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On the twisted Dirac operator

by
Adriana Turtoi

To Professor S. Ianuş on the occasion of his 70th Birthday

Abstract

In [3], [2], lower bounds are given for the eigenvalues of the so called "submanifold twisted Dirac operator" D_H and their limiting cases are discussed when the mean curvature $H \neq 0$. We showed in [6] that Hijazi-Zhang's theorem is true as well when the considered spin manifold is minimal, and therefore $H = 0$.

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1 On the twisted Dirac operator

We review our results regarding the lower bounds estimation for the eigenvalues of submanifold twisted Dirac operator when $H = 0$.

We will generally preserve the notations and the definitions of [2]. Let $(\widetilde{M}, G)$ be a Riemannian $(m+n)-$ dimensional spin manifold and let M be an immersed oriented $m-$ dimensional submanifold in $\widetilde{M}$ with the induced Riemannian structure $g = G_{|M}$. Assume that (M, g) is spin. Let NM be the normal bundle of M in $\widetilde{M}$. A spin structure also exists on NM [5].

Let $\Sigma M, \Sigma N$ and $\Sigma\widetilde{M}$ be the spinor bundles over M, NM and $\widetilde{M}$ respectively. The restricted spinor bundle $S := \Sigma\widetilde{M}_{|M}$ can be identified with $\Sigma =: \Sigma M \otimes \Sigma N$ if mn is even, [2]. If m and n are both odd, one has to take Σ as the direct sum of two copies of that bundle.

Let $(e_1, ..., e_m, \nu_1, ..., \nu_n)$ be a positively oriented local orthonormal basis of $T\widetilde{M}_{|M}$ such that $(e_1, ..., e_m)$ (resp. $(\nu_1, ..., \nu_n)$) is a positively oriented local orthonormal basis of TM (respectively of NM). If $\widetilde{\nabla}$ denotes the Levi-Civita connection of $(\widetilde{M}, G)$, then for all $X \in \Gamma(TM)$, for all $A \in \Gamma(TN)$ and $i = 1, ..., m$, the Gauss formula may be written, as

$$\widetilde{\nabla}_i(X + A) = \nabla_i(X + A) + h(e_i, X) - h^*(e_i, A) \tag{1.1}$$

where

$$\nabla_i(X + A) = \nabla^M_{e_i} X + \nabla^N_{e_i} A, \tag{1.2}$$

$h^*(e_i,.)$ is the transpose of second fundamental form h viewed as a linear map from TM to NM, and $\widetilde{\nabla}_i$ stands for $\widetilde{\nabla}_{e_i}$.

Denote also by $\widetilde{\nabla}$ and ∇ the induced spinorial covariant derivative on $\Gamma(S)$. Therefore, on $\Gamma(S)$ we have:

$$\widetilde{\nabla} = \begin{cases} (\nabla^{\Sigma M} \otimes Id + Id \otimes \nabla^{\Sigma N}) \oplus (\nabla^{\Sigma M} \otimes Id + Id \otimes \nabla^{\Sigma N}), & \text{if } m,\, n \text{ odd} \\ \nabla^{\Sigma M} \otimes Id + Id \otimes \nabla^{\Sigma N}, & \text{otherwise} \end{cases} \tag{1.3}$$

The spinorial Gauss formula is [1]

$$\widetilde{\nabla}_i \varphi = \nabla_i \varphi + \frac{1}{2} \sum_{j=1}^{m} e_j . h_{ij} . \varphi, (\forall)\varphi \in \Gamma(S).$$

The following submanifold Dirac's operators, [2], may be introduced:

$$\widetilde{D} = \sum_{i=1}^{m} e_i \cdot \widetilde{\nabla}_i, D = \sum_{i=1}^{m} e_i . \nabla_i, D_H = (-1)^n \omega_\perp \cdot D + \frac{1}{2} H . \omega_\perp, \tag{1.4}$$

where we have denoted by $H =: \sum_{i=1}^{m} h(e_i, e_i)$ the mean curvature vector field and where:

$$\omega_\perp = \begin{cases} \omega_n, & \text{for } n \text{ even,} \\ i\omega_n, & \text{for } n \text{ odd,} \end{cases}$$

ω_n denoting the complex volume form:

$$\omega_n = i^{[\frac{n+1}{2}]} \nu_1 ... \nu_n.$$

In both cases $(\omega_\perp)^2 = (-1)^n$.

We consider that (M, g) is a *minimal submanifold* of $(\widetilde{M}, G)$, therefore we have $H = 0$ and for all $\varphi \in \Gamma(S)$

$$D_H \varphi = (-1)^n \omega_\perp D \varphi, \tag{1.5}$$

Recall that there exists a Hermitian inner product on $\Gamma(S)$, denoted by $< \cdot, \cdot >$, such that Clifford multiplication by a vector of $T\widetilde{M}_{|M}$ is skew-symmetric. In the following, we write $(\cdot, \cdot) = Re(< \cdot, \cdot >)$.

For any spinor field $\varphi \in \Gamma(S)$, let $R^N_\varphi : M_\varphi \to \mathbb{R}$ be

$$R^N_\varphi := 2 \sum_{i,j=1}^{m} (e_i . e_j . Id \otimes R^N_{e_i e_j} \varphi, \frac{\varphi}{|\varphi|^2}) \tag{1.6}$$

and $M_\varphi := \{x \in M \mid \varphi(x) \neq 0\}$, where $R^N_{e_i e_j}$ be the *spinorial normal curvature tensor* [2].

Theorem 1.1. *Let (M, g) be a compact $m-$ dimensional immersed, minimal spin submanifold in the spin manifold $(\widetilde{M}, G)$. Then, denoting the scalar curvature of (M, g) by R, we have*

$$\lambda^2 \geq \inf_{M_\varphi} \frac{1}{4}(R_0 + R^N_\varphi), \tag{1.7}$$

where $R_0 = \inf_M R$ and λ is an eigenvalue of the twisted Dirac operator D_H.

Proof: Using (1.3), the Schrödinger-Lichnerowicz formula for the twisted Dirac operator D_H becomes for all spinor $\varphi \in \Gamma(S)$

$$D^2_H \varphi = \frac{R}{4}(Id \otimes Id)\varphi + \frac{1}{2} \sum_{i,j=1}^{m} e_i e_j Id \otimes R^N_{e_i e_j} \varphi - \sum_{i=1}^{m} \nabla_i \nabla_i \varphi \tag{1.8}$$

and, therefore, we obtain

$$(D_H^2\varphi,\varphi)=\frac{R+R_\varphi^N}{4}\mid\varphi\mid^2+\mid\nabla\varphi\mid^2. \tag{1.9}$$

Suppose that $\varphi\in\Gamma(S)$ is a non-zero eigenvalue spinor of the twisted Dirac operator, so that $D_H\varphi=\lambda\varphi$. Therefore, (1.9) gives

$$(\lambda^2-\frac{R+R_\varphi^N}{4})\mid\varphi\mid^2=\mid\nabla\varphi\mid^2.$$

Assuming by absurd that

$$\lambda^2-\frac{R+R_\varphi^N}{4}<0$$

then it results $\varphi=0$ in contradiction with the hypothesis. Hence the inequality is verified. □

The estimation (1.7) is not optimal. We have the following:

Theorem 1.2. *Let (M,g) be a compact $m-$ dimensional immersed minimal spin submanifold in the spin manifold $(\widetilde{M},G)$ and λ an eigenvalue of the twisted Dirac operator D_H, corresponding to the eigenspinor φ. Then, if*

$$R+R_\varphi^N>0$$

i) the following inequality holds

$$\lambda^2\geq\frac{m}{m-1}\inf_{M_\varphi}\frac{R+R_\varphi^N}{4}. \tag{1.10}$$

ii) If $\lambda=\pm\sqrt{\frac{1}{2}\frac{m}{m-1}\inf_{M_\varphi}(R+R_\varphi^N)}$ then the following equations are satisfied:

$$\nabla_X\varphi+\frac{1}{2m}\sqrt{\frac{m}{m-1}(R+R_\varphi^N)}X.\omega_\perp.\varphi=0 \tag{1.11}$$

$$\nabla_X\varphi-\frac{1}{2m}\sqrt{\frac{m}{m-1}(R+R_\varphi^N)}X.\omega_\perp.\varphi=0 \tag{1.12}$$

for all $X\in\Gamma(TM)$.

Proof: Let φ be an eigenspinor for the twisted Dirac operator D_H, so that

$$D_H\varphi=\lambda\varphi.$$

We consider the modified connection

$$\nabla_i^{\frac{\lambda}{m}}=\nabla_i+(-1)^{[\frac{n}{2}]}\frac{\lambda}{m}e_i.\omega_\perp,(\forall)i=1,...,m. \tag{1.13}$$

We can easily compute

$$\mid\nabla^{\frac{\lambda}{m}}\varphi\mid^2=\sum_{i=1}^m(\nabla_i^{\frac{\lambda}{m}}\varphi,\nabla_i^{\frac{\lambda}{m}}\varphi)=\mid\nabla\varphi\mid^2-\frac{\lambda^2}{m}\mid\varphi\mid^2. \tag{1.14}$$

The equality (1.14) is a consequence of the fact that the Clifford multiplication with $e_i, i=1,...,m$, and $\gamma_j, j=1,...,m$ is orthogonal and so

$$(\sum_{i=1}^m\nabla_i\varphi,e_i\omega_\perp\varphi)=(-1)^{[\frac{n}{2}]+1}(D_H\varphi,\varphi)=(-1)^{[\frac{n}{2}]+1}\lambda\mid\varphi\mid^2.$$

On the other hand, using Schrödinger- Lichnerowicz formula (1.8), (1.14) and (1.9) we obtain

$$((D_H-\frac{\lambda}{m})^2\varphi,\varphi)=[\frac{R+R_\varphi^N}{4}-\frac{m-1}{m^2}\lambda^2]\mid\varphi\mid^2+\mid\nabla^{\frac{\lambda}{m}}\varphi\mid^2$$

and by a direct computation

$$((D_H - \frac{\lambda}{m})^2\varphi, \varphi) = \frac{(m-1)^2}{m^2}\lambda^2 \mid \varphi \mid^2 .$$

Comparing the last two relations, we obtain

$$[\frac{R + R^N_\varphi}{4} - \frac{m-1}{m}\lambda^2] \mid \varphi \mid^2 + \mid \nabla^{\frac{\lambda}{m}} \varphi \mid^2 = 0. \tag{1.15}$$

Because $\varphi \neq 0$, this equality (1.15) implies (1.10).

ii) If $\lambda = \pm\frac{1}{2}\sqrt{\frac{m}{m-1}\inf_{M_\varphi}(R + R^N_\varphi)}$, the equality (1.15) implies that $R + R^N_\varphi$ is constant, $M_\varphi = M$ and $\mid \nabla^{\frac{\lambda}{m}} \varphi \mid^2 = 0$. So, $\nabla^{\frac{\lambda}{m}} \varphi = 0$, and the definition (1.13) implies respectively the equations (1.11), (1.12). □

Compare theorem 1.2. with Hijazi-Zhang's theorem [3], [2], which is proved when $H \neq 0$, note that this is also true for $H = 0$, as shown in formula (1.10). As an example, in [6], we showed that every spin Kähler manifold is a totally geodesic submanifold of its twistor space and we studied its twisted Killing spinors.

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University of Bucharest
Faculty of Mathematics
14 Academiei str.
70109 Bucharest, Romania
E-mail: aturtoi@gta.math.unibuc.ro