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Duality, a-invariants and canonical modules of rings arising from linear optimization problems

by

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Abstract

The aim of this paper is to study integer rounding properties of various systems of linear inequalities to gain insight about the algebraic properties of Rees algebras of monomial ideals and monomial subrings. We study the normality and Gorenstein property—as well as the canonical module and the a-invariant—of Rees algebras and subrings arising from systems with the integer rounding property. We relate the algebraic properties of Rees algebras and monomial subrings with integer rounding properties and present a duality theorem.

Key Words: a-invariant, canonical module, Gorenstein ring, normal subring, integer rounding property, Rees algebra, Ehrhart ring, bipartite graph, max-flow min-cut, clutter.
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1 Introduction

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and let $v_1, \ldots, v_q$ be the column vectors of a matrix $A = (a_{ij})$ whose entries are non-negative integers. We shall always assume that the rows and columns of A are different from zero. As usual we use the notation $x^a := x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$.

The *monomial algebras* considered here are: (a) the *Rees algebra*

$$R[It] := R \oplus It \oplus \cdots \oplus I^i t^i \oplus \cdots \subset R[t],$$

where $I = (x^{v_1}, \ldots, x^{v_q}) \subset R$ and t is a new variable, (b) the *extended Rees algebra*

$$R[It, t^{-1}] := R[It][t^{-1}] \subset R[t, t^{-1}],$$

(c) the *monomial subring*

$$K[F] = K[x^{v_1}, \ldots, x^{v_q}] \subset R$$

spanned by $F = \{x^{v_1}, \ldots, x^{v_q}\}$, (d) the *homogeneous monomial subring*

$$K[Ft] = K[x^{v_1}t, \ldots, x^{v_q}t] \subset R[t]$$

spanned by Ft, (e) the *homogeneous monomial subring*

$$K[Ft \cup \{t\}] = K[x^{v_1}t, \ldots, x^{v_q}t, t] \subset R[t]$$

spanned by $Ft \cup \{t\}$, (f) the *homogeneous monomial subring*

$$S = K[x^{w_1}t, \ldots, x^{w_r}t] \subset R[t],$$

where $w_1, \ldots, w_r$ is the set of all vectors $\alpha \in \mathbb{N}^n$ such that $0 \leq \alpha \leq v_i$ for some i, and (g) the *Ehrhart ring*

$$A(P) = K[\{x^a t^i \,|\, a \in \mathbb{Z}^n \cap iP; i \in \mathbb{N}\}] \subset R[t]$$

of a lattice polytope P.

The aim of this work is to study max-flow min-cut properties of clutters and integer rounding properties of various systems of linear inequalities—and their underlying polyhedra—to gain insight about the algebraic properties of these algebras and viceversa. Systems with integer rounding properties and clutters with the max-flow min-cut property come from linear optimization problems [23, 24]. The precise definitions will be given in Section 2.

Before stating our main results, we recall a few basic facts about the normality of monomial subrings. According to [31] the *integral closure* of $K[F]$ in its field of fractions can be expressed as

$$\overline{K[F]} = K[\{x^a \,|\, a \in \mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A}\}], \tag{1.1}$$

where $\mathcal{A} = \{v_1, \ldots, v_q\}$, $\mathbb{Z}\mathcal{A}$ is the subgroup of $\mathbb{Z}^n$ spanned by $\mathcal{A}$, and $\mathbb{R}_+\mathcal{A}$ is the cone generated by $\mathcal{A}$. The subring $K[F]$ equals $K[\mathbb{N}\mathcal{A}]$, the semigroup ring of $\mathbb{N}\mathcal{A}$. Recall that $K[F]$ is called *integrally closed* or *normal* if $K[F] = \overline{K[F]}$. Thus $K[F]$ is normal if and only if

$$\mathbb{N}\mathcal{A} = \mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A},$$

where $\mathbb{N}\mathcal{A}$ is the subsemigroup of $\mathbb{N}^n$ generated by $\mathcal{A}$. The description of the integral closure given in Eq. (1.1) can of course be applied to any of the monomial algebras considered here. In particular if $\mathcal{A}'$ is the set

$$\mathcal{A}' = \{e_1, \ldots, e_n, (v_1, 1), \ldots, (v_q, 1)\},$$

where e_i is the ith unit vector, then $\mathbb{Z}\mathcal{A}' = \mathbb{Z}^{n+1}$ and $R[It]$ is normal if and only if $\mathbb{N}\mathcal{A}' = \mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}'$. A dual characterization of the normality of $R[It]$ will be given in Proposition 2.9.

Recall that the Ehrhart ring $A(P)$ is always normal [2]. A set $\mathcal{A} \subset \mathbb{Z}^n$ is called a *Hilbert basis* if $\mathbb{N}\mathcal{A} = \mathbb{R}_+\mathcal{A} \cap \mathbb{Z}^n$. Note that if $\mathcal{A}$ is a Hilbert basis, then the ring $K[F]$ is normal.

The contents of this paper are as follows. First we use the theory of blocking and antiblocking polyhedra [1, 12, 13, 23] to describe when the systems

$$x \geq 0;\, xA \leq \mathbf{1}, \quad x \geq 0; xA \geq \mathbf{1}, \quad xA \leq \mathbf{1},$$

have the *integer rounding property* (see Definitions 2.2, 2.6, 2.23) in terms of the normality of the monomial algebras considered here. As usual, we denote the vector $(1, \ldots, 1)$ by $\mathbf{1}$. If $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are vectors, we write $a \leq b$ if $a_i \leq b_i$ for all i.

One of the main results of Section 2 is:

Theorem 2.5 *The system $x \geq 0;\, xA \leq \mathbf{1}$ has the integer rounding property if and only if the subring $S = K[x^{w_1}t, \ldots, x^{w_r}t]$ is normal.*

This result was shown in [9] when A is the incidence matrix of a clutter, i.e., when the entries of A are in $\{0, 1\}$. Recall that a *clutter* $\mathcal{C}$ with finite vertex set $X = \{x_1, \ldots, x_n\}$ is a family of subsets of X, called edges, none of which is included in another. The *incidence matrix* of a clutter $\mathcal{C}$ is the vertex-edge matrix whose columns are the characteristic vectors of the edges of $\mathcal{C}$. The *edge ideal* of a clutter $\mathcal{C}$, denoted by $I(\mathcal{C})$, is the ideal of R generated by all monomials $x_e = \prod_{x_i \in e} x_i$ such that e is an edge of $\mathcal{C}$. The *Alexander dual* of $I(\mathcal{C})$ is the ideal of R given by $I(\mathcal{C})^\vee = \cap_{e \in E}(e)$, where $E = E(\mathcal{C})$ is the edge set of $\mathcal{C}$.

The integer rounding property of some systems has already been expressed in terms of the normality of monomial algebras [8, 9]. In [8] it is shown that the system $x \geq 0; xA \geq \mathbf{1}$ has the integer rounding property if and only if $R[It]$ is normal (this was also observed by N. V. Trung if A is the incidence matrix of a clutter). Here we complement this fact by presenting a duality between the integer rounding property of the systems $x \geq 0; xA \geq \mathbf{1}$ and $x \geq 0; xA^* \leq \mathbf{1}$ valid for matrices with entries in $\{0, 1\}$, where $a_{ij}^* = 1 - a_{ij}$ is the ij-entry of A^*. This duality is extended to a duality between monomial subrings.

Altogether another main result of Section 2 is:

Theorem 2.12 *Let A be the incidence matrix of a clutter. If $v_i^* = \mathbf{1} - v_i$ and A^* is the matrix with column vectors $v_1^*, \ldots, v_q^*$, then the following are equivalent:*

- *(a) $R[I^*t]$ is normal, where $I^* = (x^{v_1^*}, \ldots, x^{v_q^*})$.*
- *(b) $S = K[x^{w_1}t, \ldots, x^{w_r}t]$ is normal.*
- *(c) $\{-e_1, \ldots, -e_n, (v_1, 1), \ldots, (v_q, 1)\}$ is a Hilbert basis.*
- *(d) $x \geq 0; xA^* \geq \mathbf{1}$ has the integer rounding property.*
- *(e) $x \geq 0; xA \leq \mathbf{1}$ has the integer rounding property.*

Then we present some interesting consequences of this duality. First of all we recover one of the main results of [34] showing that if

$$P = \{x | x \geq 0; xA \leq \mathbf{1}\}$$

is an integral polytope, i.e., P has only integer vertices, and A is a $\{0,1\}$-matrix, then the Rees algebra $R[I^*t]$ is normal (see Corollary 2.14). This result is related to perfect graphs. Indeed if P is integral, then $v_1, \ldots, v_q$ correspond to the maximal cliques (maximal complete subgraphs) of a perfect graph H [4, 21], and $v_1^*, \ldots, v_q^*$ correspond to the minimal vertex covers of the complement of H. Second we show that if A is the incidence matrix of the collection of basis of a matroid, then all systems

$$x \geq 0; xA \geq \mathbf{1}, \quad x \geq 0; xA^* \geq \mathbf{1}, \quad x \geq 0; xA \leq \mathbf{1}, \quad x \geq 0; xA^* \leq \mathbf{1}$$

have the integer rounding property (see Corollary 2.15). Third we show that if A is the incidence matrix of a graph, then $R[It]$ is normal if and only if $R[I^*t]$ is normal (see Corollary 2.16). We give an example to show that this result does not extends to arbitrary uniform clutters (see Example 2.17). If A is the incidence matrix of a graph G, we characterize when I^* is the Alexander dual of the edge ideal of the complement of G (see Proposition 2.18). If G is a triangle-free graph, we show a duality between the normality of $I = I(G)$ and that of the Alexander dual of the edge ideal of the complement of G (see Corollary 2.19). We show an example of an edge ideal of a graph whose Alexander dual is not normal (see Example 2.20). In [34] it is shown that this is never the case if the graph is perfect, i.e., the Alexander dual of the edge ideal of a perfect graph is always normal. Finally we recover one of the main results of [17] showing that if A is the incidence matrix of a clutter $\mathcal{C}$, then $\mathcal{C}$ satisfies the max-flow min-cut property if and only if the set covering polyhedron

$$Q(A) = \{x | \, x \geq 0; xA \geq \mathbf{1}\}$$

is integral and $R[It]$ is normal (see Corollary 2.22).

The last main result of Section 2 is:

Theorem 2.25 *If the system $xA \leq \mathbf{1}$ has the integer rounding property, then $K[F]$ is normal and $\mathbb{Z}^n/\mathbb{Z}\mathcal{A}$ is a torsion-free group. The converse holds if $|v_i| = d$ for all i. Here $|v_i| = \langle v_i, \mathbf{1} \rangle$.*

As a consequence of this result we prove: (i) If A is the incidence matrix of a connected graph G, then the system $xA \leq \mathbf{1}$ has the integer rounding property if and only if G is a bipartite graph (see Corollary 2.26), and (ii) Let A be the incidence matrix of a clutter $\mathcal{C}$. If $\mathcal{C}$ is uniform, i.e., all its edges have the same size, and $\mathcal{C}$ has the max-flow min-cut property (see Definition 2.21), then the system $xA \leq \mathbf{1}$ has the integer rounding property (see Corollary 2.27).

If A is the incidence matrix of a bipartite graph, a remarkable result of [9] shows that the system $x \geq 0; xA \leq \mathbf{1}$ has the integer rounding property if and only if the extended Rees algebra $R[It, t^{-1}]$ is normal.

Before stating the main results of Sections 3 and 4, we need to introduce the canonical module and the a-invariant (see Section 3 for additional details). Below we briefly explain the important role that these two objects play in the general theory. The subring S is a standard K-algebra because $\langle (w_i, 1), e_{n+1}\rangle = 1$ for all i. Here $\langle\,,\rangle$ is the standard inner product and e_i is the *ith* unit vector. If S is normal, then according to a formula of Danilov and Stanley [6] the canonical module of S is the ideal of S given by

$$\omega_S = (\{x^a t^b |\, (a,b) \in \mathbb{N}\mathcal{B} \cap (\mathbb{R}_+\mathcal{B})^{\mathrm{o}}\}), \tag{1.2}$$

where $\mathcal{B} = \{(w_1, 1), \ldots, (w_r, 1)\}$ and $(\mathbb{R}_+\mathcal{B})^{\mathrm{o}}$ is the relative interior of $\mathbb{R}_+\mathcal{B}$. This expression for the canonical module of S is central for our purposes. Recall that the *a-invariant* of S, denoted by $a(S)$, is the degree as a rational function of the Hilbert series of S [31, p. 99]. Thus we may compute a-invariants using the program *Normaliz* [3]. Let H_S and φ_S be the Hilbert function and the Hilbert polynomial of S respectively. The *index of regularity* of S, denoted by $\mathrm{reg}(S)$, is the least positive integer such that $H_S(i) = \varphi_S(i)$ for $i \geq \mathrm{reg}(S)$. The a-invariant plays a fundamental role in algebra and geometry because one has:

$$\mathrm{reg}(S) = \begin{cases} 0 & \text{if } a(S) < 0, \\ a(S) + 1 & \text{otherwise,} \end{cases}$$

see [31, Corollary 4.1.12]. If S is normal, then S is Cohen-Macaulay [19] and its a-invariant is given by

$$a(S) = -\min\{\, i \,|\, (\omega_S)_i \neq 0\}, \tag{1.3}$$

see [2, p. 141] and [31, Proposition 4.2.3].

In Section 3 we give a general technique to compute the canonical module and the a-invariant of a wide class of monomial subrings (see Theorem 3.1).

Then in Section 4 we study the canonical module and the a-invariant of monomial subrings arising from integer rounding properties. We give necessary and sufficient conditions for S to be Gorenstein and give a formula for the a-invariant of S in terms of the vertices of the polytope $P = \{x |\, x \geq 0; xA \leq \mathbf{1}\}$. For use below let $\mathrm{vert}(P)$ be the set of vertices of P and let $\ell_1, \ldots, \ell_p$ be the set of all maximal elements of $\mathrm{vert}(P)$ (maximal with respect to $\leq$). For each $1 \leq i \leq p$ there is a unique positive integer d_i such that the non-zero entries of $(-d_i\ell_i, d_i)$ are relatively prime.

The main results of Section 4 are as follows.

Theorem 4.2 *If the system $x \geq 0; xA \leq \mathbf{1}$ has the integer rounding property, then the canonical module of $S = K[x^{w_1}t, \ldots, x^{w_r}t]$ is given by*

$$\omega_S = \left(\left\{ x^a t^b \middle|\, (a,b) \begin{pmatrix} -d_1\ell_1 & \cdots & -d_p\ell_p & e_1 & \cdots & e_n \\ d_1 & \cdots & d_p & 0 & \cdots & 0 \end{pmatrix} \geq \mathbf{1} \right\} \right), \tag{1.4}$$

and the a-invariant of S is equal to $-\max_i\{\lceil 1/d_i + |\ell_i| \rceil\}$. Here $|\ell_i| = \langle \ell_i, \mathbf{1}\rangle$.

This result complements a result of [9] valid only for incidence matrices of clutters. If S is normal, the last Betti number in the homogeneous free resolution of the toric ideal P_S of S is equal to $\nu(\omega_S)$, the minimum number of generators of ω_S. This number is called the *type* of P_S. Thus by describing the canonical module of S we are in fact providing a device to compute the type of P_S. According to [28] the number of integral vertices of the polyhedron that defines ω_S (see Eq. (1.4)) is a lower bound for $\nu(\omega_S)$.

Using the description above for ω_S we then prove:

Theorem 4.3 *Assume that the system $x \geq 0$; $xA \leq \mathbf{1}$ has the integer rounding property. If S is Gorenstein and $c_0 = \max\{|\ell_i|\colon\, 1 \leq i \leq p\}$ is an integer, then $|\ell_k| = c_0$ for each $1 \leq k \leq p$ such that ℓ_k has integer entries.*

Theorem 4.4 *Assume that the system $x \geq 0; xA \leq \mathbf{1}$ has the integer rounding property. If $-a(S) = 1/d_i + |\ell_i|$ for $i = 1, \ldots, p$, then S is Gorenstein.*

As a consequence of Theorems 4.3 and 4.4 we obtain that if P is an integral polytope, i.e., it has only integral vertices, then S is Gorenstein if and only if $a(S) = -(|\ell_i| + 1)$ for $i = 1, \ldots, p$ (see Corollary 4.5).

We also examine the Gorenstein and complete intersection properties of subrings arising from systems with the integer rounding property of incidence matrices of graphs. Let G be a connected graph with n vertices and q edges and let A be its incidence matrix. Based on a computer analysis, using the program *Normaliz* [3], we conjecture a possible description of all Gorenstein subrings S in terms of the vertices of P (see Problem 4.7). If the system $xA \leq \mathbf{1}$ has the integer rounding property, then we show that $K[Ft \cup \{t\}]$ is a complete intersection if and only if G is bipartite and the number of primitive cycles of G is equal to $q - n + 1$ (see Proposition 4.9).

Let G be a bipartite graph and let A be its incidence matrix. A constructive description of all bipartite graphs such that $K[G] = K[x^{v_1}, \ldots, x^{v_q}]$ is a complete intersection is given in [15]. The Gorenstein property of $K[G]$ has been studied in [16, 18]. Thus by Lemma 4.8 and [2, Proposition 3.1.19] the Gorenstein property and the complete intersection property of $K[x^{v_1}t, \ldots, x^{v_q}t, t]$ are well understood in this particular case. The a-invariant of $K[G]$ has a combinatorial expression in terms of directed cuts and can be computed using linear programming [28]. Some other expressions for $a(K[G])$ can be found in [5, 16, 30].

2 Integer rounding properties

We continue to use the notation and definitions used in the introduction. In this section we introduce and study integer rounding properties, describe some of their properties, present a duality theorem and show several applications.

Let P be a rational polyhedron in $\mathbb{R}^n$. Recall that the *antiblocking polyhedron* of P is defined as:

$$T(P) := \{z |\, z \geq 0; \langle z, x\rangle \leq 1 \text{ for all } x \in P\}.$$

Lemma 2.1. *Let A be a matrix of order $n\times q$ with entries in $\mathbb{N}$, let $v_1,\ldots,v_q$ be the column vectors of A and let $\{w_1,\ldots,w_r\}$ be the set of all α in $\mathbb{N}^n$ such that $\alpha\leq v_i$ for some i. If $P=\{x|\, x\geq 0;\, xA\leq \mathbf{1}\}$, then*

$$T(P)=\mathrm{conv}(w_1,\ldots,w_r).$$

Proof: First we show the following equality which is interesting in its own right:

$$\mathrm{conv}(w_1,\ldots,w_r)=\mathbb{R}_+^n\cap(\mathrm{conv}(w_1,\ldots,w_r)+\mathbb{R}_+\{-e_1,\ldots,-e_n\}). \tag{2.1}$$

Clearly the left hand side is contained in the right hand side. Conversely let z be a vector in the right hand side. Then $z\geq 0$ and we can write

$$z=\lambda_1w_1+\cdots+\lambda_rw_r-\delta_1e_1-\cdots-\delta_ne_n,\quad (\lambda_i\geq 0;\ \textstyle\sum_i\lambda_i=1;\ \delta_i\geq 0). \tag{2.2}$$

Consider the vector $z'=\lambda_1w_1+\cdots+\lambda_rw_r-\delta_1e_1$. We set $T'=\mathrm{conv}(w_1,\ldots,w_r)$ and $w_i=(w_{i1},\ldots,w_{in})$. We claim that z' is in T'. We may assume that $\delta_1>0$, $\lambda_i>0$ for all i, and that the first entry w_{i1} of w_i is positive for $1\leq i\leq s$ and is equal to zero for $i>s$. From Eq. (2.2) we get $\lambda_1w_{11}+\cdots+\lambda_sw_{s1}\geq\delta_1$.

Case (I): $\lambda_1w_{11}\geq\delta_1$. Then we can write

$$z'=\frac{\delta_1}{w_{11}}(w_1-w_{11}e_1)+\left(\lambda_1-\frac{\delta_1}{w_{11}}\right)w_1+\lambda_2w_2+\cdots+\lambda_rw_r.$$

Notice that $w_1-w_{11}e_1$ is again in $\{w_1,\ldots,w_r\}$. Thus z' is a convex combination of $w_1,\ldots,w_r$, i.e., $z'\in T'$.

Case (II): $\lambda_1w_{11}<\delta_1$. Let m be the largest integer less than or equal to s such that $\lambda_1w_{11}+\cdots+\lambda_{m-1}w_{(m-1)1}<\delta_1\leq\lambda_1w_{11}+\cdots+\lambda_mw_{m1}$. Then

$$\begin{aligned} z' \;=\;& \sum_{i=1}^{m-1}\lambda_i(w_i-w_{i1}e_1)+\left[\frac{\delta_1}{w_{m1}}-\left(\sum_{i=1}^{m-1}\frac{\lambda_iw_{i1}}{w_{m1}}\right)\right](w_m-w_{m1}e_1)+\\ &\left[\lambda_m-\frac{\delta_1}{w_{m1}}+\left(\sum_{i=1}^{m-1}\frac{\lambda_iw_{i1}}{w_{m1}}\right)\right]w_m+\sum_{i=m+1}^{r}\lambda_iw_i. \end{aligned}$$

Notice that $w_i-w_{i1}e_1$ is again in $\{w_1,\ldots,w_r\}$ for $i=1,\ldots,m$. Thus z' is a convex combination of $w_1,\ldots,w_r$, i.e., $z'\in T'$. This completes the proof of the claim. Note that we can apply the argument above to any entry of z or z' thus we obtain that $z'-\delta_2e_2\in T'$. Thus by induction we obtain that $z\in T'$, as required. This completes the proof of Eq. (2.1).

Clearly one has the equality $P=\{z|\, z\geq 0;\langle z,w_i\rangle\leq 1\,\forall i\}$ because for each w_i there is v_j such that $w_i\leq v_j$. Hence by the finite basis theorem [23] we can write

$$P=\{z|\, z\geq 0;\langle z,w_i\rangle\leq 1\,\forall i\}=\mathrm{conv}(\ell_0,\ell_1,\ldots,\ell_m) \tag{2.3}$$

for some $\ell_1,\ldots,\ell_m$ in $\mathbb{Q}_+^n$ and $\ell_0=0$. From Eq. (2.3) we readily get the equality

$$\{z|\, z\geq 0;\langle z,\ell_i\rangle\leq 1\,\forall i\}=T(P). \tag{2.4}$$

Using Eq. (2.3) and noticing that $\langle \ell_i, w_j\rangle \leq 1$ for all i, j, we get

$$\mathbb{R}_+^n \cap (\mathrm{conv}(\ell_0, \ldots, \ell_m) + \mathbb{R}_+\{-e_1, \ldots, -e_n\}) = \{z \mid z \geq 0; \langle z, w_i\rangle \leq 1\, \forall i\}.$$

Hence using this equality and [23, Theorem 9.4] we obtain

$$\mathbb{R}_+^n \cap (\mathrm{conv}(w_1, \ldots, w_r) + \mathbb{R}_+\{-e_1, \ldots, -e_n\}) = \{z \mid z \geq 0; \langle z, \ell_i\rangle \leq 1\, \forall i\}. \quad (2.5)$$

Therefore by Eq. (2.1) together with Eqs. (2.4) and (2.5) we conclude that $T(P)$ is equal to $\mathrm{conv}(w_1, \ldots, w_r)$, as required. □

If $v_1, \ldots, v_q$ are $\{0,1\}$-vectors, then the equality of Lemma 2.1 follows directly from [12, Theorem 8]; see also [13].

Definition 2.2. Let A be a matrix with entries in $\mathbb{N}$. The system $x \geq 0; xA \leq \mathbf{1}$ has the *integer rounding property* if

$$\lceil \min\{\langle y, \mathbf{1}\rangle \mid y \geq 0;\ Ay \geq a\}\rceil = \min\{\langle y, \mathbf{1}\rangle \mid Ay \geq a;\ y \in \mathbb{N}^q\}$$

for each integral vector a for which $\min\{\langle y, \mathbf{1}\rangle \mid y \geq 0;\ Ay \geq a\}$ is finite.

If $a \in \mathbb{R}^n$, its *support* is given by $\mathrm{supp}(a) = \{i \mid a_i \neq 0\}$. Note that $a = a_+ - a_-$, where a_+ and a_- are two non negative vectors with disjoint support called the *positive* and *negative* part of a respectively.

Remark 2.3. *Let A be a matrix with entries in $\mathbb{N}$. The system $x \geq 0; xA \leq \mathbf{1}$ has the integer rounding property if and only if*

$$\lceil \min\{\langle y, \mathbf{1}\rangle \mid y \geq 0;\ Ay \geq a\}\rceil = \min\{\langle y, \mathbf{1}\rangle \mid Ay \geq a;\ y \in \mathbb{N}^q\}$$

for each vector $a \in \mathbb{N}^n$ for which $\min\{\langle y, \mathbf{1}\rangle \mid y \geq 0;\ Ay \geq a\}$ is finite. This follows decomposing an integral vector a as $a = a_+ - a_-$ and noticing that for $y \geq 0$ we have that $Ay \geq a$ if and only if $Ay \geq a_+$

A rational polyhedron Q is said to have the *integer decomposition property* if for each natural number k and for each integer vector a in kQ, a is the sum of k integer vectors in Q; see [24, pp. 66–82]. Recall that kQ is equal to $\{ka \mid a \in Q\}$.

The next criterion will be used to describe the integer rounding property of the system $x \geq 0; xA \leq \mathbf{1}$ in terms of the normality of a certain subring.

Theorem 2.4. ([1], [24, p. 82]) *Let A be a non-negative integer matrix and let $P = \{x \mid x \geq 0;\ xA \leq \mathbf{1}\}$. The system $xA \leq \mathbf{1}; x \geq 0$ has the integer rounding property if and only if $T(P)$ has the integer decomposition property and all maximal integer vectors of $T(P)$ are columns of A (maximal with respect to $\leq$).*

The next result was shown in [9] when A is the incidence matrix of a clutter. Its proof is similar to that of [9], but it requires some adjustments.

Theorem 2.5. *Let A be a matrix with entries in $\mathbb{N}$ and let $v_1, \ldots, v_q$ be the columns of A. If $w_1, \ldots, w_r$ is the set of all $\alpha \in \mathbb{N}^n$ such that $\alpha \leq v_i$ for some i, then the system $x \geq 0$; $xA \leq \mathbf{1}$ has the integer rounding property if and only if the subring $K[x^{w_1}t, \ldots, x^{w_r}t]$ is normal.*

Proof: Let $P = \{x | \, x \geq 0; \, xA \leq \mathbf{1}\}$ and let $T(P)$ be its antiblocking polyhedron. By Lemma 2.1 one has

$$T(P) = \operatorname{conv}(w_1, \ldots, w_r). \tag{2.6}$$

Let $\overline{S}$ be the integral closure of $S = K[x^{w_1}t, \ldots, x^{w_r}t]$ in its field of fractions. By the description of $\overline{S}$ given in Eq. (1.1) one has

$$\overline{S} = K[\{x^a t^b \,|\, (a, b) \in \mathbb{Z}\mathcal{B} \cap \mathbb{R}_+\mathcal{B}\}],$$

where $\mathcal{B} = \{(w_1, 1), \ldots, (w_r, 1)\}$. By Theorem 2.4 it suffices to prove that S is normal if and only if $T(P)$ has the integer decomposition property and all maximal integer vectors of $T(P)$ are columns of A (maximal with respect to $\leq$).

Assume that S is normal, i.e., $S = \overline{S}$. Let b be a natural number and let a be an integer vector in $bT(P)$. Then using Eq. (2.6) it is seen that (a, b) is in $\mathbb{R}_+\mathcal{B}$. Since S is normal we have $\mathbb{R}_+\mathcal{B} \cap \mathbb{Z}\mathcal{B} = \mathbb{N}\mathcal{B}$. In our situation one has $\mathbb{Z}\mathcal{B} = \mathbb{Z}^{n+1}$. Hence $(a, b) \in \mathbb{N}\mathcal{B}$ and a is the sum of b integer vectors in $T(P)$. Thus $T(P)$ has the integer decomposition property. Assume that a is a maximal integer vector of $T(P)$. It is not hard to see that $(a, 1)$ is in $\mathbb{R}_+\mathcal{B}$, i.e., $x^a t \in \overline{S} = S$. Thus $(a, 1)$ is a linear combination of vectors in $\mathcal{B}$ with coefficients in $\mathbb{N}$. Hence $(a, 1)$ is equal to $(w_j, 1)$ for some j. There exists v_i such that $a = w_j \leq v_i$. Therefore by the maximality of a, we get $a = v_i$ for some i. Thus a is a column of A as required.

Conversely assume that $T(P)$ has the integer decomposition property and that all maximal integer vectors of $T(P)$ are columns of A. Let $x^a t^b \in \overline{S}$. Then (a, b) is in the cone $\mathbb{R}_+\mathcal{B}$. Hence, using Eq. (2.6), we get $a \in bT(P)$. Thus $a = \alpha_1 + \cdots + \alpha_b$, where α_i is an integral vector of $T(P)$ for all i. Since each α_i is less than or equal to a maximal integer vector of $T(P)$, we get that $\alpha_i \in \{w_1, \ldots, w_r\}$. Then $x^a t^b \in S$. This proves that $S = \overline{S}$. $\Box$

Let A be a matrix with entries in $\mathbb{N}$. Next we study the integer rounding property of the system $x \geq 0$; $xA \geq \mathbf{1}$. The aim is to establish a duality with other systems of linear inequalities.

Definition 2.6. The system $x \geq 0; xA \geq \mathbf{1}$ has the *integer rounding property* if

$$\max\{\langle y, \mathbf{1}\rangle |\, y \geq 0; Ay \leq a; y \in \mathbb{N}^q\} = \lfloor \max\{\langle y, \mathbf{1}\rangle |\, y \geq 0; Ay \leq a\} \rfloor \tag{2.7}$$

for each integral vector a for which the right hand side is finite.

For any rational polyhedron Q in $\mathbb{R}^n$, define its *blocking polyhedron* $B(Q)$ by:

$$B(Q) := \{z \in \mathbb{R}^n |\, z \geq 0; \, \langle z, x \rangle \geq 1 \text{ for all } x \text{ in } Q\}.$$

For any matrix A with entries in $\mathbb{N}$, its *covering polyhedron* $Q(A)$ is defined by:

$$Q(A) := \{x \mid x \geq 0; xA \geq \mathbf{1}\}.$$

If A is the incidence matrix of a clutter $\mathcal{C}$, then the integral vectors of $Q(A)$ correspond to vertex covers of $\mathcal{C}$ and the integral vertices of $Q(A)$ are in one to one correspondence with the minimal vertex covers of $\mathcal{C}$ [14, Corollary 2.3].

The blocking polyhedron of $Q(A)$ can be expressed as follows.

Lemma 2.7. *If $Q = Q(A)$, then $B(Q) = \mathbb{R}^n_+ + \mathrm{conv}(v_1, \ldots, v_q)$.*

Proof: The right hand side is clearly contained in the left hand side. Conversely take z in $B(Q)$, then $\langle z, x\rangle \geq 1$ for all $x \in Q$ and $z \geq 0$. Let $\ell_1, \ldots, \ell_r$ be the vertex set of Q. In particular $\langle z, \ell_i\rangle \geq 1$ for all i. Then $\langle (z, 1), (\ell_i, -1)\rangle \geq 0$ for all i. From [17, Theorem 3.2] we get that $(z, 1)$ belongs to the cone generated by

$$\mathcal{A}' = \{e_1, \ldots, e_n, (v_1, 1), \ldots, (v_q, 1)\}.$$

Thus z is in $\mathbb{R}^n_+ + \mathrm{conv}(v_1, \ldots, v_q)$. This completes the proof of the asserted equality. $\Box$

The next criterion complements Theorem 2.4.

Theorem 2.8. ([1], [24, p. 82]) *The system $x \geq 0; xA \geq \mathbf{1}$ has the integer rounding property if and only if the blocking polyhedron $B(Q)$ of $Q = Q(A)$ has the integer decomposition property and all minimal integer vectors of $B(Q)$ are columns of A (minimal with respect to $\leq$).*

Recall that a set $\mathcal{A} \subset \mathbb{Z}^n$ is called a *Hilbert basis* if $\mathbb{N}\mathcal{A} = \mathbb{R}_+\mathcal{A} \cap \mathbb{Z}^n$. Note that if $\mathcal{A}$ is a Hilbert basis, then the semigroup ring $K[\mathbb{N}\mathcal{A}]$ is normal.

Proposition 2.9. *Let $I = (x^{v_1}, \ldots, x^{v_q})$ be a monomial ideal and let $v_i^* = \mathbf{1} - v_i$. Then $R[It]$ is normal if and only if the set*

$$\Gamma = \{-e_1, \ldots, -e_n, (v_1^*, 1), \ldots, (v_q^*, 1)\}$$

is a Hilbert basis.

Proof: Let $\mathcal{A}' = \{e_1, \ldots, e_n, (v_1, 1), \ldots, (v_q, 1)\}$. Assume that $R[It]$ is normal. Then $\mathcal{A}'$ is a Hilbert basis. Let (a, b) be an integral vector in $\mathbb{R}_+\Gamma$, with $a \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$. Then we can write

$$(a, b) = \mu_1(-e_1) + \cdots + \mu_n(-e_n) + \lambda_1(v_1^*, 1) + \cdots + \lambda_q(v_q^*, 1),$$

where $\mu_i \geq 0$ and $\lambda_j \geq 0$ for all i, j. Therefore

$$-(a, b) + b\mathbf{1} = \mu_1 e_1 + \cdots + \mu_n e_n + \lambda_1(v_1, -1) + \cdots + \lambda_q(v_q, -1),$$

where $\mathbf{1} = e_1 + \cdots + e_n$. This equality is equivalent to

$$-(a, -b) + b\mathbf{1} = \mu_1 e_1 + \cdots + \mu_n e_n + \lambda_1(v_1, 1) + \cdots + \lambda_q(v_q, 1).$$

As $\mathcal{A}'$ is a Hilbert basis we can write

$$-(a, -b) + b\mathbf{1} = \mu_1' e_1 + \cdots + \mu_n' e_n + \lambda_1'(v_1, 1) + \cdots + \lambda_q'(v_q, 1),$$

where $\mu_i' \in \mathbb{N}$ and $\lambda_j' \in \mathbb{N}$ for all i, j. Thus $(a, b) \in \mathbb{N}\Gamma$. This proves that Γ is a Hilbert basis. The converse can be shown using similar arguments. $\Box$

A *clutter* $\mathcal{C}$ with finite vertex set $X = \{x_1, \ldots, x_n\}$ is a family of subsets of X, called edges, none of which is included in another. Let $f_1, \ldots, f_q$ be the edges of $\mathcal{C}$ and let $v_k = \sum_{x_i \in f_k} e_i$ be the *characteristic vector* of f_k. The *incidence matrix* of $\mathcal{C}$ is the $n \times q$ matrix with column vectors $v_1, \ldots, v_q$.

Definition 2.10. *Let $A = (a_{ij})$ be a matrix with entries in $\{0, 1\}$. Its dual is the matrix $A^* = (a_{ij}^*)$, where $a_{ij}^* = 1 - a_{ij}$.*

The following duality is valid for incidence matrices of clutters. It will be used later to establish a duality theorem for monomial subrings.

Theorem 2.11. *Let A be the incidence matrix of a clutter and let $v_1, \ldots, v_q$ be its column vectors. If $v_i^* = \mathbf{1} - v_i$ and A^* is the matrix with column vectors $v_1^*, \ldots, v_q^*$, then the system $x \geq 0; xA \geq \mathbf{1}$ has the integer rounding property if and only if the system $x \geq 0; xA^* \leq \mathbf{1}$ has the integer rounding property.*

Proof: Consider $Q = \{x | x \geq 0; xA \geq \mathbf{1}\}$ and $P^* = \{x | x \geq 0; xA^* \leq \mathbf{1}\}$. Let $w_1^*, \ldots, w_s^*$ be the set of all $\alpha \in \mathbb{N}^n$ such that $\alpha \leq v_i^*$ for some i. Then, using Lemmas 2.7 and 2.1, we obtain that the blocking polyhedron of Q and the antiblocking polyhedron of P^* are given by

$$B(Q) = \mathbb{R}_+^n + \operatorname{conv}(v_1, \ldots, v_q) \quad \text{and} \quad T(P^*) = \operatorname{conv}(w_1^*, \ldots, w_s^*)$$

respectively.

$\Rightarrow$) By Theorem 2.4 it suffices to show that $T(P^*)$ has the integer decomposition property and all maximal integer vectors of $T(P^*)$ are columns of A^*. Let b be an integer and let a be an integer vector in $bT(P^*)$. Then we can write

$$a = b(\lambda_1 w_1^* + \cdots + \lambda_s w_s^*), \quad (\textstyle\sum_i \lambda_i = 1; \lambda_i \geq 0).$$

For each $1 \leq i \leq s$ there is $v_{j_i}^*$ in $\{v_1^*, \ldots, v_q^*\}$ such that $w_i^* \leq v_{j_i}^*$. Thus for each i we can write $\mathbf{1} - w_i^* = v_{j_i} + \delta_i$, where $\delta_i \in \mathbb{N}^n$. Therefore

$$\mathbf{1} - a/b = \lambda_1(v_{j_1} + \delta_1) + \cdots + \lambda_s(v_{j_s} + \delta_s).$$

This means that $\mathbf{1} - a/b \in B(Q)$, i.e., $b\mathbf{1} - a$ is an integer vector in $bB(Q)$. Hence by Theorem 2.8 we can write $b\mathbf{1} - a = \alpha_1 + \cdots + \alpha_b$ for some $\alpha_1, \ldots, \alpha_b$ integer vectors in $B(Q)$, and for each α_i there is v_{k_i} in $\{v_1,\ldots,v_q\}$ such that $v_{k_i}\leq\alpha_i$. Thus $\alpha_i=v_{k_i}+\epsilon_i$ for some $\epsilon_i\in\mathbb{N}^n$ and consequently:

$$a=(\mathbf{1}-v_{k_1})+\cdots+(\mathbf{1}-v_{k_b})-c=v_{k_1}^*+\cdots+v_{k_b}^*-c,$$

where $c=(c_1,\ldots,c_n)\in\mathbb{N}^n$. Notice that $v_{k_1}^*+\cdots+v_{k_b}^*\geq c$ because $a\geq 0$. If $c_1\geq 1$, then the first entry of $v_{k_i}^*$ is non-zero for some i and we can write

$$a=v_{k_1}^*+\cdots+v_{k_{i-1}}^*+(v_{k_i}^*-e_1)+v_{k_{i+1}}^*+\cdots+v_{k_b}^*-(c-e_1)$$

Since $v_{k_i}^*-e_1$ is again in $\{w_1^*,\ldots,w_s^*\}$, we can apply this argument recursively to obtain that a is the sum of b integer vectors in $\{w_1^*,\ldots,w_s^*\}$. This proves that $T(P^*)$ has the integer decomposition property. Let a be a maximal integer vector of $T(P^*)$. Since the vectors $w_1^*,\ldots,w_s^*$ have entries in $\{0,1\}$, we get $T(P^*)\cap\mathbb{Z}^n=\{w_1^*,\ldots,w_s^*\}$. Then $a=w_i^*$ for some i. As $w_i^*\leq v_j^*$ for some j, we conclude that $a=v_j^*$, i.e., a is a column of A^*, as required.

$\Leftarrow$) According to [8] the system $x\geq 0; xA\geq\mathbf{1}$ has the integer rounding property if and only if $R[It]$ is normal. Thus by Proposition 2.9 we need only show that the set $\Gamma=\{-e_1,\ldots,-e_n,(v_1^*,1),\ldots,(v_q^*,1)\}$ is a Hilbert basis. Let (a,b) be an integral vector in $\mathbb{R}_+\Gamma$, with $a\in\mathbb{Z}^n$ and $b\in\mathbb{Z}$. Then we can write

$$(a,b)=\mu_1(-e_1)+\cdots+\mu_n(-e_n)+\lambda_1(v_1^*,1)+\cdots+\lambda_q(v_q^*,1),$$

where $\mu_i\geq 0$, $\lambda_j\geq 0$ for all i,j. Hence $A^*\lambda\geq a$, where $\lambda=(\lambda_i)$. By hypothesis the system $x\geq 0; xA^*\leq\mathbf{1}$ has the integer rounding property. Then one has

$$b\geq\lceil\min\{\langle y,\mathbf{1}\rangle|\, y\geq 0;\ A^*y\geq a\}\rceil=\min\{\langle y,\mathbf{1}\rangle|\, A^*y\geq a;\ y\in\mathbb{N}^q\}=\langle y_0,\mathbf{1}\rangle$$

for some $y_0=(y_i)\in\mathbb{N}^q$ such that $|y_0|=\langle y_0,\mathbf{1}\rangle\leq b$ and $a\leq A^*y_0$. Then

$$a=y_1v_1^*+\cdots+y_qv_q^*-\delta_1e_1-\cdots-\delta_ne_n,$$

where $\delta_1,\ldots,\delta_n$ are in $\mathbb{N}$. Hence we can write

$$(a,b)=y_1(v_1^*,1)+\cdots+y_{q-1}(v_{q-1}^*,1)+(y_q+b-|y_0|)(v_q^*,1)-(b-|y_0|)v_q^*-\delta,$$

where $\delta=(\delta_i)$. As the entries of A^* are in $\mathbb{N}$, the vector $-v_q^*$ can be written as a non-negative integer combination of $-e_1,\ldots,-e_n$. Thus $(a,b)\in\mathbb{N}\Gamma$. This proves that Γ is a Hilbert basis. □

We come to one of the main result of this section. It establishes a duality for monomial subrings.

Theorem 2.12. *Let A be the incidence matrix of a clutter, let $v_1,\ldots,v_q$ be its column vectors and let $v_i^*=\mathbf{1}-v_i$. If $w_1^*,\ldots,w_s^*$ is the set of all $\alpha\in\mathbb{N}^n$ such that $\alpha\leq v_i^*$ for some i, then the following conditions are equivalent:*

(a) *$R[It]$ is normal, where $I = (x^{v_1}, \ldots, x^{v_q})$.*

(b) *$S^* = K[x^{w_1^*}t, \ldots, x^{w_s^*}t]$ is normal.*

(c) *$\{-e_1, \ldots, -e_n, (v_1^*, 1), \ldots, (v_q^*, 1)\}$ is a Hilbert basis.*

(d) *$x \geq 0; xA \geq \mathbf{1}$ has the integer rounding property.*

(e) *$x \geq 0; xA^* \leq \mathbf{1}$ has the integer rounding property.*

Proof: (a) ⇔ (c): This was shown in Proposition 2.9. (a) ⇔ (d): This is one of the main results of [8] and is valid for arbitrary monomial ideals. (b) ⇔ (e): This was shown in Theorem 2.5. (d) ⇔ (e): This follows from Theorem 2.11. □

To illustrate the usefulness of this duality, below we show various results that follow from there.

Definition 2.13. *Let $\mathcal{C}$ be a clutter on the vertex set $X = \{x_1, \ldots, x_n\}$. The edge ideal of $\mathcal{C}$, denoted by $I(\mathcal{C})$, is the ideal of R generated by all monomials $x_e = \prod_{x_i \in e} x_i$ such that e is an edge of $\mathcal{C}$. The dual I^* of an edge ideal I is the ideal of R generated by all monomials $x_1 \cdots x_n / x_e$ such that e is an edge of $\mathcal{C}$.*

Corollary 2.14. ([34, Theorem 2.10]) *Let $\mathcal{C}$ be a clutter and let A be its incidence matrix. If $P = \{x | x \geq 0; xA \leq \mathbf{1}\}$ is an integral polytope and $I = I(\mathcal{C})$, then*

(i) *$R[I^*t]$ is normal.*

(ii) *$S = K[x^{w_1}t, \ldots, x^{w_r}t]$ is normal.*

Proof: Since P has only integral vertices, by a result of Lovász [21] the system $x \geq 0; xA \leq \mathbf{1}$ is totally dual integral, i.e., the minimum in the LP-duality equation

$$\max\{\langle a, x\rangle |\, x \geq 0; xA \leq \mathbf{1}\} = \min\{\langle y, \mathbf{1}\rangle |\, y \geq 0; Ay \geq a\} \qquad (2.8)$$

has an integral optimum solution y for each integral vector a with finite minimum. In particular the system $x \geq 0; xA \leq \mathbf{1}$ satisfies the integer rounding property. Therefore $R[I^*t]$ and $K[x^{w_1}t, \ldots, x^{w_r}t]$ are normal by Theorem 2.12. □

This result is related to the theory of perfect graphs. Indeed if P is integral, the $w_i's$ correspond to the cliques (complete subgraphs) of a perfect graph H [4, 21], and the v_i^*'s correspond to the minimal vertex covers of the complement of H. The normality assertion of part (ii) is well known and it can also be shown directly using the fact that the system $x \geq 0; xA \leq \mathbf{1}$ is TDI if P is integral, where TDI stands for Totally Dual Integral (see [24]).

Corollary 2.15. *Let $B_1, \ldots, B_q$ be the collection of basis of a matroid M with vertex set X and let $v_1, \ldots, v_q$ be their characteristic vectors. If A is the matrix with column vectors $v_1, \ldots, v_q$, then all systems*

$$x \geq 0; xA \geq \mathbf{1}, \quad x \geq 0; xA^* \geq \mathbf{1}, \quad x \geq 0; xA \leq \mathbf{1}, \quad x \geq 0; xA^* \leq \mathbf{1}$$

have the integer rounding property.

Proof: Consider the basis monomial ideal $I = (x^{v_1}, \ldots, x^{v_q})$ of the matroid M. By [22, Theorem 2.1.1], the collection of basis of the dual matroid M^* of M is given by $X \setminus B_1, \ldots, X \setminus B_q$. Now, the basis monomial ideal of a matroid is normal [33, Corollary 3.8], thus the result follows at once from the duality given in Theorem 2.12. □

Corollary 2.16. *Let G be a connected graph and let $I = I(G)$ be its edge ideal. Then $R[It]$ is normal if and only if $R[I^*t]$ is normal.*

Proof: By [9] the system $x \geq 0; xA \geq \mathbf{1}$ has the integer rounding property if and only if the system $x \geq 0; xA \leq \mathbf{1}$ does. Therefore the result follows at once using Theorem 2.12. □

This result is valid even if the graph is not connected but its proof requires to use the fact that $R[It]$ is normal if and only if the extended Rees algebra $R[It, t^{-1}]$ is normal and the fact that $R[It, t^{-1}]$ is isomorphic to $S = K[x^{w_1}, \ldots, x^{w_r}]$ when I is the edge ideal of a graph (see [9]). The next example shows that Corollary 2.16 does not extends to arbitrary uniform clutters.

Example 2.17. *Consider the clutter $\mathcal{C}$ whose incidence matrix A is the transpose of the matrix:*

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Let $I = I(\mathcal{C})$ be the edge ideal of $\mathcal{C}$. Note that all edges of $\mathcal{C}$ have 7 vertices. Using Normaliz [3] it is seen that $R[It]$ is normal and that $R[I^*t]$ is not normal.

Let $\mathcal{C}$ be a clutter with vertex set X. A vertex x of $\mathcal{C}$ is called *isolated* if x does not occur in any edge of $\mathcal{C}$. A subset $C \subset X$ is a *minimal vertex cover* of $\mathcal{C}$ if: (c$_1$) every edge of $\mathcal{C}$ contains at least one vertex of C, and (c$_2$) there is no proper subset of C with the first property. If C only satisfies condition (c_1), then C is called a *vertex cover* of $\mathcal{C}$. The *Alexander dual* of $\mathcal{C}$, denoted by $\mathcal{C}^\vee$, is the clutter whose edges are the minimal vertex covers of $\mathcal{C}$. The edge ideal of $\mathcal{C}^\vee$, denoted by $I(\mathcal{C})^\vee$, is called the *Alexander dual* of $I(\mathcal{C})$. In combinatorial optimization the Alexander dual of a clutter is referred to as the *blocker* of the clutter [24].

Proposition 2.18. *Let G be a graph without isolated vertices and let G' be its complement. Then $I(G')^\vee = I(G)^*$ if and only if G is triangle free.*

Proof: $\Rightarrow$) Let $X = \{x_1, \ldots, x_n\}$ be the vertex set of G. Assume that G has a triangle $\mathcal{C}_3 = \{x_1, x_2, x_3\}$, i.e., $\{x_i, x_j\}$ are edges of G for $1 \leq i < j \leq 3$. Clearly we may assume $n \geq 4$. Notice that $C' = \{x_4, \ldots, x_n\}$ is a vertex cover of G', i.e., $x_4 \cdots x_n$ belongs to $I(G')^\vee$ and consequently it belongs to $I(G)^*$, a contradiction because $I(G)^*$ is generated by monomials of degree $n-2$.

$\Leftarrow$) Let $x^a = x_1 \cdots x_r$ be a minimal generator of $I(G')^\vee$. Then $C = \{x_1, \ldots, x_r\}$ is a minimal vertex cover of G'. Hence $X \setminus C$ is a maximal complete subgraph of G. Thus by hypothesis $X \setminus C$ is an edge of G, i.e., $x^a \in I(G)^*$. This proves the inclusion $I(G')^\vee \subset I(G)^*$. Conversely, let x^a be a minimal generator of $I(G)^*$. There is an edge $\{x_1, x_2\}$ of G such that $x^a = x_3 \cdots x_n$. Every edge of G' must intersect $C = \{x_3, \ldots, x_n\}$, i.e., $x^a \in I(G')^\vee$. $\Box$

This formula applies for instance if G is a bipartite graph.

Corollary 2.19. *Let G be a free triangle graph without isolated vertices. Then $R[I(G)t]$ is normal if and only if $R[I(G')^\vee t]$ is normal.*

Proof: It follows directly from Corollary 2.16 and Proposition 2.18. $\Box$

In [34] it is shown that the Alexander dual of the edge ideal of a perfect graph is always normal (cf. Corollary 2.14(i)). To the best of our knowledge the following is the first example of an edge ideal of a graph whose Alexander dual is not normal.

Example 2.20. *Let G be the graph consisting of two vertex disjoint odd cycles of length 5 and let G' be its complement. According to [27] the Rees algebra of $I(G)$ is not normal. Thus $R[I(G')^\vee t]$ is not normal by Corollary 2.19.*

Definition 2.21. A clutter $\mathcal{C}$ satisfies the *max-flow min-cut* (MFMC) property if both sides of the LP-duality equation

$$\min\{\langle a, x\rangle \,|\, x \geq 0; xA \geq \mathbf{1}\} = \max\{\langle y, \mathbf{1}\rangle \,|\, y \geq 0; Ay \leq a\} \tag{2.9}$$

have integral optimum solutions x and y for each non-negative integral vector a.

Corollary 2.22. ([17, Theorem 3.4]) *Let A be the incidence matrix of a clutter $\mathcal{C}$ and let $I = I(\mathcal{C})$ be its edge ideal. Then $\mathcal{C}$ satisfies the max-flow min-cut property if and only if $Q(A)$ is integral and $R[It]$ is normal.*

Proof: Notice that if $\mathcal{C}$ has the max-flow min-cut property, then $Q(A)$ is integral [23, Corollary 22.1c]. Therefore the result follows directly from Eqs. (2.7), (2.9), and Theorem 2.12. $\square$

We now turn our attention to the integer rounding property of systems of the form $xA \leq \mathbf{1}$.

Definition 2.23. Let A be a matrix with entries in $\mathbb{N}$. The system $xA \leq \mathbf{1}$ is said to have the *integer rounding property* if

$$\lceil \min\{\langle y, \mathbf{1}\rangle \,|\, y \geq 0;\ Ay = a\}\rceil = \min\{\langle y, \mathbf{1}\rangle \,|\, Ay = a;\ y \in \mathbb{N}^q\}$$

for each integral vector a for which $\min\{\langle y, \mathbf{1}\rangle \,|\, y \geq 0;\ Ay = a\}$ is finite.

The next result is just a reinterpretation of an unpublished result of Giles and Orlin [23, Theorem 22.18] that characterizes the integer rounding property in terms of Hilbert bases.

Proposition 2.24. *Let $v_1, \ldots, v_q$ be the column vectors of a non-negative integer matrix A and let $A(P)$ be the Ehrhart ring of $P = \mathrm{conv}(0, v_1, \ldots, v_q)$. Then the system $xA \leq \mathbf{1}$ has the integer rounding property if and only if*

$$K[x^{v_1}t, \ldots, x^{v_q}t, t] = A(P).$$

Proof: By [23, Theorem 22.18], we have that the system $xA \leq \mathbf{1}$ has the integer rounding property if and only if the set $\mathcal{B} = \{(v_1, 1), \ldots, (v_q, 1), (0, 1)\}$ is a Hilbert basis. Thus the proposition follows readily by noticing the equality

$$A(P) = K[\{x^a t^b | (a, b) \in \mathbb{R}_+\mathcal{B} \cap \mathbb{Z}^{n+1}\}]$$

and the inclusion $K[x^{v_1}t, \ldots, x^{v_q}t, t] \subset A(P)$. $\square$

Theorem 2.25. *Let $\mathcal{A} = \{v_1, \ldots, v_q\}$ be the set of column vectors of a matrix A with entries in $\mathbb{N}$. If the system $xA \leq \mathbf{1}$ has the integer rounding property, then*

(a) $K[F]$ is normal, where $F = \{x^{v_1}, \ldots, x^{v_q}\}$, and

(b) $\mathbb{Z}^n/\mathbb{Z}\mathcal{A}$ is a torsion-free group.

The converse holds if $|v_i| = d$ for all i.

Proof: For use below we set $\mathcal{B} = \{(v_1,1),\ldots,(v_q,1),(0,1)\}$. First we prove (a). Let $x^a \in \overline{K[F]}$. Then $a \in \mathbb{Z}\mathcal{A}$ and we can write

$$a = \lambda_1 v_1 + \cdots + \lambda_q v_q,$$

for some $\lambda_1,\ldots,\lambda_q$ in $\mathbb{R}_+$. Hence

$$(a, \lceil \textstyle\sum_i \lambda_i \rceil) = \lambda_1(v_1,1) + \cdots + \lambda_q(v_q,1) + \delta(0,1),$$

where $\delta \geq 0$. Therefore by Proposition 2.24, there are $\lambda_1',\ldots\lambda_q' \in \mathbb{N}$ and $\delta' \in \mathbb{N}$ such that

$$(a, \lceil \textstyle\sum_i \lambda_i \rceil) = \lambda_1'(v_1,1) + \cdots + \lambda_q'(v_q,1) + \delta'(0,1),$$

Thus $x^a \in K[F]$, as required. Next we show (b). From Proposition 2.24, we get

$$K[x^{v_1}t,\ldots,x^{v_q}t,t] = A(P).$$

Hence using [10, Theorem 3.9] we obtain that the group $M = \mathbb{Z}^{n+1}/\mathbb{Z}\mathcal{B}$ is torsion free. Let $\overline{a}$ be an element of $T(\mathbb{Z}^n/\mathbb{Z}\mathcal{A})$, the torsion subgroup of $\mathbb{Z}^n/\mathbb{Z}\mathcal{A}$. Thus there is a positive integer s so that

$$sa = \lambda_1 v_1 + \cdots + \lambda_q v_q$$

for some $\lambda_1,\ldots,\lambda_q$ in $\mathbb{Z}$. From the equality

$$s(a,|a|) = \lambda_1(v_1,1) + \cdots + \lambda_q(v_q,1) + (s|a| - \lambda_1 - \cdots - \lambda_q)(0,1)$$

we obtain that the image of $(a,|a|)$ in M, denoted by $\overline{(a,|a|)}$, is a torsion element, i.e., $\overline{(a,|a|)} \in T(M) = (\overline{0})$. Hence it is readily seen that $a \in \mathbb{Z}\mathcal{A}$, i.e., $\overline{a} = \overline{0}$. Altogether we have $T(\mathbb{Z}^n/\mathbb{Z}\mathcal{A}) = (0)$.

Conversely assume that $|v_i| = d$ for all i and that (a) and (b) hold. We need only show that $\mathcal{B}$ is a Hilbert basis. Let (a,b) be an integral vector in $\mathbb{R}_+\mathcal{B}$, where $a \in \mathbb{N}^n$ and $b \in \mathbb{N}$. Then we can write

$$(a,b) = \lambda_1(v_1,1) + \cdots + \lambda_q(v_q,1) + \mu(0,1), \tag{2.10}$$

for some $\lambda_1,\ldots,\lambda_q,\mu$ in $\mathbb{Q}_+$. Hence using this equality together with (b) gives that a is in $\mathbb{R}_+\mathcal{A} \cap \mathbb{Z}\mathcal{A}$. Hence $x^a \in \overline{K[F]} = K[F]$, i.e., $a \in \mathbb{N}\mathcal{A}$. Then we can write

$$a = \eta_1 v_1 + \cdots + \eta_q v_q$$

for some $\eta_1,\ldots,\eta_q$ in $\mathbb{N}^n$. Since $|v_i| = d$ for all i, one has $\sum_i \lambda_i = \sum_i \eta_i$. Therefore using Eq. (2.10), we get $\mu \in \mathbb{N}$. Consequently from the equality

$$(a,b) = \eta_1(v_1,1) + \cdots + \eta_q(v_q,1) + \mu(0,1),$$

we conclude that $(a,b) \in \mathbb{N}\mathcal{B}$. This proves that $\mathcal{B}$ is a Hilbert basis. $\square$

Corollary 2.26. *Let A be the incidence matrix of a connected graph G. Then the system $xA \leq \mathbf{1}$ has the integer rounding property if and only if G is a bipartite graph.*

Proof: $\Rightarrow$) Let $\mathcal{A} = \{v_1, \ldots, v_q\}$ be the set of columns of A. If G is not bipartite, then according to [32, Corollary 3.4] one has $\mathbb{Z}^n/\mathbb{Z}\mathcal{A} \simeq \mathbb{Z}_2$, a contradiction to Theorem 2.25(b).

$\Leftarrow$) By [32, Theorem 2.15, Corollary 3.4] we get that the ring $K[x^{v_1}, \ldots, x^{v_q}]$ is normal and that $\mathbb{Z}^n/\mathbb{Z}\mathcal{A} \simeq \mathbb{Z}$. Thus by Theorem 2.25 the system $xA \leq \mathbf{1}$ has the integer rounding property, as required. This part of the proof also follows directly from the fact that the incidence matrix of a bipartite graph is totally unimodular. Indeed, since A is totally unimodular, both problems of the LP-duality equation

$$\max\{\langle a, x\rangle \,|\, xA \leq \mathbf{1}\} = \min\{\langle y, \mathbf{1}\rangle \,|\, y \geq 0; Ay = a\}$$

have integral optimum solutions for each integral vector a for which the minimum is finite, see [23, Corollary 19.1a]. Thus the system $xA \leq \mathbf{1}$ has the integer rounding property. $\Box$

Corollary 2.27. *Let A be the incidence matrix of a clutter $\mathcal{C}$. If $\mathcal{C}$ is uniform and has the max-flow min-cut property, then the system $xA \leq \mathbf{1}$ has the integer rounding property.*

Proof: Since all edges of $\mathcal{C}$ have the same size, it suffices to observe that conditions (a) and (b) of Theorem 2.25 are satisfied because of [7, Theorem 3.6]. $\Box$

3 The canonical module and the a-invariant

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over an arbitrary field K and let $K[F] = K[x^{v_1}, \ldots, x^{v_q}]$ be a homogeneous monomial subring, i.e., there exists $0 \neq x_0 \in \mathbb{Q}^n$ satisfying $\langle x_0, v_i\rangle = 1$ for all i. Then $K[F]$ is a standard graded K-algebra with the grading induced by declaring that a monomial $x^a \in K[F]$ has degree i if and only if $\langle a, x_0\rangle = i$. Recall that the *$a$-invariant* of $K[F]$, denoted by $a(K[F])$, is the degree as a rational function of the Hilbert series of $K[F]$, see for instance [31, p. 99]. Let H and φ be the Hilbert function and the Hilbert polynomial of $K[F]$ respectively. The index of regularity of $K[F]$, denoted by $\mathrm{reg}(K[F])$, is the least positive integer such that $H(i) = \varphi(i)$ for $i \geq \mathrm{reg}(K[F])$. The a-invariant plays a fundamental role in algebra and geometry because one has: $\mathrm{reg}(K[F]) = 0$ if $a(K[F]) < 0$ and $\mathrm{reg}(K[F]) = a(K[F]) + 1$ otherwise [31, Corollary 4.1.12].

If $K[F]$ is Cohen-Macaulay and $\omega_{K[F]}$ is the canonical module of $K[F]$, then

$$a(K[F]) = -\min\{\, i \,|\, (\omega_{K[F]})_i \neq 0\}, \tag{3.1}$$

see [2, p. 141] and [31, Proposition 4.2.3]. This formula applies if $K[F]$ is normal because normal monomial subrings are Cohen-Macaulay [19]. If $K[F]$ is normal, then by a formula of Danilov and Stanley (see [2, Theorem 6.3.5] and [6]) the canonical module of $K[F]$ is the ideal given by

$$\omega_{K[F]} = (\{x^a \,|\, a \in \mathbb{N}\mathcal{A} \cap (\mathbb{R}_+\mathcal{A})^{\mathrm{o}}\}), \tag{3.2}$$

where $\mathcal{A} = \{v_1, \ldots, v_q\}$ and $(\mathbb{R}_+\mathcal{A})^{\mathrm{o}}$ is the interior of $\mathbb{R}_+\mathcal{A}$ relative to $\mathrm{aff}(\mathbb{R}_+\mathcal{A})$, the affine hull of $\mathbb{R}_+\mathcal{A}$.

The *dual cone* of $\mathbb{R}_+\mathcal{A}$ is the polyhedral cone given by

$$(\mathbb{R}_+\mathcal{A})^* = \{x \,|\, \langle x, y\rangle \geq 0;\ \forall\, y \in \mathbb{R}_+\mathcal{A}\}.$$

A set $\mathcal{H} \subset \mathbb{R}^n \setminus \{0\}$ is called an *integral basis* of $(\mathbb{R}_+\mathcal{A})^*$ if $(\mathbb{R}_+\mathcal{A})^* = \mathbb{R}_+\mathcal{H}$ and $\mathcal{H} \subset \mathbb{Z}^n$. Let $0 \neq a \in \mathbb{R}^n$. In what follows H_a^+ denotes the closed halfspace $H_a^+ = \{x | \langle x, a\rangle \geq 0\}$ and H_a stands for the hyperplane through the origin with normal vector a.

The next result gives a general technique to compute the canonical module and the a-invariant of a wide class of monomial subrings. Another technique is given in [28]. In Section 4 we give some more precise expressions for the canonical module and the a-invariant of special families of monomial subrings arising from integer rounding properties.

Theorem 3.1. *Let $c_1, \ldots, c_r$ be an integral basis of $(\mathbb{R}_+\mathcal{A})^*$ and let $b = (b_i)$ be the $\{0, -1\}$-vector given by $b_i = 0$ if $\mathbb{R}_+\mathcal{A} \subset H_{c_i}$ and $b_i = -1$ if $\mathbb{R}_+\mathcal{A} \not\subset H_{c_i}$. If $\mathbb{N}\mathcal{A} = \mathbb{Z}^n \cap \mathbb{R}_+\mathcal{A}$ and B is the matrix with column vectors $-c_1, \ldots, -c_r$, then*

(a) $\omega_{K[F]} = (\{x^a | \, a \in \mathbb{Z}^n \cap \{x | xB \leq b\}).$

(b) $a(K[F]) = -\min\,\{\langle x_0, x\rangle |\ x \in \mathbb{Z}^n \cap \{x |\, xB \leq b\}\}.$

Proof: Let $\mathcal{H} = \{c_1, \ldots, c_r\}$. By duality [23, Corollary 7.1a], we have the equality

$$\mathbb{R}_+\mathcal{A} = H_{c_1}^+ \cap \cdots \cap H_{c_r}^+. \tag{3.3}$$

Observe that $\mathbb{R}_+\mathcal{A} \cap H_{c_i}$ is a proper face if $b_i = -1$ and it is an improper face otherwise. From Eq. (3.3) we get that each facet of $\mathbb{R}_+\mathcal{A}$ has the form $\mathbb{R}_+\mathcal{A} \cap H_{c_i}$ for some i. The relative interior of the cone $\mathbb{R}_+\mathcal{A}$ is the union of its facets. Hence, using that $\mathcal{H}$ is an integral basis, we obtain the equality

$$\mathbb{Z}^n \cap (\mathbb{R}_+\mathcal{A})^{\mathrm{o}} = \mathbb{Z}^n \cap \{x |\, xB \leq b\}. \tag{3.4}$$

Now, part (a) follows readily from Eqs. (3.2) and (3.4). Part (b) follows from Eq. (3.1) and part (a). $\square$

Next we illustrate how to determine the canonical module and the a-invariant using Theorem 3.1.

Example 3.2. *Let* $F = \{x_1, x_2, x_3, x_4, x_1x_2x_5, x_2x_3x_5, x_3x_4x_5, x_1x_4x_5\}$ *and let* $\mathcal{A}$ *be the set of exponent vectors of the monomials in* F*. Notice that* $\mathcal{A}$ *is a Hilbert basis and* $\langle x_0, v\rangle = 1$ *for* $v \in \mathcal{A}$*, where* $x_0 = (1, 1, 1, 1, -1)$*. An integral basis for* $(\mathbb{R}_+\mathcal{A})^*$ *is given by*

$$\{e_1, e_2, e_3, e_4, e_5, (0, 1, 0, 1, -1), (1, 0, 1, 0, -1)\}.$$

Then it is easy to verify that $\omega_{K[F]}$ *is generated by the set of all monomials* x^a *such that* $a = (a_i)$ *is in the polyhedron* Q *defined by the system:*

$$a_i \geq 1\,\forall\, i; \quad a_1 + a_3 - a_5 \geq 1; \quad a_2 + a_4 - a_5 \geq 1.$$

The only vertex of the polyhedron Q *is* $v_0 = (1, 1, 1, 1, 1)$*. Thus the* a*-invariant of* $K[F]$ *is equal to* $-\langle x_0, v_0\rangle = -3$*.*

4 Canonical modules and integer rounding properties

In this section we give a description of the canonical module and the a-invariant for subrings arising from systems with the integer rounding property.

Let A be a matrix of order $n \times q$ with entries in $\mathbb{N}$ such that A has non-zero rows and non-zero columns. Let $v_1, \dots, v_q$ be the columns of A. For use below consider the set $w_1, \dots, w_r$ of all $\alpha \in \mathbb{N}^n$ such that $\alpha \leq v_i$ for some i. Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K and let

$$S = K[x^{w_1}t, \dots, x^{w_r}t] \subset R[t]$$

be the subring of $R[t]$ generated by $x^{w_1}t, \dots, x^{w_r}t$, where t is a new variable. As $(w_i, 1)$ lies in the hyperplane $x_{n+1} = 1$ for all i, S is a standard K-algebra. Thus a monomial x^at^b in S has degree b. In what follows we assume that S has this grading. If S is normal, then according to Eq. (3.2) the canonical module of S is the ideal given by

$$\omega_S = (\{x^at^b \,|\, (a, b) \in \mathbb{N}\mathcal{B} \cap (\mathbb{R}_+\mathcal{B})^{\mathrm{o}}\}), \tag{4.1}$$

where $\mathcal{B} = \{(w_1, 1), \dots, (w_r, 1)\}$ and $(\mathbb{R}_+\mathcal{B})^{\mathrm{o}}$ is the interior of $\mathbb{R}_+\mathcal{B}$ relative to $\mathrm{aff}(\mathbb{R}_+\mathcal{B})$, the affine hull of $\mathbb{R}_+\mathcal{B}$. In our case $\mathrm{aff}(\mathbb{R}_+\mathcal{B}) = \mathbb{R}^{n+1}$.

Let $\ell_0, \ell_1, \dots, \ell_m$ be the vertices of $P = \{x \,|\, x \geq 0; xA \leq \mathbf{1}\}$, where $\ell_0 = 0$, and let $\ell_1, \dots, \ell_p$ be the set of all maximal elements of $\ell_0, \ell_1, \dots, \ell_m$ (maximal with respect to $\leq$).

Lemma 4.1. *For each* $1 \leq i \leq p$ *there is a unique positive integer* d_i *such that the non-zero entries of* $(-d_i\ell_i, d_i)$ *are relatively prime.*

Proof: If the non-zero rational entries of ℓ_i are written in lowest terms, then d_i is the least common multiple of the denominators. □

Notation In what follows $\{\ell_1,\ldots,\ell_p\}$ is the set of maximal elements of $\{\ell_0,\ldots,\ell_m\}$ and $d_1,\ldots,d_p$ are the unique positive integers in Lemma 4.1.

The next result complements a result of [9].

Theorem 4.2. *If the system $x\geq 0; xA\leq \mathbf{1}$ has the integer rounding property, then the subring $S=K[x^{w_1}t,\ldots,x^{w_r}t]$ is normal, the canonical module of S is given by*

$$\omega_S=\left(\left\{x^at^b\middle|\,(a,b)\begin{pmatrix} -d_1\ell_1 & \cdots & -d_p\ell_p & e_1 & \cdots & e_n\\ d_1 & \cdots & d_p & 0 & \cdots & 0\end{pmatrix}\geq \mathbf{1}\right\}\right), \qquad (4.2)$$

and the a-invariant of S is equal to $-\max_i\{\lceil 1/d_i+|\ell_i|\rceil\}$. Here $|\ell_i|=\langle \ell_i,\mathbf{1}\rangle$.

Proof: Note that in Eq. (4.2) we regard $(-d_i\ell_i,d_i)$ and e_j as column vectors for all i,j. The normality of S follows from Theorem 2.5. Recall that we have the following duality (see Section 2):

$$\begin{aligned} P=\{x|\,x\geq 0;\langle x,w_i\rangle\leq 1\,\forall i\} &= \mathrm{conv}(\ell_0,\ell_1,\ldots,\ell_m),\\ \mathrm{conv}(w_1,\ldots,w_r) &= \{x|\,x\geq 0;\langle x,\ell_i\rangle\leq 1\forall i\}=T(P), \end{aligned} \qquad (4.3)$$

where $\{\ell_0,\ell_1,\ldots,\ell_m\}\subset\mathbb{Q}_+^n$ is the set of vertices of P and $\ell_0=0$. Therefore using Eq. (4.3) and the maximality of $\ell_1,\ldots,\ell_p$ we obtain

$$\mathrm{conv}(w_1,\ldots,w_r)=\{x|\,x\geq 0;\langle x,\ell_i\rangle\leq 1,\,\forall\, i=1,\ldots,p\}. \qquad (4.4)$$

We set $\mathcal{B}=\{(w_1,1),\ldots,(w_r,1)\}$. Note that $\mathbb{Z}\mathcal{B}=\mathbb{Z}^{n+1}$. From Eq. (4.4) it is seen that

$$\mathbb{R}_+\mathcal{B}=H_{e_1}^+\cap\cdots\cap H_{e_n}^+\cap H^+_{(-d_1\ell_1,d_1)}\cap\cdots\cap H^+_{(-d_p\ell_p,d_p)}. \qquad (4.5)$$

Here H_a^+ denotes the closed halfspace $H_a^+=\{x|\,\langle x,a\rangle\geq 0\}$ and H_a stands for the hyperplane through the origin with normal vector a. Notice that

$$H_{e_1}\cap\mathbb{R}_+\mathcal{B},\ldots,H_{e_n}\cap\mathbb{R}_+\mathcal{B},H_{(-d_1\ell_1,d_1)}\cap\mathbb{R}_+\mathcal{B},\ldots,H_{(-d_p\ell_p,d_p)}\cap\mathbb{R}_+\mathcal{B}$$

are proper faces of $\mathbb{R}_+\mathcal{B}$. Hence from Eq. (4.5) we get that a vector (a,b), with $a\in\mathbb{Z}^n$, $b\in\mathbb{Z}$, is in the relative interior of $\mathbb{R}_+\mathcal{B}$ if and only if the entries of a are positive and $\langle(a,b),(-d_i\ell_i,d_i)\rangle\geq 1$ for all i. Thus the required expression for ω_S, i.e., Eq. (4.2), follows using the normality of S and the Danilov-Stanley formula given in Eq. (4.1).

It remains to prove the formula for $a(S)$, the a-invariant of S. Consider the vector $(\mathbf{1},b_0)$, where $b_0=\max_i\{\lceil 1/d_i+|\ell_i|\rceil\}$. Using Eq. (4.2), it is not hard to see (by direct substitution of $(\mathbf{1},b_0)$), that the monomial $x^{\mathbf{1}}t^{b_0}$ is in ω_S. Thus from Eq. (3.1) we get $a(S)\geq -b_0$. Conversely if the monomial x^at^b is in ω_S, then again from Eq. (4.2) we get $\langle(-d_i\ell_i,d_i),(a,b)\rangle\geq 1$ for all i and $a_i\geq 1$ for all i, where $a=(a_i)$. Hence

$$bd_i\geq 1+d_i\langle a,\ell_i\rangle\geq 1+d_i\langle\mathbf{1},\ell_i\rangle=1+d_i|\ell_i|.$$

Since b is an integer we obtain $b \geq \lceil 1/d_i + |\ell_i| \rceil$ for all i. Therefore $b \geq b_0$, i.e., $\deg(x^a t^b) = b \geq b_0$. As $x^a t^b$ was an arbitrary monomial in ω_S, by the formula for the a-invariant of S given in Eq. (3.1) we obtain that $a(S) \leq -b_0$. Altogether one has $a(S) = -b_0$, as required. $\Box$

A standard graded K-algebra S is called *Gorenstein* if S is Cohen-Macaulay and ω_S is a principal ideal.

Theorem 4.3. *Assume that the system $x \geq 0$; $xA \leq \mathbf{1}$ has the integer rounding property. If $S = K[x^{w_1}t, \ldots, x^{w_r}t]$ is Gorenstein and $c_0 = \max\{|\ell_i| \colon 1 \leq i \leq p\}$ is an integer, then $|\ell_k| = c_0$ for each $1 \leq k \leq p$ such that ℓ_k has integer entries.*

Proof: We proceed by contradiction. Assume that $|\ell_k| < c_0$ for some integer $1 \leq k \leq p$ such that ℓ_k is integral. We may assume that $\ell_k = (1, \ldots, 1, 0, \ldots, 0)$ and $|\ell_k| = s$. From Eq. (4.5) it follows that the monomial $x^{\ell_k} t^{s-1}$ cannot be in S because $(\ell_k, s-1)$ does not belong to $H^+_{(-d_k\ell_k, d_k)}$. Consider the monomial $x^a t^b$, where $a = \ell_k + \mathbf{1}$, $b = b_0 + s - 1$ and $b_0 = -a(S)$. We claim that the monomial $x^a t^b$ is in ω_S. By Theorem 4.2 it suffices to show that $\langle (a, b), (-d_j\ell_j, d_j) \rangle \geq 1$ for $1 \leq j \leq p$. Thus we need only show that $\langle (a, b), (-\ell_j, 1) \rangle > 0$ for $1 \leq j \leq p$. From the proof of Theorem 4.2, it is seen that $-a(S) = \max_i\{\lfloor |\ell_i| \rfloor\} + 1$. Hence we get $b_0 = c_0 + 1$. One has the following equalities

$$\langle (a, b), (-\ell_j, 1) \rangle = -|\ell_j| - \langle \ell_k, \ell_j \rangle + b_0 + s - 1 = -|\ell_j| - \langle \ell_k, \ell_j \rangle + c_0 + s.$$

Set $\ell_j = (\ell_{j_1}, \ldots, \ell_{jn})$. From Eq. (4.5) we get that the entries of each ℓ_j are less than or equal to 1. Case (I): If $\ell_{ji} < 1$ for some $1 \leq i \leq s$, then $s - \langle \ell_k, \ell_j \rangle > 0$ and $c_0 \geq |\ell_j|$. Case (II): $\ell_{ji} = 1$ for $1 \leq i \leq s$. Then $\ell_j \geq \ell_k$. Thus by the maximality of ℓ_k we obtain $\ell_j = \ell_k$. In both cases we obtain $\langle (a, b), (-\ell_j, 1) \rangle > 0$, as required. Hence the monomial $x^a t^b$ is in ω_S. Since S is Gorenstein and ω_S is generated by $x^{\mathbf{1}} t^{b_0}$, we obtain that $x^a t^b$ is a multiple of $x^{\mathbf{1}} t^{b_0}$, i.e., $x^{\ell_k} t^{s-1}$ must be in S, a contradiction. $\Box$

Theorem 4.4. *Assume that the system $x \geq 0; xA \leq \mathbf{1}$ has the integer rounding property. If $S = K[x^{w_1}t, \ldots, x^{w_r}t]$ and $-a(S) = 1/d_i + |\ell_i|$ for $i = 1, \ldots, p$, then S is Gorenstein.*

Proof: We set $b_0 = -a(S)$ and $\mathcal{B} = \{(w_1, 1), \ldots, (w_r, 1)\}$. The ring S is normal by Theorem 2.5. Since the monomial $x^{\mathbf{1}} t^{b_0} = x_1 \cdots x_n t^{b_0}$ is in ω_S, we need only show that $\omega_S = (x^{\mathbf{1}} t^{b_0})$. Take $x^a t^b \in \omega_S$. It suffices to prove that $x^{a-\mathbf{1}} t^{b-b_0}$ is in S. Using Theorem 2.5, one has $\mathbb{R}_+\mathcal{B} \cap \mathbb{Z}^{n+1} = \mathbb{N}\mathcal{B}$. Thus we need only show that the vector $(a - \mathbf{1}, b - b_0)$ is in $\mathbb{R}_+\mathcal{B}$. From Eq. (4.5), the proof reduces to show that the vector $(a - \mathbf{1}, b - b_0)$ is in $H^+_{(-\ell_i, 1)}$ for $i = 1, \ldots, p$.

As $(a, b) \in \omega_S$, from the description of ω_S given in Theorem 4.2 we get

$$\langle (a, b), (-d_i\ell_i, d_i) \rangle = -\langle a, d_i\ell_i \rangle + bd_i \geq 1 \implies -\langle a, \ell_i \rangle \geq -b + 1/d_i$$

for $i = 1, \ldots, p$. Therefore

$$\langle (a - \mathbf{1}, b - b_0), (-\ell_i, 1) \rangle = -\langle a, \ell_i \rangle + |\ell_i| + b - b_0 \geq -b + 1/d_i + |\ell_i| + b - b_0 = 0$$

for all i, as required. □

Corollary 4.5. *If $P = \{x | \, x \geq 0; xA \leq \mathbf{1}\}$ is an integral polytope, then the monomial subring $S = K[x^{w_1}t, \ldots, x^{w_r}t]$ is Gorenstein if and only if $a(S) = -(|\ell_i| + 1)$ for $i = 1, \ldots, p$.*

Proof: Notice that if P is integral, then ℓ_i has entries in $\{0, 1\}$ for $1 \leq i \leq p$ and consequently $d_i = 1$ for $1 \leq i \leq p$. Thus the result follows from Theorems 4.3 and 4.4. □

Example 4.6. *Let G be a pentagon with vertex set $X = \{x_1, \ldots, x_5\}$, let A be the incidence matrix of G and let*

$$S = K[t, x_1t, \ldots, x_5t, x_1x_2t, x_2x_3t, x_3x_4t, x_4x_5t, x_1x_5t].$$

The system $x \geq 0; xA \leq \mathbf{1}$ has the integer rounding property and the vertex set of $P = \{x | \, x \geq 0; xA \leq \mathbf{1}\}$ is:

$$\mathrm{vert}(P) = \{0, \mathbf{1}/2, e_3 + e_5, e_2 + e_5, e_2 + e_4, e_1 + e_4, e_1 + e_3, e_1, e_2, e_3, e_4, e_5\}.$$

The maximal elements of vert(P) *are*

$$\ell_1 = \mathbf{1}/2, \ell_2 = e_3 + e_5, \ell_3 = e_2 + e_5, \ell_4 = e_2 + e_4, \ell_5 = e_1 + e_4, \ell_6 = e_1 + e_3,$$

$d_1 = 2$ and $d_i = 1$ for $2 \leq i \leq 6$. Notice that $1/d_i + |\ell_i| = 3$ for $i = 1, \ldots, 6$. Thus by Theorems 4.2 and 4.4, the ring S is Gorenstein and $a(S) = -3$.

Problem 4.7. If A is the incidence matrix of a connected graph and the system $x \geq 0$; $xA \leq \mathbf{1}$ has the integer rounding property, then the subring $S = K[x^{w_1}t, \ldots, x^{w_r}t]$ is Gorenstein if and only if $-a(S) = 1/d_i + |\ell_i|$ for $i = 1, \ldots, p$.

Note that the answer to this problem is positive if A is the incidence matrix of a bipartite graph because in this case P is an integral polytope and we may apply Corollary 4.5. If A is the incidence matrix of a connected non-bipartite graph G, E. Reyes has shown that G is unmixed if S is Gorenstein. If A is the incidence matrix of a graph, then it is seen that $d_i = 1$ or $d_i = 1/2$ for each i.

Subrings associated to the system $xA \leq \mathbf{1}$

Let A be a matrix with entries in $\mathbb{N}$ such that the system $xA \leq \mathbf{1}$ has integer rounding property. As before we assume that the rows and columns of A are different from zero and that $v_1, \ldots, v_q$ are the columns of A. In what follows we assume that $|v_i| = d$ for all i.

The following lemma is not hard to show.

Lemma 4.8. *If $|v_i| = d$ for all i. Then there are isomorphisms*

$$K[x^{v_1}t, \ldots, x^{v_q}t, t] \simeq K[x^{v_1}t, \ldots, x^{v_q}t][T] \text{ and}$$
$$K[x^{v_1}t, \ldots, x^{v_q}t] \simeq K[x^{v_1}, \ldots, x^{v_q}]$$

induced by $x^{v_i}t \mapsto x^{v_i}t$, $t \mapsto T$ and $x^{v_i}t \mapsto x^{v_i}$ respectively, where T is a new variable.

Let S be a homogeneous monomial subring and let P_S be its toric ideal. Recall that S is called a *complete intersection* if P_S is a complete intersection, i.e., P_S can be generated by $\mathrm{ht}(P_S)$ binomials, where $\mathrm{ht}(P_S)$ is the height of P_S. Let c be a cycle of a graph G. A *chord* of c is any edge of G joining two non adjacent vertices of c. A cycle without chords is called *primitive*.

Proposition 4.9. *Let G be a connected graph with n vertices and q edges and let A be its incidence matrix. If the system $xA \leq \mathbf{1}$ has the integer rounding property, then $K[x^{v_1}t, \ldots, x^{v_q}t, t]$ is a complete intersection if and only if G is bipartite and the number of primitive cycles of G is equal to $q - n + 1$.*

Proof: $\Rightarrow$) By Corollary 2.26 the graph G is bipartite. From Lemma 4.8 it follows that $K[x^{v_1}t, \ldots, x^{v_q}t, t]$ is a complete intersection if and only if $K[G] = K[x^{v_1}, \ldots, x^{v_q}]$ is a complete intersection. Therefore by [25] we get that $K[G]$ is a complete intersection if and only if the number of primitive cycles of G is equal to $q - n + 1$.

$\Leftarrow$) By [25] the ring $K[G]$ is a complete intersection. Hence $K[x^{v_1}t, \ldots, x^{v_q}t, t]$ is a complete intersection by Lemma 4.8. $\Box$

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