Para-hyperhermitian surfaces

by

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To Professor S. Ianuş on the occasion of his 70th Birthday

Abstract

In this note we discuss the problem of existence of para-hyperhermitian structures on compact complex surfaces. We construct examples of para-hypercomplex structures on Inoue surfaces of type S^- which do not admit compatible metrics.

Key Words: para-hypercomplex, para-hyperhermitian and para-hyperkähler structures, complex surfaces

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1 Introduction

Hypercomplex and hyperkähler structures have been studied for a long time and many interesting results and relations with other fields have been established. Recently there is a growing interest in their pseudo-Riemannian counterparts too due to the fact that important geometry models of string theory carry such structures [15]. The para-hyperhermitian structures arise as a pseudo-Riemannian analog of the hyperhermitian structures and it is well known [12] that in four dimensions they lead to self-dual metrics of neutral signature. There are many other similarities between these two structures, but there are also significant differences. For example, the para-hypercomplex structures, the neutral analog of hypercomplex structures, exist in any even dimension (not only in that divisible by 4) and, in contrast to the latter, they may not have compatible metrics.

An almost para-hypercomplex structure is a triple (J_1, J_2, J_3) of anti-commuting endomorphisms of the tangent bundle with $J_1^2 = -J_2^2 = -J_3^2 = -Id$. When the structures J_1, J_2, J_3 are integrable it is called para-hypercomplex. There are two natural classes of metrics compatible with such structures. The first one consists of the neutral metrics for which the structure J_1 is an isometry while J_2 and J_3 are anti-isometries. These metrics, called para-hyperhermitian in this paper, give rise to three 2-forms defined in the same way as the Kähler forms in the positive definite case. When they are closed the structure is called hypersymplectic [9], para-hyperkähler [3], hyper-parakähler [10], neutral hyperkähler [11], pseudo-hyperkähler [7], etc. The second class consists of positive definite metrics, for which the structures J_1, J_2, J_3 are isometries. Such metrics always exist and the analog of the Kähler form for

the structure J_2 is a symmetric form which is in fact a neutral metric such that J_1 and J_3 are anti-isometries but J_2 is an isometry. As pointed out in [1], neutral metrics with this property are interesting in relation with the doubled geometry models of string theory, introduced by C. Hull [8]. We should note however that the existence of para-hyperhermitian metrics leads to some additional obstructions and a purpose of this note is to clarify the problem for their existence.

The paper is organized as follows. After the preliminary definitions (Section 2) we recall in Section 3 Kamada's classification [11] of compact para- hyperkähler surfaces and relate them to the existence of parallel null vector fields. Then in Section 4 we establish some necessary conditions for the existence of a para-hyperhermitian metric with respect to a given para-hypercomplex structure on a 4-manifold and show that any two such metrics are conformally equivalent. In the last section we show that the Inoue surfaces of type S^+ have para-hyperhermitian structures and provide examples of para-hypercomlex structures on Inoue surfaces of type S^- which do not admit compatible para-hyperhermitian metrics.

2 Preliminaries

A pseudo-Riemannian metric on a smooth 4-manifold M is called *neutral* if it has signature $(+, +, -, -)$. Unlike the Riemannian case, there are topological restrictions for existence of a neutral metric on a compact manifold since it is equivalent to existence of a field of tangent 2-planes [17]. We refer to [13] for further information in this direction.

An almost para-hypercomplex structure on a smooth 4-manifold M consists of three endomorphisms J_1, J_2, J_3 of TM satisfying the relations

$$J_1^2 = -J_2^2 = -J_3^2 = -Id, \quad J_1 J_2 = -J_2 J_1 = J_3 \tag{1}$$

of the imaginary units of the paraquaternionic algebra (split quaternions). A metric g on M is called compatible with the structure $\{J_1, J_2, J_3\}$ if

$$g(J_1 X, J_1 Y) = -g(J_2 X, J_2 Y) = -g(J_3 X, J_3 Y) = g(X, Y) \tag{2}$$

(such a metric is necessarily of split signature). In this case we say that $\{g, J_1, J_2, J_3\}$ is an almost para-hyperhermitian structure. For any such a structure we define three 2-forms Ω_i setting $\Omega_i(X, Y) = g(J_i X, Y)$, $i = 1, 2, 3$. If the Nijenhuis tensors of J_1, J_2, J_3 vanish, the structure $\{g, J_1, J_2, J_3\}$ is called para-hyperhermitian. When additionally the 2-forms $\Omega_i(X, Y) = g(J_i X, Y)$ are closed, the para-hyperhermitian structure is called para-hyperkähler. It is well known [12] that the para-hyperhermitian metrics are self-dual, whereas the para-hyperkähler metrics are self-dual and Ricci-flat.

It is an observation of Hitchin [9] (see also [11]) that any para-hyperkähler structure is uniquely determined by three symplectic forms $(\Omega_1, \Omega_2, \Omega_3)$ satisfying the relations

$$-\Omega_1^2 = \Omega_2^2 = \Omega_3^2, \quad \Omega_l \wedge \Omega_m = 0, \, l \neq m.$$

A similar characterization holds for para-hyperhermitian structures [10, 11]. They are uniquely determined by three non-degenerate 2-forms $(\Omega_1, \Omega_2, \Omega_3)$ and a 1-form θ such that

$$-\Omega_1^2 = \Omega_2^2 = \Omega_3^2, \quad \Omega_l \wedge \Omega_m = 0, \, l \neq m, \quad d\Omega_l = \theta \wedge \Omega_l. \tag{3}$$

For any para-hyperhermitian structure on a 4-manifold M, the 2-form $\Omega = \Omega_2 + i\Omega_3$ is of type $(2, 0)$ with respect to the complex structure J_1, hence the canonical bundle of the complex manifold (M, J_1) is smoothly trivial. Using the well-known classification of compact complex surfaces it follows that para-hyperhermitian structures can exist only on the following surfaces: complex tori, K3 surfaces, primary Kodaira surfaces, Hopf surfaces, Inoue surfaces of type $S_M, S_N^{\pm}$ and properly elliptic surfaces of odd first Betti number. Note that except the $K3$ surfaces all these surfaces can be represented as quotients of Lie groups factored by cocompact discrete subgroups (more details will appear in [5]).

3   Para-hyperkähler surfaces

As is well known (c.f. [2]), any compact hyperkähler surface is either a complex torus with a flat metric or a $K3$-surface with Calabi-Yau metric. In the neutral case, the $(2,0)$-form $\Omega = \Omega_2 + i\Omega_3$ is holomorphic (even parallel), so the canonical bundle is holomorphically trivial. Using this fact H.Kamada [11] proved the following

Theorem 1. *If (M,g,J_1,J_2,J_3) is a compact para-hyperkähler surface, then the complex surface (M,J_1) is biholomorphic to a complex torus or a primary Kodaira surface.*

Moreover, Kamada [11, 12] obtained a description of all para-hyperkähler structures on both types of surfaces.

Theorem 2. *For any para-hyperkähler structure on a complex torus $M = \mathbb{C}^2/\Gamma$ there are complex coordinates (z_1, z_2) of $\mathbb{C}^2$, such that the structure is defined by means of the following symplectic forms:*

$$\Omega_1 = Im(dz_1 \wedge d\overline{z}_2) + (i/2)\partial\overline{\partial}\varphi,$$

$$\Omega_2 = Re(dz_1 \wedge dz_2), \quad \Omega_3 = Im(dz_1 \wedge dz_2),$$

where φ is a smooth function such that

$$4iIm(dz_1 \wedge d\overline{z}_2) \wedge \partial\overline{\partial}\varphi = \partial\overline{\partial}\varphi \wedge \partial\overline{\partial}\varphi. \tag{4}$$

Conversely, any three forms $\Omega_1, \Omega_2, \Omega_3$ of the form given above determine a para-hyperkähler structure on the torus. Moreover, its metric is flat if and only if φ is constant.

Let us note that if M is a product of two elliptic curves, then there are non-trivial solutions of the equation (4) ([12]) and it is not known if such solutions exist when M is not a product.

Before stating Kamada's result about primary Kodaira surfaces, we recall their definition.

Consider the affine transformations $\rho_i(z_1, z_2) = (z_1 + a_i, z_2 + \overline{a}_i z_1 + b_i)$ of $\mathbb{C}^2$, where $a_i, b_i, i = 1, 2, 3, 4$, are complex numbers such that $a_1 = a_2 = 0, Im(a_3\overline{a}_4) = b_1$. Then ρ_i generate a group G of affine transformations acting freely and properly discontinuously on $\mathbb{C}^2$. The quotient space $M = \mathbb{C}^2/G$ is called a primary Kodaira surface.

Theorem 3. *For any para-hyperkähler structure on a primary Kodaira surface M there are complex coordinates (z_1, z_2) of $\mathbb{C}^2$ such that the structure is defined by means of the following symplectic forms:*

$$\Omega_1 = Im(dz_1 \wedge d\overline{z}_2) + iRe(z_1)dz_1 \wedge d\overline{z}_1 + (i/2)\partial\overline{\partial}\varphi,$$

$$\Omega_2 = Re(e^{i\theta}dz_1 \wedge dz_2), \quad \Omega_3 = Im(e^{i\theta}dz_1 \wedge dz_2),$$

where θ is a real constant and φ is a smooth function on M such that

$$4i(Im(dz_1 \wedge d\overline{z}_2) + iRe(z_1)(dz_1 \wedge d\overline{z}_1)) \wedge \partial\overline{\partial}\varphi = \partial\overline{\partial}\varphi \wedge \partial\overline{\partial}\varphi \tag{5}$$

Conversely, any three forms $\Omega_1, \Omega_2, \Omega_3$ of the form given above determine a para-hyperkähler structure on M. Moreover, its metric is flat if and only if φ is constant.

Note that any primary Kodaira surface is a toric bundle over an elliptic curve and the pull-back of any smooth function on the base curve gives a solution to (5). This shows that the moduli space of para-hyperkähler structures on a primary Kodaira surface is infinite dimensional, which is in sharp contrast with the positive definite case.

Non-compact examples of para-hyperkähler mnifolds can be constructed by means of the so-called Walker manifolds.

Recall that a Walker manifold is a triple $(M, g, \mathcal{D})$, where M is a smooth manifold, g an indefinite metric, and $\mathcal{D}$ a parallel null distribution. The local structure of such manifolds has been described by A.Walker [18] and we refer to [6] for a coordinate-free version of his theorem. Of special interest are the Walker metrics on 4-manifolds for which $\mathcal{D}$ is of dimension 2 since they appear in several specific pseudo-Riemannian structures. For example, the metric of every para-hyperkähler structure is Walker, $\mathcal{D}$ being the $(+1)$-eigenbundle of either of its product structures.

According to [18], for every Walker 4-manifold $(M, g, \mathcal{D})$ with $dim \, \mathcal{D} = 2$, there exist local coordinates (x, y, z, t) around any point of M such that the matrix of g has the form

$$g_{(x,y,z,t)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix} \tag{6}$$

for some smooth functions a, b and c. Then a local orthonormal frame of TM can be defined by

$$\mathbf{e_1} = \frac{1-a}{2}\partial_x + \partial_z, \qquad \mathbf{e_2} = \frac{1-b}{2}\partial_y + \partial_t - c\partial_x,$$

$$\mathbf{e_3} = -\frac{1+a}{2}\partial_x + \partial_z, \qquad \mathbf{e_4} = -\frac{1+b}{2}\partial_y + \partial_t - c\partial_x.$$

Let $\{J_1, J_2, J_3\}$ be the (local) almost para-hypercomplex structure for which $J_1\mathbf{e_1} = \mathbf{e_2}, J_1\mathbf{e_3} = \mathbf{e_4}$, $J_2\mathbf{e_1} = \mathbf{e_3}, J_2\mathbf{e_2} = -\mathbf{e_4}$, $J_3\mathbf{e_1} = \mathbf{e_4}, J_3\mathbf{e_2} = \mathbf{e_3}$. This structure is compatible with the Walker metric g, thus we have an almost para-hyperhermitian structure, called proper in [14].

The next two results have been proved in [4].

Theorem 4. *The structure (g, J_1, J_2, J_3) is para-hyperhermitian if and only if the functions a, b and c have the form*

$$a = x^2 K + xP + \xi,$$

$$b = y^2 K + yT + \eta,$$

$$c = xyK + \tfrac{1}{2}xT + \tfrac{1}{2}yP + \gamma,$$

where the capital and Greek letters stand for arbitrary smooth functions of (z, t).

Theorem 5. *The structure (g, J_1, J_2, J_3) is para-hyperkähler if and only if the functions a, b and c do not depend on x and y.*

In particular, the above theorem shows that the neutral Kähler metrics considered by Petean [16] are all para-hyperkähler and hence self-dual and Ricci-flat.

By the Kamada results mentioned above, any compact para-hyperkähler surface admits two parallel, null and orthogonal vector fields. Conversely, Theorem 5 together with the Petean's classification of neutral Ricci-flat Kähler surfaces ([16]) leads to the following

Theorem 6. *Let (M, g) be a compact oriented neutral 4-manifold with two parallel , null and orthogonal vector fields. Then M admits a para-hyperkähler structure $\{g, J_1, J_2, J_3\}$, so (M, J_1) is biholomorphic to a complex torus or a primary Kodaira surface.*

A detailed proof of this theorem will appear in [5].

4   Existence of para-hyperhermitian metrics

In this section we discuss the problem of existence of a metric compatible with a given (almost) para-hypercomplex structure.

A (linear) para-hypercomplex structure on a vector space V is a triple $\{J_1, J_2, J_3\}$ of endomorphisms of V satisfying the relations (1). Note that such structures exist on any even-dimensional vector space. A metric g on V is called compatible with the structure $\{J_1, J_2, J_3\}$ if the identities (2) are satisfied. In this case we say that $\{g, J_1, J_2, J_3\}$ is a (linear) para- hyperhermitian structure on V.

If we are given a hypercomplex structure $\{J_1, J_2, J_3\}$ and g is any positive definite metric on V, then $h(X,Y) = g(X,Y) + g(J_1X, J_1Y) + g(J_2X, J_2Y) + g(J_3X, J_3Y)$ is a positive definite metric compatible with $\{J_1, J_2, J_3\}$. In the case of a para-hypercomplex structure, some authors suggest, by an analogy, to consider the bilinear form $h(X,Y) = g(X,Y) + g(J_1X, J_1Y) - g(J_2X, J_2Y) - g(J_3X, J_3Y)$ where g is a metric. This symmetric form is compatible with the given para-hypercomplex structure but it may be degenerate.

Example. Let e_1, e_2, e_3, e_4 be the standard bases of $\mathbb{R}^4$ and let $\{J_1, J_2, J_3\}$ be the para-hypercomplex structure on $\mathbb{R}^4$ for which $J_1e_1 = e_2, J_1e_3 = e_4$, $J_2e_1 = e_3, J_2e_2 = -e_4$, $J_3e_1 = e_4, J_3e_2 = e_3$. If g is the standard metric on $\mathbb{R}^4$, then the endomorphisms J_1, J_2, J_3 are isometries of g, hence the form $h(X,Y) = g(X,Y) + g(J_1X, J_1Y) - g(J_2X, J_2Y) - g(J_3X, J_3Y)$ is identically zero. Similarly, if g is the metric for which $e_1, ..., e_4$ is an orthogonal basis with $g(e_1, e_1) = g(e_3, e_3) = 1$, $g(e_2, e_2) = g(e_4, e_4) = -1$ (in this case J_1 and J_3 are anti-isometrices, while J_2 is an isometry).

The next observation is implicitly contained in [12].

Lemma 7. *Let $\{J_1, J_2, J_3\}$ be a para-hypercomplex structure on a vector space V. Let $V^{\pm}$ be the ± 1-eigenspace of the endomorphism J_2. Then there is a bijective correspondence between the set of non-degenerate skew-symmetric 2-forms on the space $V^{\pm}$ and the set of metrics on V compatible with the given para-hypercomplex structure.*

Proof: Let h be a non-degenerate skew-symmetric 2-form on V^+. Extend this form to a form on the whole space V setting $h(V, V^-) = h(V^-, V) = 0$. Now set $g(X,Y) = h(X, J_1Y) + h(Y, J_1X)$ for $X, Y \in V$. Then g is a symmetric bilinear form on V and $g(J_1X, J_1Y) = g(X,Y)$. Note that the spaces $V^{\pm}$ are g-isotropic since J_1 interchanges V^+ and V^-. Let $X = X^+ + X^-$, $Y = Y^+ + Y^-$ be the $V^{\pm}$-decomposition of arbitrary vectors $X, Y \in V$. Then $g(J_2X, J_2Y) = h(J_2X, J_3Y) + h(J_2Y, J_3X) = h(X^+, J_3Y^-) + h(Y^+, J_3X^-)$ since J_3 interchanges V^+ and V^-. On the other hand, $g(X,Y) = h(X^+, J_1Y^-) + h(Y^+, J_1X^-) = -h(X^+, J_1J_2Y^-) - h(Y^+, J_1J_2X^-) = -h(X^+, J_3Y^-) - h(Y^+, J_3X^-)$. Thus $g(J_2X, J_2Y) = -g(X,Y)$. It follows that $g(J_3X, J_3Y) = -g(X,Y)$ since $J_3 = J_1J_2$. Finally, the identity $g(X,Y) = h(X^+, J_1Y^-) + h(Y^+, J_1X^-)$ and the fact that h is non-degenerate on V^+ imply that g is non-degenerate.

Conversely, let g be a metric on V compatible with the para-hypercomplex structure $\{J_1, J_2, J_3\}$. Then the spaces $V^{\pm}$ are g-isotropic. It follows that $h(A, B) = \frac{1}{2}g(J_1A, B)$, $A, B \in V^+$, is a non-degenerate skew-symmetric 2-form. It is easy to check that h yields the metric g. $\square$

The proof above gives also the following

Proposition 8. *Let $\{J_1, J_2, J_3\}$ be an almost para-hypercomplex structure on a four-manifold M. Let $V^{\pm}$ be the subbundle of TM corresponding to the eigenvalue ± 1 of J_2. The manifold M admits a metric g compatible with the given para-hypercomplex structure if and only if the bundle $V^{\pm}$ is orientable.*

The bundle $V^{\pm}$ is orientable iff the linear bundle $\Lambda^2 V^{\pm}$ is trivial. It follows that if $H^1(M, \mathcal{C}^*) = 0$ where $\mathcal{C}^*$ is the sheaf of non-vanishing smooth real-valued function on M, then $V^{\pm}$ is orientable, hence $\{J_1, J_2, J_3\}$ admits a compatible metric.

It is well-known that for every vector bundle there is a double cover of its base such that the pull-back bundle is orientable. Therefore we have the following

Corollary 9. *For any almost para-hypercomplex structure on a four-manifold M there is a double cover of M such that the pull-back para-hypercomplex structure on it admits a compatible metric.*

Proposition 8 and the fact that any bundle on a simply connected manifold is orientable imply

Corollary 10. *Any almost para-hypercomplex structure on a simply connected four-manifold M admits a compatible metric.*

We should emphasize that, in contrast to the definite case, not every para-hypercomplex structure admits a compatible metric. Examples of such structures on Inoue surfaces of type S^- will be provided in the last section. Other examples can be constructed on hyperelliptic surfaces [5].

The next fact is well-known [3] and easy to prove.

Lemma 11. *Let $\{g, J_1, J_2, J_3\}$ be a para-hyperhermitian structure on a vector space V. A vector $w \in V$ is g-non-isotropic if and only if w, J_1w, J_2w, J_3w is a basis of V.*

Proof: The vectors w, J_1w, J_2w, J_3w are g-orthogonal. Thus, if w is non-isotropic, they form a basis. Conversely, suppose that w, J_1w, J_2w, J_3w is a bases. Take a vector $e_1 \in V$ with $||e_1||_g = 1$. Then $e_1, e_2 = J_1e_1, e_3 = J_2e_1, e_4 = J_3e_1$ is a g-orthonormal basis of V. Let (w_1, w_2, w_3, w_4) be the coordinates of w with respect to this basis. Then the coordinates of J_1w, J_2w, J_3w are $J_1w = (-w_2, w_1, -w_4, w_3)$, $J_2w = (w_3, -w_4, w_1, -w_3)$, $J_3w = (w_4, w_3, w_2, w_1)$. It follows that the transition matrix from the bases (w, J_1w, J_2w, J_3w) to the bases (e_1, e_2, e_3, e_4) has determinant equal to $(w_1^2 + w_2^2 - w_3^2 - w_4^2)^2 = ||w||_g^4$. Hence w is non-isotropic. $\square$

This observation implies that any metric compatible with a para-hyper-complex structure is of split signature.

Lemma 12. *Let $\{J_1, J_2, J_3\}$ be a para-hypercomplex structure on a vector space V. Let g and h be two compatible metrics. If w is an h-non-isotropic vector, then it is also g-non-isotropic and $g = \lambda h$, where $\lambda = g(w,w)/h(w,w)$.*

Proof: It is clear that the identity $g(X,Y) = g(w,w)/h(w,w) \, h(X,Y)$ holds when $X = Y = w, J_1w$, J_2w, J_3w. Hence it holds for every $X, Y \in V$ since w, J_1w, J_2w, J_3w is a basis which is g- and h-orthogonal. In particular, $g(w,w) \neq 0$. $\square$

Proposition 13. *If $\{J_1, J_2, J_3\}$ is an almost para-hypercomplex structure on a four-manifold M and g,h are two compatible metrics, then there exists a unique non-vanishing smooth function f on M such that $g = f\,h$.*

Proof: Every point of M has a neighbourhood with an h-non-isotropic vector field W on it and the proposition follows form Lemma 12 $\square$

Corrolary 9 and Proposition 13 imply the following

Proposition 14. *Every para-hypercomplex structure on a 4-manifold M determines a conformal class up to a double cover of M.*

5 Inoue surfaces of type $S^{\pm}$

It is well-known [3, 10] that the Inoue surfaces of type S^+ admit para-hyperhermitian structures. In this section, we show that, in contrast, any Inoue surface of type S^- has a para-hypercomplex structure which does not admit a compatible metric. Before that we recall the definition of the Inoue surfaces of type $S^{\pm}$.

Let p, q, r be integers, t a complex number, and $N \in SL(2, \mathbb{Z})$ a matrix with eigenvalue $\alpha > 1$ and $1/\alpha$. Denote by $\mathbb{H}$ the upper half-plane of the complex plane $\mathbb{C}$. The Inoue surface $S^+_{p,q,r,t,N}$ is obtained as a quotient of $\mathbb{H} \times \mathbb{C}$ by the action of the group generated by following transformations:

$\phi_0(z, w) \to (\alpha z, w + t)$

$\phi_i(z, w) \to (z + a_i, w + b_i z + c_i), \quad i = 1, 2$

$\phi_3(z, w) \to (z, w + A),$

where (a_1, a_2) and (b_1, b_2) are real eigenvectors of N corresponding to α and $1/\alpha$, and $A = (b_1 a_2 - b_2 a_1)/r$. Here, the constants c_i are real numbers determined by a_i, b_i, p, q, r, and the eigenvalues of N. The (1,0)-forms

$$\theta_1 = \frac{dz}{Im\, z} \quad \text{and} \quad \theta_2 = dw - \frac{Im\, w - s \ln(Im\, z)}{Im\, z} dz$$

where $s = Im\, t / \ln \alpha$ are invariant under this action and the corresponding dual (1,0)-vector fields are:

$$E_1 = (Im\, z)\frac{\partial}{\partial z} + (Im\, w - s \ln(Im\, z))\frac{\partial}{\partial w} \quad \text{and} \quad E_2 = \frac{\partial}{\partial w}.$$

It is easy to see that

$$d\theta_1 = (-1/2i)\theta_1 \wedge \overline{\theta_1}, \quad d\theta_2 = (1/2i)(\theta_1 \wedge \theta_2 - \theta_1 \wedge \overline{\theta_2} + s\theta_1 \wedge \overline{\theta_1}).$$

From here one gets

$$d(\theta_1 \wedge \theta_2) = -(Im\, \theta_1) \wedge \theta_1 \wedge \theta_2,$$

thus the $(2, 0)$-form $\Omega = \theta_1 \wedge \theta_2$ satisfies the relation $d\Omega = -Im\, \theta_1 \wedge \Omega$.

Set $\Omega_1 = Re(\theta_1 \wedge \overline{\theta}_2)$. Then one can check that

$$d\Omega_1 = -Im\, \theta_1 \wedge \Omega_1, \quad \Omega_1^2 = -(Re\, \Omega)^2 = -(Im\, \Omega)^2 = \frac{1}{2}\theta_1 \wedge \overline{\theta}_1 \wedge \theta_2 \wedge \overline{\theta}_2.$$

Therefore the triple $(\Omega_1, Re\, \Omega, Im\, \Omega)$ defines a para-hyperhermitian structure on $S^+_{p,q,r,t,N}$.

The definition of Inoue surfaces of type S^- is the same as those of type S^+, but in this case ϕ_0 is defined as $\phi_0(z, w) \to (\alpha z, -w)$. It is clear that any surface S^- is a quotient of a certain surface S^+ with $t = 0$ by the action of the involution $\sigma : S^+ \to S^+$ given by $\sigma(z, w) = (z, -w)$. Then for the $(1, 0)$-forms θ_1 and θ_2 defined above, we have $\sigma^*\theta_1 = \theta_1, \quad \sigma^*\theta_2 = -\theta_2$. Therefore $\sigma^*\Omega_1 = -\Omega_1$ and $\sigma^*\Omega = -\Omega$, hence the para-hyperhermitian structure on S^+ defined above does not descend to S^-. Nevertheless, we show below that the surface S^- admits a para-hypercomplex structure with no compatible metric. Notice first that the para-hypercomplex structure on S^+ defined by the para-hyperhermitian structure $(\Omega_1, Re\, \Omega, Im\, \Omega)$ does descend to S^-. Indeed, the map σ is an anti-isometry with respect to the metric of this structure since it preserves the complex structure and $\sigma^*(\Omega_1) = -\Omega_1$. This, the identity $\sigma^*\Omega = -\Omega$ and the fact that σ is an involution imply that σ preserves also the two product structures. Hence the para-hypercomplex structure descends to S^-. On the other hand, if we suppose that there is a metric on S^- compatible with the induced para-hypercomplex structure, its pull-back would be a metric compatible with the para-hypercomplex structure on S^+. But, according to Proposition 13, in real dimension 4, any two metrics compatible with the same para-hypercomplex structure are conformally equivalent. So there would be a nowhere vanishing real-valued function f on S^+ for which $f\Omega$ is the pull-back of the $(2, 0)$ - form on S^- associated with the para-hyperhermitian

structure there. Since $f\Omega$ is σ-invariant and $\sigma^*(\Omega) = -\Omega$, we have $f(\sigma(x)) = -f(x)$ for every $x \in S^+$. But this contradicts to the fact that f has a fixed sign. So the para-hypercomplex structure on S^- defined above does not admit a compatible metric.

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