xi_k) = 0, X, Z \in \mathcal{D}$.

We consider the (0, 4) tensor field P defined by

$$\begin{aligned} P(X, Y, Z, W) = & g(X, \varphi Z)g(Y, W) - g(X, \varphi W)g(Y, Z) \\ & - g(Y, \varphi Z)g(X, W) + g(Y, \varphi W)g(X, Z). \end{aligned} \tag{7}$$

Proposition 6. Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a generalized *f.pk*-space-form. For any $X, Y, Z, W \in \mathcal{D}$, we have:

- 1) $g(R(X, Y, Z), \varphi W) + g(R(X, Y, \varphi Z), W) = (F_2 - F_1)P(X, Y, Z, W)$,
- 2) $R(X, Y, \varphi X, \varphi Y) = R(X, Y, X, Y) + (F_1 - F_2)P(X, Y, \varphi X, Y)$,
- 3) $R(\varphi X, \varphi Y, \varphi Z, \varphi W) = R(X, Y, Z, W)$,
- 4) $R(X, \varphi X, Y, \varphi Y) = R(X, Y, X, Y) + R(X, \varphi Y, X, \varphi Y) - 2(F_2 - F_1)P(X, Y, X, \varphi Y)$.

Proposition 7. Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a generalized $f.pk$ -space-form such that $\nabla\xi_k = -\varphi$ for any $k \in \{1, \dots, s\}$. Then, M^{2n+s} is an S -manifold, $F_1 = (c + 3s)/4$, $F_2 = (c - s)/4$ and $F_{ij} = 1$ for any $i, j \in \{1, \dots, s\}$.

Proof: Since $\nabla\xi_k = -\varphi$ implies that ξ_k is Killing and $\mathcal{L}_{\xi_k}\eta^h = 0$ for any $h, k \in \{1, \dots, s\}$, we have $R(X, \xi_k, Z) = \nabla_X \nabla_Z \xi_k - \nabla_{\nabla_X Z} \xi_k = -(\nabla_X \varphi)Z$ and, from Proposition 5,

$$(\nabla_X \varphi)Z = \sum_{j=1}^s F_{kj} \{g(\varphi Z, \varphi X)\xi_j + \eta^j(Z)\varphi^2 X\}.$$

It follows that, for any $h, k \in \{1, \dots, s\}$ and $X, Z \in \Gamma(TM^{2n+s})$

$$g(\varphi Z, \varphi X) \sum_{j=1}^s F_{kj} \xi_j = g(\varphi Z, \varphi X) \sum_{j=1}^s F_{hj} \xi_j, .$$

Hence we get $F_{hj} = F_{kj} = F_{11}$ and $(\nabla_X \varphi)Z = F_{11} \{g(\varphi Z, \varphi X)\bar{\xi} + \bar{\eta}(Z)\varphi^2 X\}$. Then, putting $Z = \xi_k$, using $\nabla\xi_k = -\varphi$, we obtain $F_{11} = 1$ and $(\nabla_X \varphi)Z = g(\varphi Z, \varphi X)\bar{\xi} + \bar{\eta}(Z)\varphi^2 X$, which, together with $\mathcal{L}_{\xi_k}\eta^h = 0$ and the Killing condition on the ξ_i 's, characterizes the S -manifolds. Finally, for such manifolds we have $g(R(X, Y, Z), \varphi W) + g(R(X, Y, \varphi Z), W) = -sP(X, Y, Z, W)$, hence $F_1 - F_2 = s$, which, being $F_1 + 3F_2 = c$, gives $F_2 = (c - s)/4$ and $F_1 = (c + 3s)/4$. \square

Proposition 8. Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a generalized $f.pk$ -space-form such that any ξ_k is a parallel vector field. Then, $F_{ij} = 0$ for any $i, j \in \{1, \dots, s\}$.

Proof: Since $\nabla\xi_k = 0$, we have $R(X, \xi_k, Z) = 0$ for any $k \in \{1, \dots, s\}$ and $X, Z \in \Gamma(TM^{2n+s})$. Then, Proposition 5 implies $\sum_{j=1}^s F_{kj} \xi_j = 0$, for any $k \in \{1, \dots, s\}$, from which we obtain $F_{kj} = 0$. \square

Remark 2. If $M^{2n+s}(F_1, F_2, \mathcal{F})$ is a generalized $f.pk$ -space-form with an underlying C -structure, then any ξ_k is a parallel vector field and $F_{ij} = 0$ for any $i, j \in \{1, \dots, s\}$. Moreover, $g(R(X, Y, Z), \varphi W) + g(R(X, Y, \varphi Z), W) = 0$ and so $F_1 = F_2$ which together with $F_1 + 3F_2 = c$ gives $F_2 = F_1 = c/4$.

Proposition 9. Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a generalized $f.pk$ -space-form and suppose that $\nabla\xi_k = 0$ for $k \geq 2$ and $\nabla\xi_1 = -\varphi^2$. Then $F_{11} = -1$ and $F_{ij} = 0$ for any $(i, j) \neq (1, 1)$.

Proof: Since $\nabla\xi_k = 0$, for $k \geq 2$, arguing as in the above Proposition, we have $F_{kj} = 0$, for any $(k, j) \neq (1, 1)$. From Proposition 5 we get $R(X, Y, \xi_1) = F_{11} \{-\eta^1(Y)\varphi^2 X + \eta^1(X)\varphi^2 Y\}$, while, by direct computation, we have $R(X, Y, \xi_1) = \eta^1(Y)\varphi^2 X - \eta^1(X)\varphi^2 Y$. Thus $F_{11} = -1$ follows. \square

Definition 3. Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ be a metric $f.pk$ -manifold and \mathcal{H} a set of smooth functions H_{ij} on M^{2n+s} such that $H_{ij} = H_{ji}$ for any $i, j \in \{1, \dots, s\}$. For any $p \in M^{2n+s}$ we denote by $N_p(\mathcal{H})$ the linear space consisting of the vectors Z such that

$$R(X, Y, Z) = \sum_{i,j=1}^s H_{ij} \{ \eta^i(X)\eta^j(Z)\varphi^2 Y - \eta^i(Y)\eta^j(Z)\varphi^2 X + g(\varphi Y, \varphi Z)\eta^i(X)\xi_j - g(\varphi X, \varphi Z)\eta^i(Y)\xi_j \}.$$

We call $N(\mathcal{H})$ -distribution the distribution $p \in M^{2n+s} \rightarrow N_p(\mathcal{H})$.

It is easy to verify that the distribution $N(\mathcal{H})$ is integrable and any ξ_k belongs to it if and only if $R(X, Y, \xi_k) = \sum_{i=1}^s H_{ik} \{ \eta^i(X)\varphi^2 Y - \eta^i(Y)\varphi^2 X \}$.

Now, we are able to state the following characterization.

Theorem 1. *A metric $f.pk$ -manifold $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ is a generalized $f.pk$ -space-form if and only if*

- a) M^{2n+s} has p.c. φ -sectional curvature c ,
- b) there exists a smooth function $l \in \mathfrak{F}(M^{2n+s})$ such that for any $X, Y \in \mathcal{D}$

$$R(X, Y, \varphi X, \varphi Y) - R(X, Y, X, Y) = lP(X, Y, \varphi X, Y)$$

- c) all the vector fields ξ_k belong to an $N(\mathcal{H})$ -distribution.

Proof: Let us suppose that $M^{2n+s}(F_1, F_2, \mathcal{F})$ is a generalized $f.pk$ -space-form. Then a),b),c), are satisfied with $c = F_1 + 3F_2$, $l = F_1 - F_2$ and $\mathcal{H} = \mathcal{F}$.

Vice versa, let M^{2n+s} be a metric $f.pk$ -manifold. By a direct computation, using the first Bianchi identity, one proves the following formula for any $X, Y \in \mathcal{D}$

$$\begin{aligned} &6(R(X, Y, X, Y) + R(\varphi X, \varphi Y, \varphi X, \varphi Y)) + 20R(X, Y, \varphi X, \varphi Y) \\ &\quad - 2(R(X, \varphi Y, X, \varphi Y) + R(\varphi X, Y, \varphi X, Y) + 2R(X, \varphi Y, \varphi X, Y)) \\ &= 3(D(X + \varphi Y) + D(X - \varphi Y)) - 4(D(X) + D(Y)) \\ &\quad - D(X + Y) - D(X - Y) \end{aligned} \tag{8}$$

where $D(X) = R(X, \varphi X, X, \varphi X)$. We denote by T_1 and T_2 the (1, 3)-tensor fields given by

$$T_1(X, Y, Z) = g(\varphi X, \varphi Z)\varphi^2 Y - g(\varphi Y, \varphi Z)\varphi^2 X$$

$$T_2(X, Y, Z) = g(Z, \varphi Y)\varphi X - g(Z, \varphi X)\varphi Y + 2g(X, \varphi Y)\varphi Z$$

and with the same symbol the associated (0,4)-tensor fields, which are algebraic curvature tensor fields, as it can be easily proved. By some computations we can rewrite the right side of (8) as $8c(T_1 + T_2)(X, Y, X, Y)$ and we also obtain $(3T_1 - T_2)(X, Y, X, Y) = -3P(X, Y, \varphi X, Y)$. Using b) and (8) we get $R(X, Y, X, Y) = (\frac{c+3l}{4}T_1 + \frac{c-l}{4}T_2)(X, Y, X, Y)$, from which, for any $X, Y, Z, W \in \mathcal{D}$, we have

$$R(X, Y, Z, W) = \left(\frac{c+3l}{4}T_1 + \frac{c-l}{4}T_2 \right) (X, Y, Z, W).$$

Finally, using c), with standard technique, we obtain that the tensor fields R and $\frac{c+3l}{4}T_1 + \frac{c-l}{4}T_2$ coincide when computed on some or all ξ_k . Hence, the curvature tensor field R verifies (6) with $F_1 = (c+3l)/4$, $F_2 = (c-l)/4$ and $\mathcal{F} = \mathcal{H}$, so M^{2n+s} is a generalized $f.pk$ -space-form. \square

3 The effects of the second Bianchi identity

Definition 4. An $f.pk$ -space-form, denoted by $M^{2n+s}(F_1, F_2, \mathcal{F})(c)$, is a generalized $f.pk$ -space-form $M^{2n+s}(F_1, F_2, \mathcal{F})$ which has constant φ -sectional curvature $c \in \mathbb{R}$.

Lemma 1. *Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a generalized $f.pk$ -space-form, $n \geq 2$. Then, for any choice of*

orthonormal vector fields $X, Y, \varphi Y \in \mathcal{D}$ and for any $k, h \in \{1, \dots, s\}$, we obtain:

$$\begin{aligned} & \xi_k(F_1) + F_1\{g(\nabla_X \xi_k, X) + g(\nabla_Y \xi_k, Y)\} \\ & - \sum_{j=1}^s F_{kj}\{g(\nabla_X \xi_j, X) + g(\nabla_Y \xi_j, Y)\} = 0 \\ & F_1g(\nabla_Y \xi_k, \varphi Y) + F_2g(X, \varphi \nabla_X \xi_k) - \sum_{j=1}^s F_{kj}g(\nabla_Y \xi_j, \varphi Y) = 0 \\ & F_1g(\nabla_Y \xi_k, \varphi X) - F_2\{3g(X, (\nabla_{\xi_k} \varphi)Y) + 3g(X, \varphi \nabla_Y \xi_k) \\ & - 2g(Y, \varphi \nabla_X \xi_k)\} - \sum_{j=1}^s F_{kj}g(\nabla_Y \xi_j, \varphi X) = 0 \\ & Y(F_{kh}) - F_1\eta^h(\nabla_{\xi_k} Y) + \sum_{i=1}^s F_{ih}\eta^i([\xi_k, Y]) \\ & + \sum_{j=1}^s F_{kj}\eta^h(\nabla_Y \xi_j) = 0 \\ & \xi_k(F_2) + F_2(g(\nabla_X \xi_k, X) + g(\nabla_Y \xi_k, Y)) = 0. \end{aligned} \tag{9}$$

Proof: We apply the second identity $\mathfrak{S}_{\xi_k, X, Y}((\nabla_{\xi_k} R)(X, Y))Z = 0$ to X, Y, Z vector fields in \mathcal{D} with $\|X\| = \|Y\| = 1$ and $g(Y, X) = g(\varphi Y, X) = 0$, and choosing $Z = X$, we obtain

$$\begin{aligned} 0 = & -\xi_k(F_1)Y + \sum_{j=1}^s Y(F_{kj})\xi_j \\ & + F_1\{\eta^t([\xi_k, Y])\xi_t - g(\nabla_X \xi_k, X)Y + g(\nabla_Y \xi_k, X)X - \nabla_Y \xi_k\} \\ & + F_2\{3g(X, (\nabla_{\xi_k} \varphi)Y)\varphi X + 3g(X, \varphi \nabla_Y \xi_k)\varphi X \\ & - g(X, \varphi \nabla_X \xi_k)\varphi Y - 2g(Y, \varphi \nabla_X \xi_k)\varphi X\} \\ & + \sum_{i,j=1}^s F_{ij}\{\eta^i([\xi_k, Y])\xi_j - \delta_k^i \eta^j(\nabla_X X)Y \\ & + \delta_k^i \nabla_Y \xi_j + \delta_k^i \eta^j(\nabla_Y X)X\}. \end{aligned}$$

Taking the scalar product with $Y, \varphi Y, \varphi X$ and any ξ_h we obtain the first four equations. Analogously, putting $Z = \varphi X$, we get the last equation. \square

Lemma 2. Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a generalized f -pk-space-form, $n \geq 2$. Then, for any choice of orthonormal vector fields $X, Y, \varphi Y \in \mathcal{D}$, we have

$$\begin{aligned} & Y(F_1) + 3F_2g(X, (\nabla_X \varphi)\varphi Y) = 0 \\ & 2X(F_2) = 3F_2\{g(X, (\nabla_{\varphi Y} \varphi)Y) - g(X, (\nabla_Y \varphi)\varphi Y)\} \\ & F_1\eta^t([\varphi Y, Y]) - F_{it}\eta^i([\varphi Y, Y]) + 2F_2g((\nabla_X \varphi)X, \xi_t) = 0, \text{ for any } t. \end{aligned} \tag{10}$$

Proof: We apply the second Bianchi identity $\mathfrak{S}_{W, X, Y}((\nabla_W R)(X, Y))Z = 0$ to X, Y, Z, W vector fields in \mathcal{D} . Then, choosing $Z = X, W = \varphi Y, \|X\| = \|Y\| = 1$ and $g(Y, X) = g(\varphi Y, X) = 0$, we obtain

$$\begin{aligned} 0 = & -\varphi Y(F_1)Y - 2X(F_2)\varphi X + Y(F_1)\varphi Y - F_1\eta^k([\varphi Y, Y])\xi_k \\ & + F_2\{3g(X, (\nabla_{\varphi Y} \varphi)Y)\varphi X - 3g(X, (\nabla_Y \varphi)\varphi Y)\varphi X \\ & + g(X, (\nabla_X \varphi)\varphi Y) + g(X, (\nabla_X \varphi)Y)Y - 2(\nabla_X \varphi)X\} \\ & + \sum_{i,j=1}^s F_{ij}\eta^i([\varphi Y, Y])\xi_j. \end{aligned}$$

Taking the scalar product with $\varphi Y, \varphi X$ and any ξ_t we get (10). \square

Theorem 2. Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a generalized f -pk-space-form, $n \geq 2$, and assume $F_2 = 0$. If $g(\nabla_X \xi_k, X) = 0$, for any $X \in \mathcal{D}$ and $k \in \{1, \dots, s\}$, then $c = F_1$ is constant.

Proof: The statement follows from the first equations in (9) and (10). \square

Theorem 3. *Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a generalized f.pk-space-form, $n \geq 2$, and assume that F_2 never vanishes.*

1) *If $n = 2$, then F_1 and F_2 are constant along \mathcal{D} if and only if $d\Phi = 0$ on \mathcal{D} .*

2) *If $n \geq 3$, then F_1 and F_2 are constant along \mathcal{D} .*

Furthermore, if $g(\nabla_X \xi_k, X) = 0$ for any $X \in \mathcal{D}$ and $k \in \{1, \dots, s\}$, then, in both cases, F_1, F_2 and c are constant on M^{2n+s} .

Proof: We consider orthonormal vector fields $X, Y, \varphi Y$ in \mathcal{D} and from Lemma 2, since also $\varphi X, Y, \varphi Y$ are orthonormal, we have

$$Y(F_1) + 3F_2g(\varphi X, (\nabla_{\varphi X}\varphi)Y) = 0 \tag{11}$$

which, together the first equation in (10), implies

$$g(X, (\nabla_X\varphi)Y) - g(\varphi X, (\nabla_{\varphi X}\varphi)Y) = 0.$$

Using b) and c) of Proposition 3, we get

$$g(N(X, Y), X) = 0. \tag{12}$$

Moreover, interchanging X and Y in the second equation in (10), we obtain

$$2Y(F_2) = 3F_2\{g(Y, (\nabla_{\varphi X}\varphi)X) - g(Y, (\nabla_X\varphi)Y)\}$$

and

$$Y(F_1 + F_2) = 0, \quad Y(F_1 - F_2) = 9F_2d\Phi(X, Y, \varphi X). \tag{13}$$

It follows that $d\Phi(X, Y, \varphi X) = 0$ with $g(Y, X) = g(Y, \varphi X) = 0$, is a necessary and sufficient condition to obtain that F_1, F_2, c are constant along \mathcal{D} . Moreover, for $n = 2$ such a condition is equivalent to $d\Phi = 0$ on \mathcal{D} and 1) is proved.

Now, to prove 2), fixed $X, Y \in \mathcal{D}$ and $\{X_1, \dots, X_{2n}\}$ as local orthonormal basis of \mathcal{D} , we apply the Bianchi identity $\mathfrak{S}_{X, X_h, Y}((\nabla_X R)(X_h, Y))\varphi X_h = 0$. Then, summing over h and using the first condition in (13), we get

$$\begin{aligned} & (1 - 2n)X(F_2)\varphi Y + (2n - 1)Y(F_2)\varphi X - \varphi Y(F_2)X + \varphi X(F_2)Y \\ & - 2g(Y, \varphi X) \sum_h X_h(F_2)X_h \\ & + F_1 \sum_i \{(\nabla_X \eta^i)\varphi Y - (\nabla_Y \eta^i)\varphi X + (\nabla_{\varphi X} \eta^i)Y - (\nabla_{\varphi Y} \eta^i)X\} \xi_i \\ & - \sum_{i,q} F_{iq} \{(\nabla_X \eta^i)\varphi Y - (\nabla_Y \eta^i)\varphi X + (\nabla_{\varphi X} \eta^i)Y - (\nabla_{\varphi Y} \eta^i)X\} \xi_q \\ & + F_2 \{2n((\nabla_Y \varphi)X - (\nabla_X \varphi)Y) + \sum_h g((\nabla_Y \varphi)X - (\nabla_X \varphi)Y, X_h)X_h \\ & - 2 \sum_h g((\nabla_{X_h} \varphi)X, Y)X_h + 2g(Y, \varphi X) \sum_h (\nabla_{X_h} \varphi)\varphi X_h \\ & + \sum_h g((\nabla_{X_h} \varphi)\varphi X_h, Y)\varphi X - \sum_h g((\nabla_{X_h} \varphi)\varphi X_h, X)\varphi Y\} = 0. \end{aligned} \tag{14}$$

Assuming $g(Y, X) = g(Y, \varphi X) = 0, \|Y\| = \|X\| = 1$ and taking the scalar product with Y , we get $\varphi X(F_2) + (2n - 1)F_2g((\nabla_Y \varphi)X, Y) = 0$. Moreover, we apply (11) replacing Y and X with φX and $-\varphi Y$, respectively, and since $\varphi X(F_2) = -\varphi X(F_1)$ we get $\varphi X(F_2) = -3F_2g((\nabla_Y \varphi)X, Y)$. Then, we obtain $(2n - 4)F_2g((\nabla_Y \varphi)X, Y) = 0$. It follows that, if $n \geq 3$,

$$g((\nabla_Y \varphi)Y, X) = 0 \tag{15}$$

for $g(Y, X) = g(Y, \varphi X) = 0$. Hence $\varphi X(F_2) = 0$ and we obtain the constancy of F_2, F_1, c along \mathcal{D} . Finally, the first and last equation in (9) and the hypothesis $g(\nabla_X \xi_k, X) = 0$ imply $\xi_k(F_1) = \xi_k(F_2) = 0$. Hence $\xi_k(c) = 0$, concluding the proof. \square

Remark 3. If $n = 2$, by Proposition 3 b) and (12) we get $g(N(X, Y), Z) = 0$ for any vector fields $X, Y, Z \in \Gamma(TM^{4+s})$. Furthermore, if the ξ_k 's are Killing vector fields, and, for $n = 2$, $d\Phi = 0$ on \mathcal{D} , then M^{2n+s} is an $f.pk$ -space-form. Moreover, the condition $g(\nabla_X \xi_k, X) = 0$ for any $X \in \mathcal{D}$ and $k \in \{1, \dots, s\}$, which influences the dependence of F_1 and F_2 on the ξ_k 's, implies that the ξ_k 's are Killing vector fields on \mathcal{D} .

Theorem 4. Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a generalized $f.pk$ -space-form, $n \geq 2$, and assume that F_2 never vanishes. If either $n = 2$ and $d\Phi = 0$ on \mathcal{D} , or $n \geq 3$, then F_1, F_2 are constant along \mathcal{D} . Furthermore, we have

$$g((\nabla_X \varphi)Y, Z) = 0. \tag{16}$$

for any $X, Y, Z \in \mathcal{D}$ and \mathcal{D} is a CR-integrable distribution.

Proof: Let us suppose $n = 2$ and $d\Phi = 0$ on \mathcal{D} . Then (16) follows from c) of Proposition 3 and $g(N(X, Y), Z) = 0$ for any $X, Y, Z \in \Gamma(TM^{4+s})$. Now, we assume $n \geq 3$. Since $g((\nabla_Y \varphi)Y, Y) = g((\nabla_Y \varphi)Y, \varphi Y) = 0$, taking account of (15), we have $g((\nabla_Y \varphi)Y, X) = 0$, for any $X \in \mathcal{D}$. By polarization, for any $X, Y, Z \in \mathcal{D}$ we get

$$g((\nabla_Y \varphi)Z, X) + g((\nabla_Z \varphi)Y, X) = 0 \tag{17}$$

Then, in (14), we take the scalar product with any $Z \in \mathcal{D}$, use (17) and $F_2 \neq 0$, obtain $g((\nabla_Z \varphi)Y, X) - (2n + 1)g((\nabla_Y \varphi)Z, X) = 0$. Then, using (17), we get (16). \square

As concerns the functions F_{hk} , we have the following result.

Theorem 5. Let M^{2n+s} be a generalized $f.pk$ -space-form, $n \geq 2$ such that $g(\nabla_X \xi_k, X) = 0$ for any $X \in \mathcal{D}$ and $k \in \{1, \dots, s\}$. Then, for any $Y \in \mathcal{D}$ and $h, k, t \in \{1, \dots, s\}$, we have

$$\begin{aligned} & Y(F_{kh}) - F_1 \eta^h(\nabla_{\xi_k} Y) + \sum_{i=1}^s F_{ih} \eta^i([\xi_k, Y]) \\ & + \sum_{j=1}^s F_{kj} \eta^h(\nabla_Y \xi_j) = 0 \\ & \xi_h(F_{kt}) - \xi_k(F_{ht}) - \sum_{j=1}^s F_{kj} \eta^j(\nabla_{\xi_h} \xi_t) + \sum_{j=1}^s F_{hj} \eta^j(\nabla_{\xi_k} \xi_t) \\ & - \sum_{j=1}^s F_{jt} \eta^j([\xi_h, \xi_k]) = 0. \end{aligned}$$

Proof: The first equation is the fourth equation in (9). To determine the dependence of the functions F_{hk} on the ξ_j 's, we apply the second Bianchi identity $\mathfrak{S}_{W, \xi_h, \xi_k}((\nabla_W R)(\xi_h, \xi_k))Z = 0$ with W, Z arbitrary vector fields in \mathcal{D} and $h, k \in \{1, \dots, s\}$. Then, we consider the component in \mathcal{D}^\perp and using Proposition 3 a), we obtain

$$\begin{aligned} 0 = & (\xi_h(F_{kj}) - \xi_k(F_{hj}))g(Z, W)\xi_j \\ & + \sum_{i,j=1}^s F_{ij} \{g(Z, \nabla_W \xi_h)\delta_k^i \xi_j - g(Z, \nabla_W \xi_k)\delta_h^i \xi_j \\ & + g(Z, W)(\delta_k^i(\nabla_{\xi_h} \xi_j))^{\mathcal{D}^\perp} - \delta_h^i(\nabla_{\xi_k} \xi_j)^{\mathcal{D}^\perp} - \eta^i([\xi_h, \xi_k]\xi_j)\}. \end{aligned}$$

Taking the scalar product with any ξ_t we get

$$\begin{aligned} 0 = & (\xi_h(F_{kt}) - \xi_k(F_{ht}))g(Z, W) + F_{it} \{g(Z, \nabla_W \xi_h)\delta_k^i \\ & - g(Z, \nabla_W \xi_k)\delta_h^i\} - g(Z, W)F_{kj} \eta^j(\nabla_{\xi_h} \xi_t) \\ & + g(Z, W)F_{hj} \eta^j(\nabla_{\xi_k} \xi_t) - g(Z, W)F_{jt} \eta^j([\xi_h, \xi_k]). \end{aligned}$$

Finally, putting $Z = W$, $\|W\| = 1$, we complete the proof. \square

We end this section remarking that the different behavior in the cases $n = 2$ and $n \geq 3$ is, in a certain sense, related to the results due to Tricerri and Vanhecke ([21]) and to Olszak ([20]). It is clear that there exist $f.pk$ -manifolds such that $d\Phi = 0$ on \mathcal{D} . This happens, for example, for \mathcal{K} -manifolds and for $f.pk$ -manifolds of Kenmotsu type. However it can simply happen that $d\eta^k = 0$ for all $k \in \{1, \dots, s\}$ and this implies that the distribution \mathcal{D} is integrable. Then we can consider a $2n$ -dimensional integral submanifold \bar{M} and using the Gauss equation we get that the second fundamental form is given by $\alpha(X, Y) = -\sum_{k=1}^s g(Y, \nabla_X \xi_k) \xi_k$ and the curvature of \bar{M} satisfies the following formula, for X, Y, Z, W tangent to \bar{M}

$$\begin{aligned} \bar{R}(X, Y, Z, W) = & R(X, Y, Z, W) + \sum_{k=1}^s \{g(Z, \nabla_X \xi_k)g(W, \nabla_Y \xi_k) \\ & - g(Z, \nabla_Y \xi_k)g(W, \nabla_X \xi_k)\}. \end{aligned} \tag{18}$$

Now, let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a generalized $f.pk$ -space-form and suppose that $F_2 \neq 0$ everywhere and any ξ_k is a parallel vector field. Then, by Proposition 8 we have $F_{ij} = 0$ for any $i, j \in \{1, \dots, s\}$. Furthermore, $d\eta^k = 0$ and M^{2n+s} is locally a Riemannian product of an integral submanifold of \mathcal{D} and \mathbb{R}^s . Moreover, $\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) = (F_1\pi_1 + F_2\pi_2)(X, Y, Z, W)$, where $\pi_1(X, Y, Z) = g(Z, Y)X - g(Z, X)Y$ and $\pi_2(X, Y, Z) = g(Z, \varphi Y)\varphi X - g(Z, \varphi X)\varphi Y + 2g(X, \varphi Y)\varphi Z$. Hence (\bar{M}, J, \bar{g}) , with $J = \varphi|_{\bar{M}}$, $\bar{g} = g|_{\bar{M}}$ is a $2n$ -dimensional generalized complex space-form. Therefore, if $n \geq 3$, by the theorem of Tricerri and Vanhecke \bar{M} is a Kählerian space-form. Then $d\Phi = 0$ on \mathcal{D} and, by Theorem 3, $M^{2n+s}(F_1, F_2, \mathcal{F})$ is an $f.pk$ -space-form. Furthermore, if $d\Phi$ vanishes everywhere, the underlying structure is an almost \mathcal{C} -structure with Kählerian leaves and, being $\nabla \xi_k = 0$, it is a \mathcal{C} -structure ([15]).

On the contrary, for $n = 2$, Olszak proved that there exist generalized complex space-forms with $F_2 \neq 0$ everywhere and non constant, characterized as Bochner flat globally conformal Kähler manifold. This allows to construct an example of $(4 + s)$ -dimensional generalized $f.pk$ -space-form with p.c., non constant, φ -sectional curvature, which turns out to be the local model of such $(4 + s)$ -dimensional generalized $f.pk$ -space-forms. In fact, one considers a 4-dimensional generalized complex space-form (\bar{M}, J, \bar{g}) with $\bar{R} = f_1\pi_1 + f_2\pi_2$, f_2 nowhere vanishing and non constant, and the Riemannian product $M^{4+s} = \bar{M} \times \mathbb{R}^s$, with metric $g = \bar{g} + g_0$, g_0 being the standard metric on \mathbb{R}^s . We put $\xi_k = \frac{\partial}{\partial x^k}$ and define a $(1, 1)$ -tensor field φ putting $\varphi \bar{X} = J\bar{X}$ for $\bar{X} \in \Gamma(T\bar{M})$, $\varphi \xi_k = 0$ for any $k \in \{1, \dots, s\}$ and the 1-forms η^k by $\eta^k(X) = g(X, \xi_k)$. It is easy to verify that $(M^{4+s}, \varphi, \xi_k, \eta^k, g)$ is a metric $f.pk$ -manifold, the ξ_k 's are parallel and the curvature satisfies (6) with any $F_{ij} = 0$, $F_1 = f_1$ and $F_2 = f_2$ nowhere vanishing and non constant.

4 \mathcal{K} -manifolds and generalized $f.pk$ -space-forms

We begin recalling some known results.

Lemma 3. ([14]) Let M^{2n+s} be an $f.pk$ -manifold with structure $(\varphi, \xi_i, \eta^i, g)$. If M^{2n+s} is normal then we have: $[\xi_i, \xi_j] = 0$, $2d\eta^j(X, \xi_i) = -(\mathcal{L}_{\xi_i} \eta^j)X = 0$, $\mathcal{L}_{\xi_i} \varphi = 0$, and $d\eta^i(\varphi X, Y) = -d\eta^i(X, \varphi Y)$, for any $i, j \in \{1, \dots, s\}$ and $X, Y \in \Gamma(TM^{2n+s})$.

Since a \mathcal{K} -manifold is a normal $f.pk$ -manifold $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ with $d\Phi = 0$, the s vector fields ξ_i are Killing ([3]). Moreover, (1) reduces to

$$g((\nabla_X \varphi)Y, Z) = \sum_{j=1}^s (d\eta^j(\varphi Y, X)\eta^j(Z) - d\eta^j(\varphi Z, X)\eta^j(Y)). \tag{19}$$

Hence $\nabla_{\xi_j} \varphi = 0$, for any $j \in \{1, \dots, s\}$, and $\nabla \varphi = 0$, if each $d\eta^j$ vanishes.

Theorem 6. ([9]) *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ be an $f.pk$ -manifold. Then M^{2n+s} is a \mathcal{K} -manifold if and only if $\mathcal{L}_{\xi_i} \eta^j = 0$, for any $i, j \in \{1, \dots, s\}$ and there exists a family of symmetric tensor fields of type $(1, 1)$, A_i , $i \in \{1, \dots, s\}$ which commute with φ and satisfy*

$$(\nabla_X \varphi)Y = \sum_{i=1}^s \{g(A_i X, Y)\xi_i - \eta^i(Y)A_i X\}.$$

In [9] it is also proved that the tensor fields of type $(1, 1)$, A_i , defined by $A_i(X) = \varphi(\nabla_X \xi_i)$, satisfy the conditions in the previous theorem.

Theorem 7. *Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a generalized $f.pk$ -space-form, with a \mathcal{K} -structure and $n \geq 2$. If $F_2 = 0$ or F_2 never vanishes, then M^{2n+s} has constant φ -sectional curvature $c = F_1 + 3F_2$ and the functions in \mathcal{F} depend on the direction of the ξ_k 's, only, according to $\xi_h(F_{kt}) = \xi_k(F_{ht})$, for any $h, k, t \in \{1, \dots, s\}$.*

Proof: The statement follows from Theorem 5, since the structure is normal and the ξ_k 's are Killing. □

Theorem 8. *Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a generalized $f.pk$ -space-form, with a \mathcal{K} -structure, $n \geq 2$, and $F_2 \neq 0$ everywhere. Then M^{2n+s} has constant φ -sectional curvature $c = F_1 + 3F_2$ and the functions in \mathcal{F} verify the following conditions:*

$$F_{kh} = \frac{tr(A_h \circ A_k)}{2n}, \quad F_{kk} = \frac{tr(A_k^2)}{2n} \geq 0, \quad F_{hk}^2 = F_{hh}F_{kk}.$$

Moreover, if $n \geq 3$, then $l = F_1 - F_2 = \sum_{k=1}^s F_{kk}$ is a constant.

Proof: We apply the second Bianchi identity $\mathfrak{S}_{\xi_h, X, Y}((\nabla_{\xi_h} R)(X, Y))\xi_k = 0$ with X, Y unit vector fields in \mathcal{D} . and $h, k \in \{1, \dots, s\}$. Then, we obtain

$$0 = F_{hk}\eta^j([X, Y])\xi_j + F_{ij}\{\eta^i([\xi_h, X])\delta_k^j Y - \eta^i([\xi_h, Y])\delta_k^j X - \eta^k([X, Y])\delta_h^i \xi_j + \eta^k(\nabla_X \xi_j)\delta_h^i Y - \eta^k(\nabla_Y \xi_j)\delta_h^i X\}$$

and taking the scalar product with any ξ_t we have

$$F_{hk}\eta^t([X, Y]) + F_{ht}\{g(\nabla_X \xi_k, Y) - g(\nabla_Y \xi_k, X)\} = 0.$$

Since the ξ_i 's are Killing, we get $g(X, F_{hk}\nabla_Y \xi_t - F_{ht}\nabla_Y \xi_k) = 0$. Being $X \in \mathcal{D}$, we can write $g(\varphi X, F_{hk}\varphi(\nabla_Y \xi_t) - F_{ht}\varphi(\nabla_Y \xi_k)) = 0$, and we obtain $F_{hk}A_t = F_{ht}A_k$, for any $h, k, t \in \{1, \dots, s\}$, which also implies

$$\sum_{i=1}^s F_{ij}A_i = \sum_{i=1}^s F_{ii}A_j. \tag{20}$$

Moreover, in this context, using (20), the second equation in (9) becomes

$$(F_1 - \sum_{j=1}^s F_{jj})g(A_k Y, Y) - F_2 g(X, A_k X) = 0.$$

We remark that (9), and then the above equation, holds for any unit vector fields $X, Y \in \mathcal{D}$ such that $g(X, Y) = g(X, \varphi Y) = 0$. Now, each A_k , being a symmetric operator, is diagonalizable. Moreover each ξ_i is an eigenvector with eigenvalue zero and if $X \in \mathcal{D}$ is an eigenvector, then φX is an eigenvector corresponding to the same eigenvalue, since A_k commutes with φ . Therefore, we can construct a φ -basis $\{X_1, \dots, X_n, \varphi X_1, \dots, \varphi X_n, \xi_1, \dots, \xi_s\}$ of eigenvectors. For any $i, h \in \{1, \dots, n\}, i \neq h$, we have

$$\begin{aligned} (F_1 - \sum_{j=1}^s F_{jj})g(A_k X_h, X_h) &= F_2 g(X_i, A_k X_i) \\ (F_1 - \sum_{j=1}^s F_{jj})g(A_k X_i, X_i) &= F_2 g(X_h, A_k X_h) \end{aligned} \tag{21}$$

Taking the trace we get $(F_1 - F_2 - \sum_{j=1}^s F_{jj})tr(A_k) = 0$. Now we consider $\Delta = (F_1 - \sum_{j=1}^s F_{jj})^2 - F_2^2 = (F_1 - F_2 - \sum_{j=1}^s F_{jj})(F_1 + F_2 - \sum_{j=1}^s F_{jj})$ and observe that $\Delta = 0$. In fact if $\Delta \neq 0$, for any $k \in \{1, \dots, s\}$ locally we get $A_k = 0$ which implies $\nabla_X \xi_k \in \mathcal{D}^\perp$ and $\nabla \xi_k = 0$, so $d\eta^k = 0$. It follows that, locally, M^{2n+s} is a \mathcal{C} -manifold and by Proposition 8 and Remark 2 we have $F_{ij} = 0$, $F_1 = F_2$, $F_1 - F_2 - \sum_{j=1}^s F_{jj} = 0$, obtaining a contradiction.

Hence we have $\Delta = 0$ and the factors can not be both zero since F_2 nowhere vanishes. We claim that, if $n \geq 3$, we have

$$F_1 - F_2 - \sum_{j=1}^s F_{jj} = 0. \tag{22}$$

Assuming $F_1 - F_2 - \sum_{j=1}^s F_{jj} \neq 0$, we get $F_1 + F_2 - \sum_{j=1}^s F_{jj} = 0$ and, from (21), $g(A_k X_h, X_h) = -g(X_i, A_k X_i)$, $h \neq i$. So for the eigenvalues we have $\lambda_1 = -\lambda_2 = -\lambda_3 = \dots = -\lambda_n$ and $\lambda_2 = -\lambda_1 = -\lambda_3 = \dots = -\lambda_n$, which, for $n \geq 3$, gives $A_k = 0$, for any $k \in \{1, \dots, s\}$. As before one obtains a contradiction, namely $F_1 - F_2 - \sum_{j=1}^s F_{jj} = 0$.

We remark that for $n = 2$ we obtain, $A_k = \text{diag}(\lambda, -\lambda, \lambda, -\lambda, 0, \dots, 0)$, with respect to the φ -basis of eigenvectors $\{X_1, X_2, \varphi X_1, \varphi X_2, \xi_1, \dots, \xi_s\}$, and $tr(A_k^2) = 4\lambda^2$.

In any case, for $n \geq 2$, using Proposition 4, $\nabla_X \xi_k = -\varphi(A_k X)$ and Theorem 6, we get for any $X \in \mathcal{D}$ and $k, h \in \{1, \dots, s\}$,

$$F_{kh} = g(A_k X, A_h X) = g((A_h \circ A_k)X, X).$$

It follows that $F_{kh} = \frac{1}{2n} tr(A_h \circ A_k)$, $F_{kk} = \frac{1}{2n} tr(A_k^2)$, $A_k^2 = -\frac{1}{2n} tr(A_k^2)\varphi^2$, $F_{kk} \geq 0$, $F_{hk}^2 = F_{hh}F_{kk}$, and, if $n \geq 3$, $l = F_1 - F_2 = \sum_{k=1}^s F_{kk} \geq 0$ is a non negative constant, since F_1, F_2 are constant on M^{2n+s} . □

Remark 4. A \mathcal{K} -manifold is an \mathcal{S} -manifold if and only if $A_i = -\varphi^2$. Then, in the hypothesis of the previous Theorem, we get $A_i^2 = \varphi^4 = -\varphi^2 = A_i$ and, comparing with $A_i^2 = -F_{ii}\varphi^2 = F_{ii}A_i$, we obtain $F_{kk} = 1$, $F_{hk} = 1$ and $F_1 - F_2 = s$, according to Proposition 7

A \mathcal{K} -manifold is a \mathcal{C} -manifold if and only if $A_i = 0$. Then, in the hypothesis of the previous Theorem, we get $A_i^2 = 0$ and comparing with $A_i^2 = -F_{ii}\varphi^2$ we obtain $F_{kk} = 0$, $F_{hk} = 0$ and $F_1 - F_2 = 0$, according to Remark 2

Also \mathcal{K} -structures satisfying the conditions $d\eta^i = \Phi$ for some indexes $i \in \{1, \dots, s\}$ and $d\eta^i = 0$ for the others have been studied in [9]. Such manifolds are subject to a theorem of local decomposition saying that a \mathcal{K} -manifold of dimension $2n + s$, $s \geq 2$, which admits r closed 1-forms among the η^i 's, $1 \leq r < s$, whereas the remaining $t = s - r$ coincide with Φ , can be viewed locally as a product of an \mathcal{S} -manifold of dimension $2n + t$ and a flat manifold of dimension r . Hence, the behavior of the φ -sectional curvature depends on the \mathcal{S} -factor.

5 f -manifolds of Kenmotsu type and generalized $f.pk$ -space forms

In [11, 12], we introduced the classes of (almost) Kenmotsu $f.pk$ -manifolds. By definition, a metric $f.pk$ -manifold M^{2n+s} , $s \geq 1$, with $f.pk$ -structure $(\varphi, \xi_i, \eta^i, g)$, is said to be an almost *Kenmotsu $f.pk$ -manifold* if the 1-forms η^i 's are closed and $d\Phi = 2\eta^1 \wedge \Phi$. A Kenmotsu $f.pk$ -manifold is a normal almost Kenmotsu $f.pk$ -manifold. We recall that M^{2n+s} is a Kenmotsu $f.pk$ -manifold if and only if for any vector fields X, Y , one has:

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi_1 - \eta^1(Y)\varphi(X), \tag{23}$$

Moreover, in a Kenmotsu $f.pk$ -manifold we have

$$\nabla \xi_1 = -\varphi^2, \quad \nabla \xi_j = 0, \quad 2 \leq j \leq s, \tag{24}$$

and each ξ_j , with $j \neq 1$, is a Killing vector field. The distribution \mathcal{D} is integrable and its leaves are $2n$ -dimensional totally umbilical Kähler manifolds with mean curvature vector field $H = -\xi_1$. The

distribution $\mathcal{D}^\perp = \langle \xi_1, \dots, \xi_s \rangle$ is integrable, with totally geodesic flat leaves. When $s \geq 2$ we can consider the distribution $\mathcal{D}' = \mathcal{D} \oplus \langle \xi_1 \rangle$, which is integrable and its leaves are totally geodesic Kenmotsu manifolds, and the distribution $\langle \xi_2, \dots, \xi_s \rangle$, which is integrable with totally geodesic flat leaves. The existence of the described involutive distributions also allows to state that Kenmotsu $f.pk$ -manifolds are locally classified as warped products $M^s \times_f N^{2n}$ where N^{2n} is a Kähler manifold, M^s is a flat manifold with coordinates (t^1, \dots, t^s) , and $f = c'e^{t^1}$ for some positive constant c' . Furthermore, if $s \geq 2$, they can be viewed, locally, as Riemannian products of a Kenmotsu manifold and an $(s - 1)$ -dimensional flat manifold. This implies that the behavior of the φ -sectional curvature depends on the Kenmotsu factor, only. Namely, we have (cf. [11])

Theorem 9. *A Kenmotsu $f.pk$ -manifold $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ has p.c. φ -sectional curvature c if and only if, for any vector fields X, Y, Z ,*

$$\begin{aligned}
 R(X, Y, Z) = & \frac{c-3}{4}(g(\varphi X, \varphi Z)\varphi^2 Y - g(\varphi Y, \varphi Z)\varphi^2 X) \\
 & + \frac{c+1}{4}(2g(X, \varphi Y)\varphi Z + g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X) \\
 & - \eta^1(Z)(\eta^1(X)\varphi^2 Y - \eta^1(Y)\varphi^2 X) \\
 & - (\eta^1(X)g(\varphi Y, \varphi Z) - \eta^1(Y)g(\varphi X, \varphi Z))\xi_1.
 \end{aligned} \tag{25}$$

Theorem 10. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $n \geq 2$, be a Kenmotsu $f.pk$ -manifold with pointwise constant φ -sectional curvature c .*

a) *If c is constant, then $c = -1$ and M^{2n+s} is locally a warped product $M^s \times_f N^{2n}$, M^s and N^{2n} being both flat manifolds and f a positive function.*

b) *For any $p \in M^{2n+s}$ with $c(p) \neq -1$ there exists a neighborhood $W = B^s \times F$ of p such that $c|_W \neq -1$ everywhere, $g|_W = g_0 + f^2\bar{g}$, $c + 1 = kf^{-2}$, k constant. Moreover (B^s, g_0) is flat and $(F, J = \varphi|_F, \bar{g} = g|_F)$ is a Kähler manifold with constant holomorphic sectional curvature $c + 1$.*

Since (24) is a necessary condition for $f.pk$ -Kenmotsu manifolds, we discuss the influence of such condition on the underlying $f.pk$ -structure of a generalized $f.pk$ -space-form.

Theorem 11. *Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a generalized $f.pk$ -space-form and suppose that $\nabla \xi_k = 0$ for $k \geq 2$ and $\nabla \xi_1 = -\varphi^2$. If $n \geq 3$ and $F_2 \neq 0$ everywhere, then the underlying $f.pk$ -structure is of Kenmotsu type if and only if $\mathcal{L}_{\xi_k} \varphi = 0$, for any $k \in \{1, \dots, s\}$. Moreover, $M^{2n+s}(F_1, F_2, \mathcal{F})$ is a generalized $f.pk$ -space-form with p.c., non constant, φ -sectional curvature and it is locally classified as in b) of Theorem 10.*

Proof: By Proposition 9 we know that $F_{kj} = 0$ for $(k, j) \neq (1, 1)$, and $F_{11} = -1$. Obviously, $d\eta^k = 0$ for $k \geq 2$ and one can easily verify that $d\eta^1 = 0$, so the distribution \mathcal{D} is integrable. Then we can consider a $2n$ -dimensional integral submanifold \bar{M} and using the Gauss equation we get that the second fundamental form is given by $\alpha(X, Y) = -g(Y, \nabla_X \xi_1)\xi_1$. By (18) and (6) we have

$$\bar{R}(X, Y, Z) = R(X, Y, Z) - \pi_1(X, Y, Z) = ((F_1 + 1)\pi_1 + F_2\pi_2)(X, Y, Z) \tag{26}$$

for X, Y, Z tangent to \bar{M} . Hence (\bar{M}, J, \bar{g}) , with $J = \varphi|_{\bar{M}}$, $\bar{g} = g|_{\bar{M}}$ is a $2n$ -dimensional generalized complex space-form with functions $f_1 = F_1 + 1$, $f_2 = F_2$ and p.c. holomorphic sectional curvature $\bar{c} = 1 + c|_{\bar{M}}$.

From Lemma 1 we get $\xi_k(F_1) = \xi_k(F_2) = 0$, for $k \geq 2$, $\xi_1(F_1) = -2(F_1 + 1)$, $\xi_1(F_2) = -2F_2$, which imply $\xi_k(c) = 0$ for $k \geq 2$, and $\xi_1(c) = -2(c + 1)$. Now, assuming $F_2 \neq 0$ everywhere, the structure is subject to condition 1) and 2) of Theorem 3 and in both cases, we have

$$g((\nabla_X \varphi)Y, Z) = 0, \quad g(N(X, Y), Z) = 0, \quad X, Y, Z \in \mathcal{D}.$$

By direct computation we have $g(N(X, Y), \xi_k) = -d\eta^k(\varphi X, \varphi Y) = 0$ and N vanishes on \mathcal{D} and the integral submanifolds are Kähler. Therefore, if $n \geq 3$, by the theorem of Tricerri and Vanhecke \bar{M} is a Kählerian space-form and F_2 must be constant on each \bar{M} .

Computing $(\nabla_X \varphi)Y$, we obtain that equation (23) is satisfied with the only possible exception for $X = \xi_k$, $k \in \{1, \dots, s\}$, and $Y \in \mathcal{D}$. Namely, in such a case, we have $(\nabla_{\xi_k} \varphi)Y = (\mathcal{L}_{\xi_k} \varphi)Y$ and $g(\varphi \xi_k, X)\xi_1 - \eta^1(X)\varphi \xi_k = 0$. It follows that the structure is of Kenmotsu type if and only if $\mathcal{L}_{\xi_k} \varphi = 0$. Finally, we apply Theorem 10 and observe that case a) has to be excluded since $c = -1$ gives $F_2 = 0$. \square

References

- [1] P. ALEGRE, D.E. BLAIR AND A. CARRIAZO, Generalized Sasakian-space-forms, *Israel J. Math.* **141** (2004), 157–183.
- [2] P. ALEGRE AND A. CARRIAZO, Structures on generalized Sasakian-space-forms, *Differential Geom. Appl.* **26** (2008), 656–666.
- [3] D.E. BLAIR, Geometry of manifolds with structural group $U(n) \times O(s)$, *J. Differ. Geom.* **4** (1970), 155–167.
- [4] D.E. BLAIR, Contact manifolds in Riemannian geometry, *Lecture Notes in Math.* **509**, Springer-Verlag, 1976.
- [5] D.E. BLAIR, Riemannian geometry of Contact and Symplectic Manifolds, Birkhäuser, Boston, 2002.
- [6] P. BUEKEN AND L. VANHECKE, Curvature characterizations in contact geometry, *Riv. Mat. Univ. Parma* (4), **14** (1988), 303–313.
- [7] J.L. CABRERIZO, L.M. FERNÁNDEZ AND M. FERNÁNDEZ, The curvature tensor fields on f -manifolds with complemented frames, *An. St. Univ. Al. I. Cuza, Iași, Matematica*, **36** (1990), 151–161.
- [8] L. DI TERLIZZI, J.J. KONDERAK, A.M. PASTORE AND R. WOLAK, \mathcal{K} -structures and foliations, *Annales Univ. Sci. Budapest*, **44** (2001), 171–182.
- [9] L. DI TERLIZZI AND A.M. PASTORE, Some results on \mathcal{K} -manifolds, *Balkan J. Geom. Appl.* **7**, No.1 (2002), 43–62.
- [10] K. DUGGAL, S. IANUS AND A.M. PASTORE, Maps interchanging f -structures and their harmonicity, *Acta Appl. Math.* **67**, 1 (2001), 91–115.
- [11] M. FALCITELLI AND A.M. PASTORE, f -structures of Kenmotsu type, *Mediterr. j. math.* **3**, No.3-4 (2006), 549–564.
- [12] M. FALCITELLI AND A.M. PASTORE, Almost Kenmotsu f -manifolds, *Balkan J. Geom. Appl.* **12**, No.1 (2007), 32–43.
- [13] S.I. GOLDBERG, On the existence of manifolds with an f -structure, *Tensor, New Ser.* **26** (1972), 323–329.
- [14] S.I. GOLDBERG AND K. YANO, Globally framed f -manifolds, *Ill. J. Math.* **15** (1971), 456–474.
- [15] S. IANUS AND A.M. PASTORE, Harmonic maps and f -structures with parallelizable kernel, in *New Developments in Differential Geometry Proceedings of the Conference on Differential Geometry, Budapest 1996* (1999), 143–154.
- [16] K. KENMOTSU, A class of almost contact Riemannian manifolds, *Tôhoku Math. J. II Ser.* **24** (1972), 93–103.

- [17] U.K. KIM, Conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms, *Note Mat.* **26**, no.1 (2006) 55–67.
- [18] S. KOBAYASHI AND K. NOMIZU, *Foundations of Differential Geometry*, Vol. I, II, Intersciences Publishers, New-York, 1963, 1969.
- [19] M. KOBAYASHI AND S. TSUHIYA, Invariant submanifolds of an f -manifold with complemented frames, *Kodai Math. Semin. Rep.* **24** (1972), 430–450.
- [20] Z. OLSZAK, On the existence of generalized complex space forms, *Israel J. Math.* **65** (1989), 214–218.
- [21] F. TRICERRI AND L. VANHECKE, Curvature tensors on almost Hermitian manifolds, *Trans. Am. Math. Soc.* **267**, N.2 (1981), 365–398.
- [22] K. YANO, On a structure defined by a tensor field f satisfying $f^3 + f = 0$, *Tensor, New Ser.* **14** (1963), 99–109.
- [23] K. YANO AND M. KON, *Structures on manifolds*, Series in Pure Math., **3**, World Scientific, Singapore, 1984.

Received: 6.03.2009.

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