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Generalized complex structures on complex 2-tori

by
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To Professor S. Ianuş on the occasion of his 70th Birthday

Abstract

We compute the deformations, in the sense of generalized complex structures, of the standard complex structure on a complex 2-torus. We get a smooth complete family depending on six complex parameters and, in particular, we obtain the well-known smooth complete family of complex deformations depending on four parameters.

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1 Introduction

Nigel Hitchin introduced in the paper [5] new geometrical structures which unify many classical structures. He defines a generalized complex structure to be a complex structure, not on the tangent bundle T of a manifold, but on $T \oplus T^*$, unifying in this way the complex geometry and the symplectic geometry in some sense. This new geometrical structure is also, in some sense, the complex analogue of the Dirac structure introduced by Courant and Weinstein [2], [3], in order to unify Poisson geometry with symplectic geometry. The study of generalized complex structures was continued by Gualtieri (see [4]).

In this paper we start with the (classical) complex structure on a 2-torus and we compute (as in [1]) by using properties of the Lie algebroids, the family of deformations of this complex structure in the sense of generalized complex structures (see [4]).

By solving the generalized Maurer-Cartan equation we get the main result of the paper, which shows that obtained family of deformations is a smooth locally complete family depending on six complex parameters. In particular, we get the family (depending on four complex parameters) of deformations of (classical) complex structures on a 2-torus (see, for example [6]), as well as examples of generalized complex structures of complex type, which are not (classical) complex structures on a 2-torus.

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2 Generalized complex structures on manifolds

A generalized complex structure on a manifold M (see [5], [4]) is defined to be a complex structure J ($J^2 = -1$) not on the tangent bundle T_M, but on the sum $T_M \oplus T_M^*$ of the tangent and cotangent bundles, which is required to be orthogonal with respect to the natural inner product on sections $X + \sigma, Y + \tau \in \mathcal{C}^\infty(T_M \oplus T_M^*)$ defined by

$$\langle X + \sigma, Y + \tau \rangle = \frac{1}{2}(\sigma(Y) + \tau(X)).$$

This is only possible if $\dim_{\mathbb{R}} M = 2n$, which we suppose. In addition, the $(+i)$-eigenbundle

$$L \subset (T_M \oplus T_M^*) \otimes \mathbb{C}$$

of J is required to be involutive with respect to the Courant bracket, a skew bracket operation on smooth sections of $T_M \oplus T_M^*$ defined by

$$[X + \sigma, Y + \tau] = [X, Y] + \mathcal{L}_X \tau - \mathcal{L}_Y \sigma - \frac{1}{2} d(i_X \tau - i_Y \sigma),$$

where $\mathcal{L}_X$ and i_X denote the Lie derivative and interior product operations on forms.

Since J is orthogonal with respect to $\langle \cdot, \cdot \rangle$, the $(+i)$-eigenbundle L is a maximal isotropic subbundle of $(T_M \oplus T_M^*) \otimes \mathbb{C}$ of real index zero (i.e. $L \cap \bar{L} = \{0\}$). In fact, a generalized complex structure on M is completely determined by a maximal isotropic subbundle of $(T_M \oplus T_M^*) \otimes \mathbb{C}$ of real index zero, which is Courant involutive (see [5], [4]).

For such a subbundle we have the decomposition

$$(T_M \oplus T_M^*) \otimes \mathbb{C} = L \oplus \bar{L},$$

and we may use the inner product $\langle \cdot, \cdot \rangle$ to identify $\bar{L} \equiv L^*$. Let

$$\pi_T : (T_M \oplus T_M^*) \otimes \mathbb{C} \to T_M \otimes \mathbb{C}$$

be the projection and let $E = \pi_T(L)$. Then, the type $k \in \{0, 1, \ldots, n\}$ of the generalized complex structure at $x \in M$ is defined as the codimension of $E_x \subset T_x \otimes \mathbb{C}$.

To deform J we will vary L in the Grassmannian of maximal isotropic. Any maximal isotropic having zero intersection with $\bar{L}$ (this is an open set containing L) can be uniquely described as the graph of a homomorphism $\varepsilon : L \to \bar{L}$ satisfying

$$(*) \qquad \langle \varepsilon(X), Y \rangle + \langle X, \varepsilon(Y) \rangle = 0, \ \forall\, X, Y \in \mathcal{C}^\infty(L)$$

or equivalently, $\varepsilon \in \mathcal{C}^\infty(\wedge^2 L^*)$. Therefore, the new isotropic is given by $L_\varepsilon = (1 + \varepsilon)L$. As the deformed J is to remain real, we must have $\bar{L}_\varepsilon = (1 + \bar{\varepsilon})\bar{L}$. The subbundle L_ε has zero intersection with $\bar{L}_\varepsilon$ if and only if the endomorphism on $L \oplus L^*$, described by

$$A_\varepsilon = \begin{pmatrix} 1 & \bar{\varepsilon} \\ \varepsilon & 1 \end{pmatrix}$$

is invertible; this is the case for ε in an open set around zero (see [4]). So, providing ε is small enough, $J_\varepsilon = A_\varepsilon J A_\varepsilon^{-1}$ is a new generalized almost complex structure. By [8], J_ε is integrable if and only if $\varepsilon \in \mathcal{C}^\infty(\wedge^2 L^*)$ satisfies the generalized Maurer-Cartan equation

$$(**) \qquad d_L \varepsilon + \frac{1}{2}[\varepsilon, \varepsilon] = 0.$$

3 Deformations of generalized complex structures on 2-tori

Let $N = \mathbb{C}^2/\Lambda$ be a complex 2-torus, where $\mathbb{C}^2$ denotes the space of two complex variables (z, w) and $\Lambda \subset \mathbb{C}^2$ is an integral lattice of rank 4.

We shall identify $\mathbb{C}^2$ with $\mathbb{R}^4$, the space of four real variables (x, y, u, v) by $z = x + iy$, $w = u + iv$. From the point of view of differential structure, a complex 2-torus is a parallelizable manifold, i.e. the tangent bundle T_N is globally generated by invariant vector fields $\{X, Y, U, V\}$ with all Poisson brackets zero. The complex structure endomorphism J is acting on T_N by

$$JX = Y, \quad JY = -X, \quad JU = V, \quad JV = -U.$$

Let

$$T = \frac{1}{2}(X - iY), \quad W = \frac{1}{2}(U - iV).$$

Then, the tangent bundle T_N is globally generated by $\{T, W, \bar{T}, \bar{W}\}$ and the cotangent bundle T_N^* is globally generated by the dual basis of 1-forms $\{\omega, \rho, \bar{\omega}, \bar{\rho}\}$.

We have

$$JT = iT, \quad JW = iW, \quad J\bar{T} = -i\bar{T}, \quad J\bar{W} = -i\bar{W}.$$

It follows that the subbundle $T_{0,1}$ and $T_{1,0}$ of the tangent bundle T_N are globally generated by $\{\bar{T}, \bar{W}\}$, respectively $\{T, W\}$ and the dual bundle $T^*_{1,0}$ is globally generated by $\{\omega, \rho\}$.

The standard complex structure on a 2-torus N can be seen as a generalized complex structure given by the subbundle $L \subset (T_N \oplus T_N^*) \otimes \mathbb{C}$ of the form

$$L = \{\bar{T}, \bar{W}, \omega, \rho\}^{\sim} = (T_{0,1} \oplus T^*_{1,0}) \otimes \mathbb{C},$$

which is maximal isotropic and Courant involutive (see [4]).

Using the inner product we can identify $\bar{L}$ with L^* by the isomorphism:

$$\theta : \bar{L} \tilde{\to} L^*, \quad \theta(T) = \frac{1}{2}\omega^*, \quad \theta(W) = \frac{1}{2}\rho^*, \quad \theta(\bar{\omega}) = \frac{1}{2}\bar{T}^*, \quad \theta(\bar{\rho}) = \frac{1}{2}\bar{W}^*.$$

In the following we shall study the deformations of this generalized complex structure on a 2-torus N.

In order to obtain the deformations of the generalized complex structure we must consider the linear maps $\varepsilon : L \to \bar{L}$, which verify the condition (*) or, equivalently, we can consider the linear maps $\tilde{\varepsilon} = \theta \circ \varepsilon : L \to L^*$.

The map $\tilde{\varepsilon}$ is given by the following matrix:

$$\tilde{\varepsilon} = \frac{1}{2}\begin{pmatrix} 0 & t_{32} & -t_{11} & -t_{21} \\ -t_{32} & 0 & -t_{12} & -t_{22} \\ t_{11} & t_{12} & 0 & t_{14} \\ t_{21} & t_{22} & -t_{14} & 0 \end{pmatrix}, \quad t_{ij} \in \mathbb{C},$$

and $\tilde{\varepsilon} \in \mathcal{C}^\infty(\wedge^2 L^*)$.

Now, as in [1], we shall solve the generalized Maurer-Cartan equation (**), where $[\tilde{\varepsilon}, \tilde{\varepsilon}]$ is the Schouten bracket and the derivative

$$d_L : \mathcal{C}^\infty(\wedge^2 L^*) \to \mathcal{C}^\infty(\wedge^3 L^*),$$

for a Lie algebroid is given in this case by the formula

$$d_L\tilde{\varepsilon}(X_0, X_1, X_2) = a(X_0)\tilde{\varepsilon}(X_1, X_2) - a(X_1)\tilde{\varepsilon}(X_0, X_2) + a(X_2)\tilde{\varepsilon}(X_0, X_1) -$$

$$-\tilde{\varepsilon}([X_0, X_1], X_2) + \tilde{\varepsilon}([X_0, X_2], X_1) - \tilde{\varepsilon}([X_1, X_2], X_0);$$

the anchor map $a : \mathcal{C}^\infty(L) \to \mathcal{C}^\infty(T_N)$ is the projection on the tangent bundle T_N and $X_0, X_1, X_2 \in \mathcal{C}^\infty(L)$.

We shall use the notation:

$$X_0 = u_1\bar{T} + u_2\bar{W} + u_3\omega + u_4\rho,$$

$$X_1 = \alpha_1\bar{T} + \alpha_2\bar{W} + \alpha_3\omega + \alpha_4\rho,$$

$$X_2 = \beta_1\bar{T} + \beta_2\bar{W} + \beta_3\omega + \beta_4\rho,$$

where $u_i, \alpha_i, \beta_i \in \mathcal{C}^\infty(N),\ \forall i = 1, 2, 3, 4.$

We have

$$\tilde{\varepsilon}(X_1, X_2) = \tfrac{1}{2}(\beta_1(t_{32}\alpha_2 - t_{11}\alpha_3 - t_{21}\alpha_4) + \beta_2(-t_{32}\alpha_1 - t_{12}\alpha_3 - t_{22}\alpha_4)+$$

$$+\beta_3(t_{11}\alpha_1 + t_{12}\alpha_2 + t_{14}\alpha_4) + \beta_4(t_{21}\alpha_1 + t_{22}\alpha_2 - t_{14}\alpha_3)).$$

Since $a(X_0) = u_1\bar{T} + u_2\bar{W}$, by direct computation as in [1], we get:

Lemma 3.1

$$a(X_0)\tilde{\varepsilon}(X_1, X_2) = \tfrac{1}{2}(t_{11}u_1(\bar{T}(\alpha_1)\beta_3 + \bar{T}(\beta_3)\alpha_1 - \bar{T}(\alpha_3)\beta_1 - \bar{T}(\beta_2)\alpha_3)+$$

$$+t_{12}u_1(\bar{T}(\alpha_2)\beta_3 + \bar{T}(\beta_3)\alpha_2 - \bar{T}(\alpha_3)\beta_2 - \bar{T}(\beta_2)\alpha_3)+$$

$$+t_{21}u_1(\bar{T}(\alpha_1)\beta_4 + \bar{T}(\beta_4)\alpha_1 - \bar{T}(\alpha_4)\beta_1 - \bar{T}(\beta_1)\alpha_4)+$$

$$+t_{22}u_1(\bar{T}(\alpha_2)\beta_4 + \bar{T}(\beta_4)\alpha_2 - \bar{T}(\alpha_4)\beta_2 - \bar{T}(\beta_2)\alpha_4)+$$

$$+t_{14}u_1(\bar{T}(\alpha_4)\beta_3 + \bar{T}(\beta_3)\alpha_4 - \bar{T}(\alpha_3)\beta_4 - \bar{T}(\beta_4)\alpha_3)+$$

$$+t_{32}u_1(\bar{T}(\alpha_2)\beta_1 + \bar{T}(\beta_1)\alpha_2 - \bar{T}(\alpha_1)\beta_2 - \bar{T}(\beta_2)\alpha_1)+$$

$$+t_{11}u_2(\bar{W}(\alpha_1)\beta_3 + \bar{W}(\beta_3)\alpha_1 - \bar{W}(\alpha_3)\beta_1 - \bar{W}(\beta_1)\alpha_3)+$$

$$+t_{12}u_2(\bar{W}(\alpha_2)\beta_3 + \bar{W}(\beta_3)\alpha_2 - \bar{W}(\alpha_3)\beta_2 - \bar{W}(\beta_2)\alpha_3)+$$

$$+t_{21}u_2(\bar{W}(\alpha_1)\beta_4 + \bar{W}(\beta_4)\alpha_1 - \bar{W}(\alpha_4)\beta_1 - \bar{W}(\beta_1)\alpha_4)+$$

$$+t_{22}u_2(\bar{W}(\alpha_2)\beta_4 + \bar{W}(\beta_4)\alpha_2 - \bar{W}(\alpha_4)\beta_2 - \bar{W}(\beta_2)\alpha_4)+$$

$$+t_{14}u_2(\bar{W}(\alpha_4)\beta_3 + \bar{W}(\beta_3)\alpha_4 - \bar{W}(\alpha_3)\beta_4 - \bar{W}(\beta_4)\alpha_3)+$$

$$+t_{32}u_2(\bar{W}(\alpha_2)\beta_1 + \bar{W}(\beta_1)\alpha_2 - \bar{W}(\alpha_1)\beta_2 - \bar{W}(\beta_2)\alpha_1)).$$

Analogous formulae can be written for the terms $a(X_1)\tilde{\varepsilon}(X_0, X_2)$ and $a(X_2)\tilde{\varepsilon}(X_0, X_1)$.

Lemma 3.2 *For the Courant bracket we have:*

$$[X_0, X_1] = (X(\alpha_1) - Y(u_1))\bar{T} + (X(\alpha_2) - Y(u_2))\bar{W}+$$

$$+(X(\alpha_3) - Y(u_3))\omega + (X(\alpha_4) - Y(u_4))\rho.$$

where $X_0 = X + \sigma,\ X_1 = Y + \tau,$

$$X = u_1\bar{T} + u_2\bar{W},\ Y = \alpha_1\bar{T} + \alpha_2\bar{W} \in \mathcal{C}^\infty(T_N)$$

and

$$\sigma = u_3\omega + u_4\rho,\ \tau = \alpha_3\omega + \alpha_4\rho \in \mathcal{C}^\infty(T_N^*).$$

Proof: By direct computation we have

$$\mathcal{L}_X\tau = X(\alpha_3)\omega + X(\alpha_4)\rho, \quad \mathcal{L}_Y\sigma = Y(\alpha_3)\omega + Y(u_4)\rho,$$

and

$$i_X\tau = 0, \quad i_Y\sigma = 0.$$

By using the definition of the Courant bracket the result follows.

We have analogous formulae for $[X_0, X_2]$ and $[X_1, X_2]$. □

Now, a tedious but direct computation gives:

Lemma 3.3

$$\tilde{\varepsilon}([X_0, X_2], X_1) = \tfrac{1}{2}(t_{11}(-\alpha_1 u_1\bar{T}(\beta_3) - \alpha_1 u_2\bar{W}(\beta_3) + \alpha_1\beta_1\bar{T}(u_3)+$$
$$+\alpha_1\beta_2\bar{W}(u_3) + \alpha_3 u_1\bar{T}(\beta_1)+$$
$$+\alpha_3 u_2\bar{W}(\beta_1) - \alpha_3\beta_1\bar{T}(u_1) - \alpha_3\beta_2\bar{W}(u_1)) + t_{12}(-\alpha_2 u_1\bar{T}(\beta_3)-$$
$$-\alpha_2 u_2\bar{W}(\beta_3) + \alpha_2\beta_1\bar{T}(u_3) + \alpha_2\beta_2\bar{W}(u_3)+$$
$$+\alpha_3 u_1\bar{T}(\beta_2) + \alpha_3 u_2\bar{W}(\beta_2) - \alpha_3\beta_1\bar{T}(u_2)-$$
$$-\alpha_3\beta_2\bar{W}(u_2)) + t_{21}(-\alpha_1 u_1\bar{T}(\beta_4) - \alpha_1 u_2\bar{W}(\beta_4) + \alpha_1\beta_1\bar{T}(u_4)+$$
$$+\alpha_1\beta_2\bar{W}(u_4) + \alpha_4 u_1\bar{T}(\beta_1) + \alpha_4 u_2\bar{W}(\beta_1) - \alpha_4\beta_1\bar{T}(u_1)-$$
$$-\alpha_4\beta_2\bar{W}(u_1)) + t_{22}(-\alpha_2 u_1\bar{T}(\beta_4) - \alpha_2 u_2\bar{W}(\beta_4) + \alpha_2\beta_1\bar{T}(u_4)+$$
$$+\alpha_2\beta_2\bar{W}(u_4) + \alpha_4 u_1\bar{T}(\beta_2) + \alpha_4 u_2\bar{W}(\beta_2) - \alpha_4\beta_1\bar{T}(u_2)-$$
$$-\alpha_4\beta_2\bar{W}(u_2)) + t_{14}(\alpha_3 u_1\bar{T}(\beta_4) + \alpha_3 u_2\bar{W}(\beta_4) - \alpha_3\beta_1\bar{T}(u_4)-$$
$$-\alpha_3\beta_2\bar{W}(u_4) - \alpha_4 u_1\bar{T}(\beta_3) - \alpha_4 u_2\bar{W}(\beta_3) + \alpha_4\beta_1\bar{T}(u_3)+$$
$$+\alpha_4\beta_2\bar{W}(u_3) + t_{32}(\alpha_1 u_1\bar{T}(\beta_2)+$$
$$+\alpha_1 u_2\bar{W}(\beta_2) - \alpha_1\beta_1\bar{T}(u_2) - \alpha_1\beta_2\bar{W}(u_2) - \alpha_2 u_1\bar{T}(\beta_1)-$$
$$-\alpha_2 u_2\bar{W}(\beta_1) + \alpha_2\beta_1\bar{T}(u_1) + \alpha_2\beta_2\bar{W}(u_1))).$$

There are analogous formulae for the terms $\tilde{\varepsilon}([X_0, X_1], X_2)$ and $\tilde{\varepsilon}([X_1, X_2], X_0)$. By using all the above formulae, we get

Theorem 3.4 *For the differential d_L we have:*

$$(d_L\tilde{\varepsilon})(X_0, X_1, X_2) = 0,\ \forall\ X_0, X_1, X_2 \in \mathcal{C}^\infty(L).$$

As in [1], by similar computation we get:

Theorem 3.5 *For the Schouten bracket we have:*

$$[\tilde{\varepsilon}, \tilde{\varepsilon}] = 0.$$

From the two theorems, we get the following:

Corollary 3.6 *The solutions of the generalized Maurer-Cartan equation are given by*

$$\bar{\varepsilon} = 4(t_{32}\bar{\omega} \wedge \bar{\rho} - t_{11}\bar{\omega} \wedge T - t_{21}\bar{\rho} \wedge T - t_{21}\bar{\omega} \wedge W - t_{22}\bar{\rho} \wedge W + t_{14}T \wedge W)$$

Now, we need the following result:

Lemma 3.7 *The image of the differential*

$$d_L : \mathcal{C}^{\infty}(L^*) \to \mathcal{C}^{\infty}(\wedge^2 L^*)$$

is zero.

The proof is similar to the proof of Proposition 4.8 in [1].

By all the above results we get the main result:

Theorem 3.8 *The deformations in the sense of generalized complex structures of the standard complex structure on a 2-torus N are given by*

$$\tilde{\varepsilon} = t_{32}\bar{\omega} \wedge \bar{\rho} - t_{11}\bar{\omega} \wedge T - t_{12}\bar{\rho} \wedge T - t_{21}\bar{\omega} \wedge W - t_{22}\bar{\rho} \wedge W + t_{14}T \wedge W,$$

where $(t_{32}, t_{11}, t_{12}, t_{21}, t_{22}, t_{14}) \in \mathbb{C}^6$.

In particular, taking the parameters $t_{32} = 0$ and $t_{14} = 0$ we get the classical deformations of complex structures (see [6]).

We have the following:

Corollary 3.9 *The family of deformations of generalized complex structures on a complex 2-torus N, given by*

$$\widetilde{\varepsilon} = t_{32}\bar{\omega} \wedge \bar{\rho} - t_{11}\bar{\omega} \wedge T - t_{12}\bar{\rho} \wedge T - t_{21}\bar{\omega} \wedge W - t_{22}\bar{\rho} \wedge W + t_{14}T \wedge W,$$

with $(t_{32}, t_{11}, t_{12}, t_{21}, t_{22}, t_{14}) \in U \subset \mathbb{C}^6$, where U is an open neighborhood of $0 \in \mathbb{C}^6$, is a smooth locally complete family.

Proof: By Theorem 3.5 we have $[\widetilde{\varepsilon}, \widetilde{\varepsilon}] = 0$. From the definition of the obstruction map ϕ by Theorem 5.4, [4] (see also [7]) we get $\phi = 0$. Then, applying again Theorem 5.4 in [4] it follows that the above family of deformations is a smooth locally complete family in a open neighborhood U of $0 \in \mathbb{C}^6$. □

We have the following result:

Proposition 3.10 *Let N be a complex 2-torus. The type of the generalized complex structure given by the subbundle $L_\varepsilon = (1+\varepsilon)L \subset (T_N \oplus T_N^*) \otimes \mathbb{C}$ is $k_\varepsilon = 0$ (symplectic type) or $k_\varepsilon = 2$ (complex type).*

Proof: The type k_ε of a generalized complex structure is the codimension in any fibre of the projection of L_ε on $T_N \otimes \mathbb{C}$ by the canonical map

$$\pi_T : (T_N \oplus T_N^*) \otimes \mathbb{C} \to T_N \otimes \mathbb{C}.$$

Since

$$L = \{\bar{T}, \bar{W}, \omega, \rho\}^{\tilde{}},$$

we get:

$$\begin{aligned}(1+\varepsilon)(\bar{T}) &= \bar{T} + t_{11}T + t_{21}W - t_{32}\bar{\rho}\\ (1+\varepsilon)(\bar{W}) &= \bar{W} + t_{12}T + t_{22}W - t_{32}\bar{\omega}\\ (1+\varepsilon)(\omega) &= -t_{14}W + \omega - t_{11}\bar{\omega} - t_{12}\bar{\rho}\\ (1+\varepsilon)(\rho) &= t_{14}T + \rho - t_{21}\bar{\omega} - t_{22}\bar{\rho}.\end{aligned}$$

It follows that the projection of L_ε on $T_N \otimes \mathbb{C}$ is globally generated by

$$\{\bar{T} + t_{11}T + t_{21}W, \bar{W} + t_{12}T + t_{22}W, -t_{14}W, t_{14}T\}.$$

If $t_{14} \neq 0$, then the type $k_\varepsilon = 0$ (symplectic type) and, if $t_{14} = 0$,then $k_\varepsilon = 2$ (complex type). □

Remark If, in the case $t_{14} = 0$, we have also $t_{32} = 0$, we get classical deformations of complex structures. If, in the case $t_{14} = 0$, we have $t_{32} \neq 0$, we get examples of generalized complex structures of complex type, which are not classical complex structures.

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