Bull. Math. Soc. Sci. Math. Roumanie
Tome 52(100) No. 3, 2009, 333–346

Some new geometric structures of natural lift type on the tangent bundle

by
V.Oproiu and N.Papaghiuc*

To Professor S. Ianuş on the occasion of his 70th Birthday

Abstract

We present some aspects from the study of geometric structures of natural lift type on the tangent bundle of a Riemannian manifold. After presenting, in the first two sections, the basic constructions leading to the almost Hermitian structures of diagonal type, we find the conditions under which such structures are almost Kählerian or Kählerian. In section 4 we present the results concerning the almost anti-Hermitian structures. Next we study the general natural almost Hermitian structures, obtaining similar results. Finally, in section 6 we obtain sufficient conditions for a general natural Kaehler structure on the tangent bundle to be a Kaehler Einstein structure.

Key Words: tangent bundle, Riemannian metric, general natural lift, Einstein structure.
2000 Mathematics Subject Classification: Primary 53C55, Secondary 53C15, 53C05.

1 Introduction

It is well known (see [28]) that the tangent bundle TM of a Riemannian manifold (M, g) has a structure of almost Kaehlerian manifold with an an almost complex structure determined by the isomorphic vertical and horizontal distributions VTM, HTM on TM (the last one being determined by the Levi Civita connection on M) and the Sasaki metric on TM.However, this structure is Kaehler only in the case where the base manifold is locally Euclidean, so the Sasaki metric is rather rigid an it should be interesting to get another Riemannian or pseudo-Riemannian metrics on TM, having some better properties. Among some other authors who studied the differential geometry of the tangent bundle we can quote [6],[7], [8].

First, recall that E.Calabi defined a new Riemannian metric on the cotangent bundle of a Kaehler manifold, by using a special generating function defined by a smooth real valued function depending on the density energy only and has obtained a new almost complex structure, which together with the original one determines a structure of hyper-Kaehler manifold of positive constant Q-sectional curvature. Inspired by this idea, in 1998 the present authors considered a regular Lagrangian on

*Passed away on November 5, 2008

a Riemannian manifold (M, g) defined by a smooth function L depending on the energy density only [20]. A quite interesting result is that the usual nonlinear connection determined by the Euler-Lagrange equations associated to L does coincide with the nonlinear connection defined by the Levi Civita connection of g, thus the horizontal distribution HTM obtained is the standard one. Next we have obtained a Riemannian metric G on the tangent bundle TM (defined by using some natural lifts of g) such that the vertical and horizontal distributions VTM, HTM are orthogonal to each other but they are no longer isometric. Then we have considered an almost complex structure J on TM related to the above Riemannian metric G such that (TM, G, J) is an almost Kaehlerian manifold (Theorem 2 in [20]). From the integrability condition of the almost complex structure J we have obtained an important result. If (M, g) has positive constant sectional curvature then we may obtain a certain smooth function L on the subset T_0M of the nonzero tangent vectors to M such that the structure (T_0M, G, J) is Kaehlerian (Theorem 3 in [20]). Next, we have obtained the Levi Civita connection $\tilde{\nabla}$ of G and its curvature tensor field showing that the Kaehlerian manifold (T_0M, G, J) cannot be an Einstein manifold and cannot have constant holomorphic sectional curvature.

The possibility to consider vertical, complete and horizontal lifts on TM and the using of some natural lifts of g lead to other interesting geometric structures, studied in the last years, and to some interesting relations with problems in Lagrangian and Hamiltonian mechanics.

The first author has studied some properties of a natural lift G, of diagonal type, of the Riemannian metric g (so that it is no longer obtained from a Lagrangian) and a natural almost complex structure J of diagonal type on TM (see [18], [14], [13], and see also [20], [21]). The condition for (TM, G, J) to be a Kähler Einstein manifold leads to the conditions for (M, g) to have constant sectional curvature, and for (TM, G, J) to have constant sectional holomorphic curvature or to be a locally symmetric space. Note that in [14], [18], the first author excludes some important cases which appeared, in a certain sense, as singular cases. Note also that one of the cases excluded in [14] is just the case where the Riemannian metric G is defined by the Lagrangian L, case studied in [20]. Other singular cases excluded by the first author in [14], [13] have been studied by the second author in [26], [27].

A family of natural almost anti-Hermitian structures (G, J) on the tangent bundle TM of a Riemannian manifold (M, g) is defined by a semi-Riemannian metric G (it is a lift of natural type of g to TM) such that the vertical and horizontal distributions VTM, HTM are maximally isotropic and the almost complex structure J is a usual natural lift of g of diagonal type interchanging VTM and HTM. It has been studied by the present authors in [22]. They have obtained the conditions under which this almost-Hermitian structure belongs to one of the eight classes of anti-Hermitian manifolds obtained in the classification given Ganchev and Borisov in [4]. Also, in [21], the present authors consider another semi-Riemannian metric G on TM such that the vertical and horizontal distributions are orthogonal to each other, study the conditions under which the above almost complex structure J defines, together with G, an almost anti-Hermitian structure on TM and obtain the conditions under which this structure belongs to one of the eight classes of anti-Hermitian manifolds obtained in the classification in [4].

In the paper [15], the first author has presented a general expression of the natural almost complex structures on TM. In the definition of the natural almost complex structure J of general type there are involved eight parameters (smooth functions of the density energy on TM). However, from the condition for J to define an almost complex structure, four of the above parameters can be expressed as (rational) functions of the other four parameters. A Riemannian metric G which is a natural lift of general type of the metric g depends on other six parameters. From the conditions for G to be Hermitian with respect to J, one gets that these six parameters can be expressed with the help of the first eight parameters involved in definition of J and two proportionality factors. From the integrability condition for J, we get (beside the condition for the base manifold to be of constant sectional curvature) that two other parameters involved in the definition of J can be expressed as functions of other two essential parameters and their first order derivatives. Thus a natural Hermitian structure (G, J) of general type depends on four essential parameters (two essential parameters are involved in the definition of the integrable almost complex structure J and two are proportionality factors). From the condition for (G, J) to be almost Kählerian, we get that the second proportionality factor is the derivative of the first one. The family of natural Kählerian structures (G, J) of general type on TM depends on three essential coefficients (two are involved in the expression of J, and the third one is the first proportionality coefficient).

In the last section of this present paper we study some properties of the curvature tensor field of the natural Kählerian structure (G, J) of general type on TM. Namely, we are interested in finding the conditions under which the Kählerian manifold (TM, G, J) is Einstein. We found three cases. In the first case the first proportionality factor is expressed as a rational function of the first two essential parameters, and the value of the constant sectional curvature of the base manifold (M, g). It follows that (TM, G, J) has constant holomorphic sectional curvature. In the second case we obtained a second order homogeneous equation in the proportionality factor λ and its first order derivative λ'. Expressing λ' as a function of λ and the remained parameters, we were able to show, after quite long computations, that the manifold $T_0M \subset TM$ consisting of nonzero tangent vectors to M is a Kähler Einstein space. In the third case we obtained an expression which cannot be zero (in fact it is always positive), so this case lead to no Kähler Einstein structure.

The manifolds, tensor fields and other geometric objects we consider in this paper are assumed to be differentiable of class C^∞ (i.e. smooth). We use the computations in local coordinates in a fixed local chart but many results may be expressed in an invariant form by using the vertical and horizontal lifts. The well known summation convention is used throughout this paper, the range of the indices h, i, j, k, l, r being always $\{1, \dots, n\}$. Some quite long computations have been done by using the Mathematica package RICCI for doing tensor calculations.

2 Preliminary results

Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \longrightarrow M$. Recall that TM has a structure of a $2n$-dimensional smooth manifold, induced from the smooth manifold structure of M. This structure is obtained by using local charts on TM induced from usual local charts on M. If $(U, \varphi) = (U, x^1, \dots, x^n)$ is a local chart on M, then the corresponding induced local chart on TM is $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n, y^1, \dots, y^n)$, where the local coordinates $x^i, y^j;\ i, j = 1, \dots, n$, are defined as follows. The first n local coordinates x^i of a tangent vector $y \in \tau^{-1}(U)$ are the local coordinates in the local chart (U, φ) of its base point, i.e. $x^i = x^i \circ \tau$, by an abuse of notation. The last n local coordinates $y^j,\ j = 1, \dots, n$, of $y \in \tau^{-1}(U)$ are the vector space coordinates of y with respect to the natural basis in $T_{\tau(y)}M$ defined by the local chart (U, φ). Due to this special structure of differentiable manifold for TM, it is possible to introduce the concept of M-tensor field on it. The M-tensor fields are defined by their components with respect to the induced local charts on TM (hence they are defined locally), but they can be interpreted as some (partial) usual tensor fields on TM. However, the essential quality of an M-tensor field on TM is that the local coordinate change rule of its components with respect to the change of induced local charts is the same as the local coordinate change rule of the components of a usual tensor field on M with respect to the change of local charts on M. More precisely, an M-tensor field of type (p, q) on TM is defined by sets of n^{p+q} components (functions depending on x^i and y^i), with p upper indices and q lower indices, assigned to induced local charts $(\tau^{-1}(U), \Phi)$ on TM, such that the local coordinate change rule of these components (with respect to induced local charts on TM) is that of the local coordinate components of a tensor field of type (p, q) on the base manifold M (with respect to usual local charts on M), when a change of local charts on M (and hence on TM) is performed (see [11] for further details);

e.g., the components y^i, $i=1,\dots,n$, corresponding to the last n local coordinates of a tangent vector y, assigned to the induced local chart $(\tau^{-1}(U),\Phi)$ define an M-tensor field of type $(1,0)$ on TM. A usual tensor field of type (p,q) on M may be thought of as an M-tensor field of type (p,q) on TM. If the considered tensor field on M is covariant only, the corresponding M-tensor field on TM may be identified with the induced (pullback by τ) tensor field on TM. Some useful M-tensor fields on TM may be obtained as follows. Let $u:[0,\infty)\longrightarrow\mathbf{R}$ be a smooth function and let $\|y\|^2=g_{\tau(y)}(y,y)$ be the square of the norm of the tangent vector $y\in\tau^{-1}(U)$. If δ^i_j are the Kronecker symbols (in fact, they are the local coordinate components of the identity tensor field I on M), then the components $u(\|y\|^2)\delta^i_j$ define an M-tensor field of type $(1,1)$ on TM. Similarly, if $g_{ij}(x)$ are the local coordinate components of the metric tensor field g on M in the local chart (U,φ), then the components $u(\|y\|^2)g_{ij}$ define a symmetric M-tensor field of type $(0,2)$ on TM. The components $g_{0i}=y^kg_{ki}$ define an M-tensor field of type $(0,1)$ on TM.

Denote by $\dot\nabla$ the Levi Civita connection of the Riemannian metric g on M. Then we have the direct sum decomposition

$$TTM=VTM\oplus HTM$$

of the tangent bundle to TM into the vertical distribution $VTM=\mathrm{Ker}\,\tau_*$ and the horizontal distribution HTM defined by $\dot\nabla$. The set of vector fields $(\frac{\partial}{\partial y^1},\dots,\frac{\partial}{\partial y^n})$ on $\tau^{-1}(U)$ defines a local frame field for VTM and for HTM we have the local frame field $(\frac{\delta}{\delta x^1},\dots,\frac{\delta}{\delta x^n})$, where

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-\Gamma^h_{0i}\frac{\partial}{\partial y^h},\quad \Gamma^h_{0i}=y^k\Gamma^h_{ki},$$

and $\Gamma^h_{ki}(x)$ are the Christoffel symbols of g.

The set $(\frac{\partial}{\partial y^1},\dots,\frac{\partial}{\partial y^n},\frac{\delta}{\delta x^1},\dots,\frac{\delta}{\delta x^n})$ defines a local frame on TM, adapted to the direct sum decomposition (1). Remark that

$$\frac{\partial}{\partial y^i}=(\frac{\partial}{\partial x^i})^V,\quad \frac{\delta}{\delta x^i}=(\frac{\partial}{\partial x^i})^H,$$

where X^V and X^H denote the vertical and horizontal lift of the vector field X on M respectively. We can use the vertical and horizontal lifts in order to obtain invariant expressions for some results in this paper. However, we should prefer to work in local coordinates since the formulas are obtained easier and, in a certain sense, they are more natural.

Consider the energy density of the tangent vector y with respect to the Riemannian metric g

$$t=\frac12\|y\|^2=\frac12 g_{\tau(y)}(y,y)=\frac12 g_{ik}(x)y^iy^k,\quad y\in\tau^{-1}(U).$$

Obviously, we have $t\in[0,\infty)$ for all $y\in TM$. Denote by $C=y^i\frac{\partial}{\partial y^i}$ the Liouville vector field on TM and by $\widetilde{C}=y^i\frac{\delta}{\delta x^i}$ the similar horizontal vector field on TM.

3 Almost Hermitian structures of diagonal type

Let $a_1,a_2,b_1,b_2:[0,\infty)\longrightarrow R$ be four real smooth functions and define a diagonal natural 1-st order almost complex structure J on TM, by using these coefficients and the Riemannian metric g, just like as the natural 1-st order lifts of g to TM are obtained in [9]. The expression of J in adapted local frame is given by [15],

$$J\delta_i=a_1(t)\partial_i+b_1(t)g_{0i}C,\quad J\partial_i=-a_2(t)\delta_i-b_2(t)g_{0i}\widetilde{C}.$$

where we have used the following simpler (but less clear) notations:

$$\frac{\partial}{\partial y^i}=\partial_i,\quad \frac{\delta}{\delta x^i}=\delta_i.$$

Proposition 1,[15]. *The operator J defines an almost complex structure on TM if and only if*

$$a_1a_2 = 1, \qquad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1.$$

We assume that $a_1 > 0, a_2 > 0, a_1 + 2tb_1 > 0, a_2 + 2tb_2 > 0$ for all $t \in [0, \infty)$.

Consider a natural Riemannian metric of diagonal type G on TM, induced by g having the expression in local adapted frames defined by the M-tensor fields

$$G^{(1)}_{ij} = G(\partial_i, \partial_j) = c_1 g_{ij} + d_1 g_{0i} g_{0j},$$

$$G^{(2)}_{ij} = G(\delta_i, \delta_j) = c_2 g_{ij} + d_2 g_{0i} g_{0j},$$

$$G(\partial_i, \delta_j) = G(\delta_j, \partial_i) = 0.$$

The associated ($2n \times 2n$-matrix with respect to the adapted local frame has two $n \times n$-blocks on the first diagonal:

$$G = \begin{pmatrix} G^{(1)}_{ij} & 0 \\ 0 & G^{(2)}_{ij} \end{pmatrix}.$$

The coefficients c_1, c_2, d_1, d_2 are smooth functions depending on the energy density $t \in [0.\infty)$ and the conditions for G to be positive definite are given by

$$c_1 > 0, \quad c_2 > 0, \quad c_1 + 2td_1 > 0, \quad c_2 + 2td_2 > 0.$$

The conditions under which the metric G is almost Hermitian with respect to the almost complex structure J considered above, i.e.

$$G(JX, JY) = G(X, Y),$$

for all vector fields X, Y on TM are given by

$$c_1 = \lambda a_1, \quad c_2 = \lambda a_2,$$

$$c_1 + 2td_1 = (\lambda + 2t\mu)(a_1 + 2tb_1),$$

$$c_2 + 2td_2 = (\lambda + 2t\mu)(a_2 + 2tb_2),$$

where $\lambda = \lambda(t)$ and $\lambda + 2t\mu = \lambda(t) + 2t\mu(t)$ are positive smooth function of $t \in [0, \infty)$.

Hence we have

Theorem 2,[15]. *Let J be a diagonal natural almost complex structure on TM defined by g and the functions a_1, a_2, b_1, b_2. Then the family of diagonal natural, Riemannian metrics G on TM defined by g and the functions c_1, c_2, d_1, d_2 with the property that (TM, G, J) is an almost Hermitian manifold must satisfy the conditions*

$$\frac{c_1}{a_1} = \frac{c_2}{a_2} = \lambda, \quad \frac{c_1 + 2td_1}{a_1 + 2tb_1} = \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \lambda + 2t\mu,$$

where λ, μ are smooth functions depending an t and $\lambda > 0, \lambda + 2t\mu > 0$.

In [15], the first author obtains some classes of natural almost Hermitian structures (G, J) on TM, these classes are obtained from the well known classification of the almost Hermitian structures in sixteen classes (see [5]). The results concerning this classification are given by Theorems 4, 5, 6 and 7 in [15].

Some very important particular cases of the almost Hermitian structure obtained in Theorem 2 have been studied further by the present authors. So, if we consider the case when $c_1(t) = a_1(t) = u(t)$,$b_1(t) = d_1(t) = v(t)$, $b_2(t) = d_2(t) = w(t)$, $a_2(t) = c_2(t) = \frac{1}{u(t)}$, $\lambda(t) = 1$ and $\mu(t) = 0$, where $w(t) = -\frac{v}{u(u+2tv)}$, then all the conditions from Proposition 1 and Theorem 2 are satisfied, and we obtain the almost Hermitian structure defined and studied by the first author in [18]. in this case we have

Theorem 3, [18]. *In the above conditions, the almost Hermitian manifold (TM, G, J) is an almost Kaehlerian manifold. Moreover, the almost complex structure J on TM is integrable (i.e. (TM, G, J) is a Kaehler manifold) if (M, g) has constant sectional curvature c and the function v is given by*

$$v = \frac{c - uu'}{2tu' - u}.$$

Also, in [18] the first author obtains the existence of a Kaehler-Einstein structure even in the case where (M, g) has positive constant sectional curvature, but only on a tube around the zero section in TM (Theorem 4.2). He also obtains on TM a structure of Kaehler manifold with constant holomorphic sectional curvature (Theorem 5.1).

The particular case of the above situation, when the function $u(t) = 1$, for all $t \in [0, \infty)$, has been studied by the first author in [14]. In this case we have

Theorem 4,[14]. *(TM, G, J) is an almost Kaehlerian manifold and the almost complex structure J is integrable if and only if (M, g) has constant sectional curvature c and $v = -c$. If $c < 0$ is obtained a Kaehler structure on whole TM. In the case where the constant c is positive is obtained a Kaehler structure in the tube around the zero section in TM defined by $t < \frac{1}{2c}$.*

Moreover, in this case it is proved

Theorem 5. *If (M, g) has negative constant sectional curvature then its tangent bundle has a structure of Kaehler Einstein manifold. If (M, g) has positive constant sectional curvature then a tube around the zero section in TM has a structure of Kaehler Einstein manifold.*

Another important case which appear as a singular case is when $v(t) = 0$ for all $t \in [0, \infty)$ which implies $w(t) = 0$. This case has been studied by the second author in [27], [26]. In this case the author has obtained

Theorem 6,[27]. *Assume that the function $u(t)$ satisfies the condition $u(0)u'(0) \neq 0$. Then the almost complex structure J on TM or on a tube around the zero section in TM is integrable if and only if the base manifold (M, g) has constant sectional curvature c and the function $u(t)$ satisfies the ordinary differential equation $uu' = c$, i.e. $u(t) = \sqrt{2ct + A}$, where A is an arbitrary real constant.*

Another result obtained in [27](see also [26]) is given by

Theorem 7. *If $n \neq 2$, then the Kaehlerian manifold (TM, G, J) cannot be an Einstein manifold and cannot have constant holomorphic sectional curvature. If $n = 2$, then the Kaehlerian manifold (TM, G, J) is Ricci flat.*

Remark that the singular case when $v(t) = u'(t)$ is studied by the present authors in [20].

4 Almost anti-Hermitian structures of diagonal type TM

In this section we consider the same almost complex structure J on TM defined as above in Proposition 1, section 3, therefore this Proposition 1 is still valid. We assume that the coefficients which define J satisfy the conditions $a_1 > 0, a_2 > 0, a_1 + 2tb_1 > 0, a_2 + 2tb_2 > 0$ for all $t \in [0, \infty)$. Also, we consider the semi-Riemannian metric G defined in the local adapted frames by the same M-tensor fields $G^{(1)}_{ij}, G^{(2)}_{ij}$ as above in Theorem 2, section 3, i.e. G is of diagonal type, but it is no longer a Riemannian metric. The conditions for G to be anti-Hermitian with respect to J, are obtained from the property

$$G(JX, JY) = -G(X, Y),$$

for all vector fields X, Y on TM (see [4], [23] for details). Such a metric is called, sometimes, a Norden metric.

The above property of the semi-Riemannian metric G, implies that it has the signature (n,n). Using the expressions in the adapted local frames we can obtain the explicit conditions for G to be almost anti-Hermitian. We have

Proposition 8. *The semi-Riemannian metric G is anti-Hermitian with respect to the almost complex structure J on (TM), i.e. (TM, G, J) is an almost anti-Hermitian manifold, if and only if the coefficients c_1, c_2, d_1, d_2 satisfy the following relations*

$$a_2c_1 + a_1c_2 = 0, \quad (a_1 + 2tb_1)(c_2 + 2tb_2) + (a_2 + 2tb_2)(c_1 + 2td_1) = 0.$$

Remark. From the conditions for (TM, G, J) to be almost anti-Hermitian manifold, we have the following essential relations

$$\begin{cases} a_2 = \frac{1}{a_1}, \quad b_2 = -\frac{b_1}{a_1(a_1+2tb_1)} \\ c_2 = -\frac{c_1}{a_1^2}, \quad d_2 = \frac{2a_1b_1c_1+2tb_1^2c_1-a_1^2d_1}{a_1^2(a_1+2tb_1)^2}. \end{cases}$$

Hence, only the coefficients a_1, b_1, c_1, d_1 can be considered as being essential.

In [21] we get the local coordinate expression of the Levi Civita connection of G, next we study the existence of some classes of almost anti-Hermitian structures (G, J), defined as above on TM, and are interested in the cases where such structures belong to some classes form the classification given by Ganchev and Borisov in [4](see Theorems 7, 8, 9, 10, 12, 14, 15 and Corollaries 11, 13, 16 in [21]). In fact we get specific examples in each of the seven classes of almost anti-Hermitian manifolds.

In [22], we have considered the same almost complex structure J on TM as above, but we considered the property of the semi-Riemannian metric G in the sense that the vertical and horizontal distributions VTM,HTM are maximally isotropic. Remark that Proposition 1 is also valid. Moreover, it is proved

Proposition 9. *Let (M, g) be an $n(> 2)$-dimensional connected Riemannian manifold. The almost complex structure J defined by* (3) *on TM is integrable if and only if (M, g) has constant sectional curvature c and the function b_1 is given by*

$$b_1 = \frac{a_1a_1' - c}{a_1 - 2ta_1'}.$$

Remark. The above relations allow us to express two of the coefficients $a_1,\ a_2, b_1,\ b_2$ as functions of the other two; e.g. we have

$$a_2 = \frac{1}{a_1}, \quad b_2 = \frac{-a_2b_1}{a_1 + 2tb_1} = \frac{-b_1}{a_1(a_1 + 2tb_1)}.$$

Remark. In the case where the almost complex structure J is integrable, we have:

$$b_2 = \frac{c - a_1a_1'}{a_1(a_1^2 - 2ct)}.$$

Next, we have considered a particular 1-st order natural lift G of g to TM defining a semi-Riemannian metric of signature (n,n) on TM.

This lift is defined by two real valued smooth functions $u, v : [0,\infty) \to \mathbf{R}$ and the expression of G in local adapted frames is defined by the conditions

$$G(\delta_i, \delta_j) = 0, \quad G(\partial_i, \partial_j) = 0,$$

$$G(\partial_i, \delta_j) = G(\delta_i, \partial_j) = ug_{ij} + vg_{0i}g_{0j}.$$

Remark that G is defined, essentially, by the symmetric M-tensor field $G_{ij} = ug_{ij} + vg_{0i}g_{0j}$ of type $(0,2)$. The condition for G to be nondegenerate is assured if

$$u \neq 0, \qquad u + 2tv \neq 0.$$

The condition for G to be anti-Hermitian with respect to the almost complex structure J, considered above, is given by

$$G(JX, JY) = -G(X,Y),$$

for all vector fields X, Y on TM.

The above property of the semi-Riemannian metric G and the almost complex structure J can be checked easily by a straightforward computation in the adapted local frames. Hence we have [22]

Proposition 10. *The semi-Riemannian metric G is anti-Hermitian with respect to the almost complex structure J, i.e. (TM, G, J) is an almost anti-Hermitian manifold.*

In [22] we have considered the almost anti-Hermitian manifold (TM, G, J) defined in Proposition 10 and we have obtained specific examples for all seven classes of almost anti-Hermitian manifolds obtained in the classification from [4] (see Theorems 7, 8, 9, 10, 11, 12 in [22].

Remark that the family of general natural almost anti-Hermitian structures on the tangent bundle of a Riemannian manifold and the integrability conditions for these structures have been studied by the first author in [16].

5 General natural almost Hermitian structures on TM

Consider the real valued smooth functions $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ defined on $[0, \infty) \subset \mathbf{R}$. A natural 1-st order almost complex structure on TM, defined by the Riemannian metric g, is obtained just like the natural 1-st order lifts of g to TM are obtained in [9].

Theorem 11. *The natural tensor field J of type $(1,1)$ on TM, given by*

$$\begin{cases} J\frac{\delta}{\delta x^i} = a_1(t)\frac{\partial}{\partial y^i} + b_1(t)g_{0i}C + a_4(t)\frac{\delta}{\delta x^i} + b_4(t)g_{0i}\widetilde{C}, \\ \\ J\frac{\partial}{\partial y^i} = a_3(t)\frac{\partial}{\partial y^i} + b_3(t)g_{0i}C - a_2(t)\frac{\delta}{\delta x^i} - b_2(t)g_{0i}\widetilde{C}, \end{cases} \tag{5.1}$$

defines an almost complex structure on TM, if and only if $a_4 = -a_3, b_4 = -b_3$ and the coefficients a_1, a_2, a_3, b_1, b_2 and b_3 are related by

$$a_1 a_2 = 1 + a_3^2\ , \quad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1 + (a_3 + 2tb_3)^2. \tag{5.2}$$

Remark. From the conditions (2) we have that the coefficients $a_1, a_2, a_1 + 2tb_1, a_2 + 2tb_2$ cannot vanish and have the same sign. We assume that $a_1 > 0,\ a_2 > 0,\ a_1 + 2tb_1 > 0,\ a_2 + 2tb_2 > 0$ for all $t \geq 0$.

Remark. The relations (2) allow us to express two of the coefficients $a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3$ as functions of the other four; e.g. we have:

$$a_2 = \frac{1 + a_3^2}{a_1}, \quad b_2 = \frac{2a_3b_3 - a_2b_1 + 2tb_3^2}{a_1 + 2tb_1}. \tag{5.3}$$

The integrability conditions obtained in [15] are given in the theorem:

Theorem 12. *Let (M, g) be an $n(> 2)$-dimensional connected Riemannian manifold. The almost complex structure J defined by on TM is integrable if and only if (M, g) has constant sectional curvature* *c and the coefficients b_1, b_2, b_3 are given by:*

$$\begin{cases} b_1 = \frac{2c^2ta_2^2+2cta_1a_2'+a_1a_1'-c+3ca_3^2}{a_1-2ta_1'-2cta_2-4ct^2a_2'}, \\ \\ b_2 = \frac{2ta_3'^2-2ta_1'a_2'+ca_2^2+2cta_2a_2'+a_1a_2'}{a_1-2ta_1'-2cta_2-4ct^2a_2'}, \\ \\ b_3 = \frac{a_1a_3'+2ca_2a_3+4cta_2'a_3-2cta_2a_3'}{a_1-2ta_1'-2cta_2-4ct^2a_2'}. \end{cases} \tag{5.4}$$

Remark. The second relation in (3) (or in (2)) is identically fulfilled by the expressions b_1, b_2, b_3 in (4).

Remark. Equivalently, the derivatives of the coefficients a_1, a_2, a_3 can be expressed as functions of a_1, a_2, a_3 and b_1, b_2, b_3.

$$\begin{cases} a_1' = \frac{1}{a_1+2tb_1}(a_1b_1 + c - 3ca_3^2 - 4cta_3b_3), \\ \\ a_2' = \frac{1}{a_1+2tb_1}(2a_3b_3 - a_2b_1 - ca_2^2), \\ \\ a_3' = \frac{1}{a_1+2tb_1}(a_1b_3 - 2ca_2a_3 - 2cta_2b_3). \end{cases} \tag{5.5}$$

In the paper [15], the first author studied the conditions under which a Riemannian metric G of natural type on TM, defined by

$$\begin{cases} G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = c_1g_{ij} + d_1g_{0i}g_{0j} = G^{(1)}_{ij}, \\ \\ G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = c_2g_{ij} + d_2g_{0i}g_{0j} = G^{(2)}_{ij}, \\ \\ G(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}) = G(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) = c_3g_{ij} + d_3g_{0i}g_{0j} = G^{(3)}_{ij}, \end{cases} \tag{5.6}$$

is almost Hermitian with respect to the general almost complex structure J, i.e.

$$G(JX, JY) = G(X, Y),$$

for all vector fields X, Y on TM. He proved the following result

Theorem 13. *The family of natural, Riemannian metrics G on TM such that (TM, G, J) is an almost Hermitian manifold, is given by (6), provided that the coefficients $c_1, c_2, c_3, d_1, d_2,$ and d_3 are related to the coefficients $a_1, a_2, a_3, b_1, b_2,$ and b_3 by the following proportionality relations*

$$\frac{c_1}{a_1} = \frac{c_2}{a_2} = \frac{c_3}{a_3} = \lambda$$

$$\frac{c_1 + 2td_1}{a_1 + 2tb_1} = \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \frac{c_3 + 2td_3}{a_3 + 2tb_3} = \lambda + 2t\mu,$$

where the proportionality coefficients $\lambda > 0$ and $\lambda + 2t\mu > 0$ are functions depending on t.

Remark. In the case where $a_3 = 0$, it follows that $c_3 = d_3 = 0$ and we obtain the almost Hermitian structure considered in [30]. Remark that the functions used in [30] are slightly different of the functions used in the present paper. Moreover, if $\lambda = 1$ and $\mu = 0$, we obtain the almost Kählerian structure considered in [18].

In [17], the first author has proved

Theorem 14. *The almost Hermitian structure (TM, G, J) is almost Kählerian if and only if*

$$\mu = \lambda'. \tag{5.7}$$

Thus the family of general almost Kählerian structures on TM depends on five essential coefficients $a_1, a_3, b_1, b_3, \lambda$. Combining the results from the theorems 12, 13 and 14, we obtain that a general natural Kählerian structure (G, J) on TM is defined by three essential coefficients a_1, a_3, λ. However, these coefficients must satisfy the supplementary conditions $a_1 > 0, a_1 + 2tb_1 > 0, \lambda > 0, \lambda + 2t\mu > 0$. Examples of such structures can be found in [30] (see also [18]).

A very important result obtained in [3] is given by the following theorem:

Theorem 15. *The Kählerian manifold (TM, G, J) with G and J obtained as natural lifts of general type of the Riemannian metric g on the Riemannian manifold (M, g), has constant holomorphic sectional curvature k if and only if the parameter λ is expressed by*

$$\lambda = \frac{4a_1 c}{k(a_1^2 + 2ct + 2a_3^2 ct)}. \tag{5.8}$$

6 General natural Einstein Kähler structures on the tangent bundle

The Levi-Civita connection ∇ of the Riemannian manifold (TM, G) is obtained in [3].

Theorem 16. *The Levi-Civita connection ∇ of G has the following expression in the local adapted frame $(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n})$*

$$\nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = Q^h_{ij} \frac{\partial}{\partial y^h} + \widetilde{Q}^h_{ij} \frac{\delta}{\delta x^h}, \ \nabla_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = (\Gamma^h_{ij} + \widetilde{P}^h_{ji}) \frac{\partial}{\partial y^h} + P^h_{ji} \frac{\delta}{\delta x^h}$$

$$\nabla_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = P^h_{ij} \frac{\delta}{\delta x^h} + \widetilde{P}^h_{ij} \frac{\delta}{\delta y^h}, \ \nabla_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = (\Gamma^h_{ij} + \widetilde{S}^h_{ij}) \frac{\delta}{\delta y^h} + S^h_{ij} \frac{\partial}{\partial y^h},$$

where Γ^h_{ij} are the Christoffel symbols of the connection $\dot{\nabla}$ and M-tensor fields appearing as coefficients in the above expressions are given as

$$\begin{cases} Q^h_{ij} = \frac{1}{2}(\partial_i G^{(2)}_{jk} + \partial_j G^{(2)}_{ik} - \partial_k G^{(2)}_{ij}) H^{kh}_{(2)} + \frac{1}{2}(\partial_i G^{(3)}_{jk} + \partial_j G^{(3)}_{ik}) H^{kh}_{(3)}, \\ \widetilde{Q}^h_{ij} = \frac{1}{2}(\partial_i G^{(2)}_{jk} + \partial_j G^{(2)}_{ik} - \partial_k G^{(2)}_{ij}) H^{kh}_{(3)} + \frac{1}{2}(\partial_i G^{(3)}_{jk} + \partial_j G^{(3)}_{ik}) H^{kh}_{(1)}, \\ \\ P^h_{ij} = \frac{1}{2}(\partial_i G^{(3)}_{jk} - \partial_k G^{(3)}_{ij}) H^{kh}_{(3)} + \frac{1}{2}(\partial_i G^{(1)}_{jk} + R^l_{0jk} G^{(2)}_{li}) H^{kh}_{(1)}, \\ \widetilde{P}^h_{ij} = \frac{1}{2}(\partial_i G^{(3)}_{jk} - \partial_k G^{(3)}_{ij}) H^{kh}_{(2)} + \frac{1}{2}(\partial_i G^{(1)}_{jk} + R^l_{0jk} G^{(2)}_{li}) H^{kh}_{(3)}, \\ \\ S^h_{ij} = -\frac{1}{2}(\partial_k G^{(2)}_{ij} + R^l_{0ij} G^{(2)}_{lk}) H^{kh}_{(2)} + c_3 R_{i0jk} H^{kh}_{(3)}, \\ \widetilde{S}^h_{ij} = -\frac{1}{2}(\partial_k G^{(1)}_{ij} + R^l_{0ij} G^{(2)}_{lk}) H^{kh}_{(3)} + c_3 R_{i0jk} H^{kh}_{(1)}. \end{cases}$$

Here R^h_{kij} are the components of the curvature tensor field of the Levi Civita connection $\dot{\nabla}$ of the base manifold (M, g).

Taking into account the above expressions and by using the above formulas we can obtain the detailed expressions of $P^h_{ij}, Q^h_{ij}, S^h_{ij}, \widetilde{P}^h_{ij}, \widetilde{Q}^h_{ij}, \widetilde{S}^h_{ij}$.

The curvature tensor field K of the connection ∇ is defined by the well known formula

$$K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \Gamma(TM).$$

By using the local adapted frame $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}),\ i, j = 1, \ldots, n$ we obtain, after a standard straightforward computation

$$K\Big(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\Big) \frac{\delta}{\delta x^k} = XXXX^h_{kij} \frac{\delta}{\delta x^h} + XXXY^h_{kij} \frac{\partial}{\partial y^h},$$

$$K(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})\frac{\partial}{\partial y^k}=XXYX^h_{kij}\frac{\delta}{\delta x^h}+XXYY^h_{kij}\frac{\partial}{\partial y^h},$$
$$K(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})\frac{\delta}{\delta x^k}=YYXX^h_{kij}\frac{\delta}{\delta x^h}+YYXY^h_{kij}\frac{\partial}{\partial y^h},$$
$$K(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k}=YYYX^h_{kij}\frac{\delta}{\delta x^h}+YYYY^h_{kij}\frac{\partial}{\partial y^h},$$
$$K(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j})\frac{\delta}{\delta x^k}=YXXX^h_{kij}\frac{\delta}{\delta x^h}+YXXY^h_{kij}\frac{\partial}{\partial y^h},$$
$$K(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j})\frac{\partial}{\partial y^k}=YXYX^h_{kij}\frac{\delta}{\delta x^h}+YXYY^h_{kij}\frac{\partial}{\partial y^h},$$

where the M-tensor fields appearing as coefficients are given by

$$XXXX^h_{kij}=\widetilde{S}^h_{il}\widetilde{S}^l_{jk}+P^h_{li}S^l_{jk}-\widetilde{S}^h_{jl}\widetilde{S}^l_{ik}-P^h_{lj}S^l_{ik}+R^h_{kij}+R^l_{0ij}P^h_{lk}$$
$$XXXY^h_{kij}=\widetilde{S}^l_{jk}S^h_{il}+\widetilde{P}^h_{li}S^l_{jk}-\widetilde{S}^l_{ik}S^h_{jl}-\widetilde{P}^h_{lj}S^l_{ik}+\widetilde{P}^h_{lk}R^l_{0ij}-$$
$$-\frac{1}{2}\dot\nabla_iR^r_{0jk}G^{(2)}_{rl}H^{(3)}_{hl}+c_3\dot\nabla_iR_{j0kh},$$
$$XXYX^h_{kij}=\widetilde{P}^l_{kj}P^h_{li}+P^l_{kj}\widetilde{S}^h_{il}-\widetilde{P}^l_{ki}P^h_{lj}-P^l_{ki}\widetilde{S}^h_{jl}+R^l_{0ij}\widetilde{Q}^h_{lk},$$
$$XXYY^h_{kij}=\widetilde{P}^l_{kj}\widetilde{P}^h_{li}+P^l_{kj}S^h_{il}-\widetilde{P}^l_{ki}\widetilde{P}^h_{lj}-P^l_{ki}S^h_{jl}+R^l_{0ij}Q^h_{lk}+R^h_{kij},$$
$$YYXX^h_{kij}=\partial_iP^h_{jk}-\partial_jP^h_{ik}+\widetilde{P}^l_{jk}\widetilde{Q}^h_{il}+P^l_{jk}P^h_{il}-\widetilde{P}^l_{ik}\widetilde{Q}^h_{jl}-P^l_{ik}P^h_{jl},$$
$$YYXY^h_{kij}=\partial_i\widetilde{P}^h_{jk}-\partial_j\widetilde{P}^h_{ik}+\widetilde{P}^l_{jk}Q^h_{il}+P^l_{jk}\widetilde{P}^h_{il}-\widetilde{P}^l_{ik}Q^h_{jl}-P^l_{ik}\widetilde{P}^h_{jl},$$
$$YYYX^h_{kij}=\partial_i\widetilde{Q}^h_{jk}-\partial_j\widetilde{Q}^h_{ik}+Q^l_{jk}\widetilde{Q}^h_{il}+\widetilde{Q}^l_{jk}P^h_{il}-Q^l_{ik}\widetilde{Q}^h_{jl}-\widetilde{Q}^l_{ik}P^h_{jl},$$
$$YYYY^h_{kij}=\partial_iQ^h_{jk}-\partial_jQ^h_{ik}+Q^l_{jk}Q^h_{il}+\widetilde{Q}^l_{jk}\widetilde{P}^h_{il}-Q^l_{ik}Q^h_{jl}-\widetilde{Q}^l_{ik}\widetilde{P}^h_{jl},$$
$$YXXX^h_{kij}=\partial_i\widetilde{S}^h_{jk}+S^l_{jk}\widetilde{Q}^h_{il}+\widetilde{S}^l_{jk}P^h_{il}-\widetilde{P}^l_{ik}P^h_{lj}-P^l_{ik}\widetilde{S}^h_{jl}-\dot\nabla_jR^r_{0ik}G^{(2)}_{rl}H^{(3)}_{hl},$$
$$YXXY^h_{kij}=\partial_iS^h_{jk}+S^l_{jk}Q^h_{il}+\widetilde{S}^l_{jk}\widetilde{P}^h_{il}-\widetilde{P}^l_{ik}\widetilde{P}^h_{lj}-P^l_{ik}S^h_{jl}-\dot\nabla_jR^r_{0ik}G^{(2)}_{rl}H^{(1)}_{hl},$$
$$YXYX^h_{kij}=\partial_iP^h_{kj}+\widetilde{P}^l_{kj}\widetilde{Q}^h_{il}+P^l_{kj}P^h_{il}-Q^l_{ik}P^h_{lj}-\widetilde{Q}^l_{ik}\widetilde{S}^h_{jl},$$
$$YXYY^h_{kij}=\partial_i\widetilde{P}^h_{kj}+\widetilde{P}^l_{kj}Q^h_{il}+P^l_{kj}\widetilde{P}^h_{il}-Q^l_{ik}\widetilde{P}^h_{lj}-\widetilde{Q}^l_{ik}S^h_{jl}.$$

Remark that, due to the condition for (M,g) to have constant sectional curvature, we have $\dot\nabla R^h_{kij}=0$, and the above formulas become simpler.

The condition under which the Kählerian manifold (TM,G,J) is an Einstein manifold, i.e. $Ric(X,Y)=LG(X,Y)$ has been studied by the present authors in [24]. We obtain that the following equation must be fulfilled:

$$-(a_1^2a_1'\lambda+2a_1c\lambda+2a_1a_3^2c\lambda+a_1^3\lambda'-2a_1'c\lambda t-2a_1'a_3^2c\lambda t+4a_1a_3a_3'c\lambda t+$$
$$2a_1c\lambda't+2a_1a_3^2c\lambda't)(a_1^5a_1'\lambda^2+2a_1^4a_3^2c\lambda^2+a_1^6\lambda\lambda'-a_1^4{a_1'}^2\lambda^2t-$$
$$4a_1^3a_1'c\lambda^2t-4a_1^3a_1'a_3^2c\lambda^2t+4a_1^4a_3a_3'c\lambda^2t-4a_1^4c\lambda\lambda't+4a_1^4a_3^2c\lambda\lambda't+a_1^6\lambda'^2t+$$
$$4a_1^2{a_1'}^2c\lambda^2t^2+4a_1^2{a_1'}^2a_3^2c\lambda^2t^2-8a_1^3a_1'a_3a_3'c\lambda^2t^2+4a_1a_1'c^2\lambda^2t^2+8a_1a_1'a_3^2c^2\lambda^2t^2+$$
$$4a_1a_1'a_3^4c^2\lambda^2t^2-8a_1^2a_3a_3'c^2\lambda^2t^2-8a_1^2a_3^3a_3'c^2\lambda^2t^2+4a_1^2c^2\lambda\lambda't^2+$$
$$8a_1^2a_3^2c^2\lambda\lambda't^2+4a_1^2a_3^4c^2\lambda\lambda't^2-4a_1^4c\lambda'^2t^2+4a_1^4a_3^2c\lambda'^2t^2-$$
$$4{a_1'}^2c^2\lambda^2t^3-8{a_1'}^2a_3^2c^2\lambda^2t^3-4{a_1'}^2a_3^4c^2\lambda^2t^3+16a_1a_1'a_3a_3'c^2\lambda^2t^3+$$
$$16a_1a_1'a_3^3a_3'c^2\lambda^2t^3-16a_1^2a_3^2{a_3'}^2c^2\lambda^2t^3+4a_1^2c^2\lambda'^2t^3+8a_1^2a_3^2c^2\lambda'^2t^3+$$
$$4a_1^2a_3^4c^2\lambda'^2t^3)(a_1^3-2a_1^2a_1't-2a_1ct-2a_1a_3^2ct+4a_1'ct^2+4a_1'a_3^2ct^2-8a_1a_3a_3'ct^2)=0.$$

and we have the following three cases, according to the vanishing of each of the three factors from this equation.

Case I) (The vanishing of the first factor). In this case we obtain the following expression of λ (see [24]

$$\lambda = \frac{2a_1c(n+1)}{L(a_1^2 + 2ct + 2a_3^2ct)}. \tag{6.1}$$

We see that this expression of λ is (up to some change of constants) the same with the expression of λ in (8), in the case of the natural Kählerian structures of general type on the tangent bundles of constant holomorphic sectional curvature. Thus we can state the following result [24]:

Theorem 17. *Let (TM, G, J) be the Kählerian manifold, with G and J obtained as natural lifts of general type of the Riemannian metric g on the base manifold M. Assume that the parameter λ is expressed by* (9)*, where L is a nonzero real constant. Then (TM, G, J) is an Einstein Kähler manifold, i.e. $Ric(X, Y) = LG(X, Y)$, for all vector fields X and Y on TM.*

Remark. Taking into account of Theorem 15 in we remark that the expression (9) of λ implies that the Kählerian manifold (TM, G, J) has constant holomorphic sectional curvature $k = \frac{2L}{n+1}$.

Remark. In the particular subcase where $a_3 = 0$ and $\lambda = 1$ we obtain Theorems 4.2 and 5.1 from [14].

Case II) (The vanishing of the second factor). The relation obtained in this case can be thought of as an equation of second order in λ'. This equation can be solved easily and we get some quite interesting results related to the property of (TM, G, J) to be Kähler Einstein. First of all we shall use the following notation

$$A = \sqrt{a_1^4 - 4a_1^2ct + 4a_1^2a_3^2ct + 4c^2t^2 + 8a_3^2c^2t^2 + 4a_3^4c^2t^2}$$

and we show that the expression under square root is > 0, i.e. A is real.

Then the solutions of the equation from the case II) are

$$\lambda' = \lambda(-\frac{1}{2t} \pm$$
$$\frac{a_1^3 - 2a_1^2a_1't - 2a_1ct - 2a_1a_3^2ct + 4a_1'ct^2 + 4a_1'a_3^2ct^2 - 8a_1a_3a_3'ct^2}{2a_1tA}).$$

Consider only the case of the solution with + and in this case we obtain that the expression of λ is given by (see [24])

$$\lambda = \frac{n(a_1^2 + 2ct + 2a_3^2ct - A)}{4a_1Lt}. \tag{6.2}$$

By using this expression of λ in the condition for (T_0M, G, J) to be an Einstein manifold, we obtain, after quite long computations the property $Ric - LG = 0$. Hence we state

Theorem 18. *Let (T_0M, G, J) be the Kählerian manifold, with G and J obtained as natural lifts of general type of the Riemannian metric g on the base manifold M. Assume that the parameter λ is expressed by* (10)*, where L is a nonzero real constant. Then (T_0M, G, J) is an Einstein Kähler manifold.*

Remark. In the particular case where $a_1 = 1$, $a_3 = 0$ and assuming that $\lambda = 1$, we obtain the main result obtained by the first author in [13] (Theorem 5).

In the case III) (the vanishing of the last factor), replacing a_1', a_3' from (5) we obtain that the last factor becomes

$$\frac{a_1^4 - 4a_1^2ct + 4a_1^2a_3^2ct + 4c^2t^2 + 8a_3^2c^2t^2 + 4a_3^4c^2t^2}{a_1 + 2tb_1} = \frac{A^2}{a_1 + 2tb_1} > 0$$

Therefore, in this case, the first member of the considered equation is always positive, i.e. the Kählerian manifold (TM, G, J) cannot be Einstein.

Remark. In the particular subcase, where $a_3 = 0$ and $c > 0$, it follows that $a_1 = \sqrt{2ct}$ and the obtained Kählerian structure is that defined by the Lagrangian L on T_0M studied by the present authors in [20]. Therefore we obtain Proposition 7 in [20], i.e (T_0M, G, J) cannot be an Einstein manifold.

General remark. From the above results it follows that TM endowed with a Kählerian structure (G, J) of general natural lift type is Einstein if and only if it has constant holomorphic sectional curvature. The subset $T_0M \subset TM$ becomes Einstein if and only if the parameter λ is given by (10).

References

[1] Abbassi, M.T.K., Sarih, M., *On some hereditary properties of Riemannian g-natural metrics on tangent bundles of Riemannian manifolds,* Diff. Geom. And its Appl. 22 (2005), 19-47.

[2] Bejan, C. L., Oproiu, V., *Tangent bundles of quasi-constant holomorphic sectional curvatures*, Balkan J. Geom. Applic., 11 (2006), 11-22

[3] Druţă, S., V. Oproiu., *General natural Kähler structures of constant holomorphic sectional curvature on tangent bundles*, An. St. Univ. "Al.I.Cuza" Iasi, Matematica, 53 (2007), 149-166.

[4] Ganchev, G.T., Borisov, D.V. *Note on the almost complex manifolds with Norden metric*, Compt. Rend. Acad. Bulg. 39 (1986), 31-34.

[5] Gray, L., Hervella, M. *The sixteen classes o f almost Hermitian manifolds and their linear invariants.* Ann. di Mat. Pura ad Applic., (IV), 123 (1980), 35-58.

[6] Ianuş, S. *On certain distributions on the tangent bundle.* (Romanian) Stud. Cerc. Mat. 23 (1971), 873-880.

[7] Ianuş, S. *Quelques remarques sur l'espace fibré tangent d'une variété cosymplectique.* (French) An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.) 17 (1971), 465-471.

[8] Ianuş, S., Udrişte, C. *On the tangent bundle of a differentiable manifold.* (Romanian) Stud. Cerc. Mat. 22 (1970), 599–611.

[9] Kowalski, O., Sekizawa, M., *Natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles - a classification,* Bull. Tokyo Gakugei Univ. (4), 40 (1988), 1-29.

[10] Krupka, D., Janyška, J., *Lectures on Differential Invariants,* Folia Fac. Sci. Nat. Univ. Purkinianae Brunensis, 1990.

[11] Mok, K.P., Patterson, E.M., Wong, Y.C., *Structure of symmetric tensors of type (0,2) and tensors of type (1,1) on the tangent bundle*, Trans. Amer. Math. Soc., 234 (1977), 253-278.

[12] Oproiu, V., *A generalization of natural almost Hermitian structures on the tangent bundles.* Math. J. Toyama Univ., 22 (1999), 1-14.

[13] Oproiu, V., *A locally symmetric Kaehler Einstein structure on the tangent bundle of a space form*, Beiträge Algebra Geom/Contributions to Algebra and Geometry, 40 (1999), 363-372.

[14] Oproiu, V., *A Kaehler Einstein structure on the tangent bundle of a space form*, Int. J. Math. Math. Sci. 25 (3) (2001), 183-195.

[15] Oproiu, V., *Some classes of natural almost Hermitian structures on the tangent bundle*, Publ. Math. Debrecen, 62 (2003) 561-576.

[16] Oproiu V., *General natural almost Hermitian and anti-Hermitian structures on the tangent bundle*, Bull. Math. Soc. Sci. Roum. 43 (91) (2000), 325-340.

[17] Oproiu, V., *A generalization of natural almost Hermitian structures on the tangent bundle*, Math. J. Toyama Univ., 22 (1999), 1-14.

[18] Oproiu, V., Some new geometric structures on the tangent bundles, Publ. Math. Debrecen, 55/3-4 (1999), 261-281.

[19] Oproiu, V.*Bochner flat tangent bundles* An.St.Univ. "Al.I.Cuza" Iasi, Matematica, 52 (2006), 25-36.

[20] Oproiu, V., Papaghiuc, N., *A Kaehler structure on the nonzero tangent bundle of a space form,* Diff. Geom. Appl. 11 (1999), 1-14.

[21] Oproiu, V., Papaghiuc, N. *Some classes of almost anti-Hermitian structures on the tangent bundle*,Mediterranean Journal of Mathematics 1 (3) (2004), 269-282.

[22] Oproiu, V., Papaghiuc, N. *Classes of almost ant-Hermitian structures on the tangent bundle of a Riemannian manifold*, An. St. Univ. "Al.I Cuza",Iaşi, t. L, sIa, f.1, (2004), 175-190.

[23] Oproiu, V., Papaghiuc, N.,*Some examples of almost complex manifolds with Norden metric*, Publ. Math, debrecen, 41 (1992), 199-211.

[24] Oproiu, V., Papaghiuc, M., *General natural Einstein Kähler structures on tangent bundles*, to appear Diff. Geom. Appl. 27 (2009).

[25] Oproiu, V., Poroşniuc, D.,*A class of Kaehler Einstein structures on the cotangent bundle of a space form*, Publ. Math. Debrecen 66 (2005), 457-478.

[26] Papaghiuc, N., *On an Einstein structure on the tangent bundle of a space form*, Publ. Math. Debrecen, 55. (1999), 349-361.

[27] Papaghiuc, N., *Another Kaehler structure on the tangen bundle of a space form*, Demonstratio Mathematica, vol. XXXI, No.4, (1998), 855-866.

[28] Sasaki, S., *On the differential geometry of the tangent bundle of Riemannian manifolds*, Tôhoku Math. J., 10 (1958), 238-354.

[29] Tahara, M., Vanhecke, L., Watanabe, Y., *New structures on tangent bundles*, Note di Matematica (Lecce), 18 (1998), 131-141.

[30] Tahara, M., Watanabe, Y., *Natural almost Hermitian, Hermitian and Kähler metrics on the tangent bundles*, Math. J. Toyama Univ., 20 (1997), 149-160.

Received: 20.04.2009.

V.Oproiu
Faculty of Mathematics
University "Al.I.Cuza", Iaşi
700 506, România
and
Institute of Mathematics
"O.Mayer", Romanian Academy,
Iaşi Branch
E-mail: `voproiu@uaic.ro`