Bull. Math. Soc. Sci. Math. Roumanie
Tome 51(99) No. 4, 2008, 325–337

Eigenvalue problems for some nonlinear perturbations of the Laplace operator

by
Mihai Mihăilescu

Abstract

The goal of this paper is to prove existence results for some eigenvalue problems in which is involved a class of nonlinear operators which perturb the Laplace operator. Our proofs rely essentially on the Banach fixed point theorem and on a minimization technique.

Key Words: Nonlinear perturbation of the Laplace operator, eigenvalue problem, principal frequency.
2000 Mathematics Subject Classification: Primary 35P30, Secondary 35J60, 35J20.

1 Introduction and main results

The study of eigenvalue problems of various differential operators captured an enormous interest in the last decades. A large variety of papers pointed out different phenomena which can occur on the spectrum of certain differential operators. We just remember the recent advances in [3, 4, 5, 7, 8, 9, 10, 11, 12, 15, 16]. The goal of this paper is to point out certain results on an eigenvalue problem in which we perturb the Laplace operator in a sense that will be described later. More exactly, in this paper we are concerned with the study of an eigenvalue problem of the type

$$\begin{cases} -\text{div}(\mathcal{A}(\nabla u)) = \lambda u, & \text{for} \quad x \in \Omega \\ u = 0, & \text{for} \quad x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbf{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary, $\lambda > 0$ and $\mathcal{A} : \mathbf{R}^N \to \mathbf{R}^N$ is a function which represents a perturbation of the Laplace operator that will be specified later in the paper.

In the case when $\mathcal{A}(\xi) = \xi$ for all $\xi \in \mathbf{R}^N$ problem (1) goes back to the classical problem

$$\begin{cases} -\Delta u = \lambda u, & \text{for} \quad x \in \Omega \\ u = 0, & \text{for} \quad x \in \partial\Omega. \end{cases} \tag{2}$$

A real number λ is called an *eigenvalue* of problem (2) if there exists $u \in H_0^1(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \nabla u \nabla \varphi \, dx = \lambda \int_\Omega u\varphi \, dx, \quad \forall\, \varphi \in H_0^1(\Omega).$$

The function u is called an *eigenfunction* associated with the eigenvalue λ.

It is known (see e.g. Brezis [1], Theorem IX.31) that for problem (2) there exists a nondecreasing sequence of positive eigenvalues $\{\lambda_n\}$ such that $\lambda_n \to \infty$ as $n \to \infty$. More exactly, the spectrum of the Laplace operator is *discrete* on bounded domains with smooth boundary. Furthermore, λ_1 is the minimum of the Rayleigh quotient

$$\lambda_1 = \min_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx}{\displaystyle\int_\Omega u^2 \, dx},$$

and is called *the principal frequency*. The associated eigenfunction u describes the shape of a membrane when it vibrates emitting its gravest tone, cf. Polya and Szego [13]. It is known that λ_1 is simple, i.e. all the associated eigenfunctions u are merely constant multiples of each other (see e.g. Gilbarg and Trudinger [2]). Moreover, the first eigenfunction never changes signs in Ω. On the other hand, higher eigenvalues are not simple (see e.g. Polya and Szego [13]).

This time we study problem (1) when $\mathrm{div}(\mathcal{A}(\nabla u))$ is a perturbation of the Laplace operator. Thus, throughout this paper we assume that $\mathcal{A}$ is of the type

$$\mathcal{A}(\xi) = (\mathcal{A}_1(\xi), ..., \mathcal{A}_N(\xi)), \quad \forall\, \xi = (\xi_1, ..., \xi_N) \in \mathbf{R}^N,$$

with $\mathcal{A}_1, ..., \mathcal{A}_N : \mathbf{R}^N \to \mathbf{R}$ and for any $i \in \{1, ..., N\}$ either $\mathcal{A}_i(\xi) = \cos(\xi_i) + 2\xi_i$ or $\mathcal{A}_i(\xi) = \sin(\xi_i) + 2\xi_i$, for all $\xi = (\xi_1, ..., \xi_N) \in \mathbf{R}^N$.

Definition 1. *We say that $\lambda \in \mathbf{R}$ is an eigenvalue of problem (1) if there exists $u \in H_0^1(\Omega) \setminus \{0\}$ such that*

$$\int_\Omega \mathcal{A}(\nabla u) \nabla \varphi \, dx = \lambda \int_\Omega u\varphi \, dx, \quad \forall\, \varphi \in H_0^1(\Omega).$$

We prove that the operators of the type described above possess a *continuous family* of positive eigenvalues in a right neighborhood of the origin excepting the case when $\mathcal{A}_i(x) = \sin(x_i) + 2x_i$ for all $i \in \{1, ..., N\}$ and all $x \in \Omega$. However, even in that case we can prove the existence of at least one positive eigenvalue. The main results of our study are given by the following theorems:

Theorem 1. *Assume that there exists $i_0 \in \{1, ..., N\}$ such that $\mathcal{A}_{i_0}(x) = \cos(x_{i_0}) + 2x_{i_0}$ for all $x \in \mathbf{R}^N$. Then any $\lambda \in (0, \lambda_1)$ is an eigenvalue of problem (1), where λ_1 is the first eigenvalue of the Laplace operator.*

Theorem 2. *Assume that for any $i \in \{1, ..., N\}$ we have $\mathcal{A}_i(x) = \sin(x_i) + 2x_i$ for all $x \in \mathbf{R}^N$. Then there exists at least a positive eigenvalue λ of problem (1), such that $\lambda \geq \lambda_1$ where λ_1 is the first eigenvalue of the Laplace operator.*

Theorem 3. *Assume the hypotheses of Theorem 1 are fulfilled. Then there exists at least a positive eigenvalue λ of problem (1), such that $\lambda \geq \lambda_1$ where λ_1 is the first eigenvalue of the Laplace operator.*

Throughout this paper we denote by $\langle ., . \rangle$ the scalar product on the Sobolev space $H_0^1(\Omega)$ and by $\|.\|$ the corresponding norm on $H_0^1(\Omega)$, i.e.

$$\langle u, v \rangle = \int_\Omega \nabla u \nabla v \; dx, \quad \|u\| = \left(\int_\Omega |\nabla u|^2 \; dx \right)^{1/2}.$$

We also denote by $\|.\|_{L^2}$ the norm on the Lebesgue space $L^2(\Omega)$, i.e.

$$\|u\|_{L^2} = \left(\int_\Omega u^2 \; dx \right)^{1/2}.$$

2 Proof of Theorem 1

In order to prove Theorem 1 we use a method borrowed from the proof of a nonlinear version of the Lax-Milgram Theorem (see Zeidler [18], Section 2.15). Our proof will use as main tool the Banach fixed point theorem (see Zeidler [18], Section 1.6).

First, we define the operators $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbf{R}$ by

$$a(u, v) = \int_\Omega \mathcal{A}(\nabla u) \nabla v \; dx, \quad \forall \; u, v \in H_0^1(\Omega)$$

and $b_\lambda : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbf{R}$ by

$$b_\lambda(u, v) = \lambda \int_\Omega uv \; dx, \quad \forall \; u, v \in H_0^1(\Omega).$$

It is enough to show that for any $\lambda \in (0, \lambda_1)$ there exists $u \in H_0^1(\Omega)$ such that

$$a(u, v) = b_\lambda(u, v), \quad \forall \; u, v \in H_0^1(\Omega).$$

We point out certain properties of the operators a respectively b_λ.

Proposition 1. *The operator a verifies the following properties:*
(i) for any $w \in H_0^1(\Omega)$ the application $v \to a(w, v)$ is linear and continuous on $H_0^1(\Omega)$;
(ii) $\|u - v\|^2 \leq a(u, u - v) - a(v, u - v)$, for all u, $v \in H_0^1(\Omega)$;
(iii) $|a(u, w) - a(v, w)| \leq 3 \cdot \|u - v\| \cdot \|w\|$, for all u, $v \in H_0^1(\Omega)$.

Proof: For any $i \in \{1, ..., N\}$ we set $\gamma_i(x_i) = \mathcal{A}_i(x) - 2x_i$, for all

$$x = (x_1, ..., x_N) \in \mathbf{R}^N.$$

(i) We fix $w \in H_0^1(\Omega)$. It is clear that the application $v \to a(w, v)$ is linear. On the other hand, using Hölder's inequality we have

$$\begin{aligned}
|a(w,v)| &= \left|\int_\Omega \mathcal{A}(\nabla w)\nabla v \, dx\right| \\
&= \left|\sum_{i=1}^N \int_\Omega \gamma_i\left(\frac{\partial w}{\partial x_i}\right)\frac{\partial v}{\partial x_i}\, dx + 2\int_\Omega \nabla w \nabla v \, dx\right| \\
&\leq \sum_{i=1}^N \int_\Omega \left|\gamma_i\left(\frac{\partial w}{\partial x_i}\right)\right|\left|\frac{\partial v}{\partial x_i}\right| \, dx + 2\|w\| \cdot \|v\| \\
&\leq \sum_{i=1}^N \int_\Omega \left|\frac{\partial v}{\partial x_i}\right| \, dx + 2\|w\| \cdot \|v\| \\
&\leq (c + 2\|w\|)\|v\|
\end{aligned}$$

where c is a positive constant. It follows that $v \to a(w, v)$ is continuous.

(ii) We have

$$\begin{aligned}
& a(u, u-v) - a(v, u-v) \\
= & \int_\Omega (\mathcal{A}(u) - \mathcal{A}(v))\nabla(u-v) \, dx \\
= & \sum_{i=1}^N \int_\Omega \left(\gamma_i\left(\frac{\partial u}{\partial x_i}\right) - \gamma_i\left(\frac{\partial v}{\partial x_i}\right)\right)\left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right) \, dx + 2 \cdot \|u-v\|^2.
\end{aligned}$$

Using the mean value theorem and taking into account that $|\gamma_i^{'}(y)| \leq 1$ for all $y \in \mathbf{R}$ and all $i \in \{1, ..., N\}$ we deduce that

$$\begin{aligned}
a(u, u-v) - a(v, u-v) &= \sum_{i=1}^N \int_\Omega \gamma_i^{'}(\theta_i)\left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right)^2 \, dx + 2 \cdot \|u-v\|^2 \\
&\geq -\|u-v\|^2 + 2 \cdot \|u-v\|^2 = \|u-v\|^2
\end{aligned}$$

where $\theta_i(x) = \mu_i(x)\frac{\partial u}{\partial x_i}(x) + (1 - \mu_i(x))\frac{\partial v}{\partial x_i}(x)$ for all $i \in \{1, ..., N\}$ and for all $x \in \Omega$ with $\mu_i(x) \in [0, 1]$ for all $i \in \{1, ..., N\}$ and all $x \in \Omega$.

(iii) Using the same arguments and notations as above we have

$$\begin{aligned}
&|a(u,w)-a(v,w)| \\
= &\left|\sum_{i=1}^{N}\int_\Omega\left(\gamma_i\left(\frac{\partial u}{\partial x_i}\right)-\gamma_i\left(\frac{\partial v}{\partial x_i}\right)\right)\frac{\partial w}{\partial x_i}\,dx+2\int_\Omega \nabla(u-v)\nabla w\,dx\right| \\
\leq &\sum_{i=1}^{N}\int_\Omega |\gamma_i^{'}(\theta_i)|\cdot\left|\frac{\partial u}{\partial x_i}-\frac{\partial v}{\partial x_i}\right|\cdot\left|\frac{\partial w}{\partial x_i}\right|\,dx+2\int_\Omega |\nabla(u-v)|\cdot|\nabla w|\,dx \\
\leq &\sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u}{\partial x_i}-\frac{\partial v}{\partial x_i}\right|\cdot\left|\frac{\partial w}{\partial x_i}\right|\,dx+2\int_\Omega |\nabla(u-v)|\cdot|\nabla w|\,dx \\
\leq &\ 3\cdot\|u-v\|\cdot\|w\|.
\end{aligned}$$

The proof of Proposition 1 is complete. □

Proposition 2. *For any $\lambda\in(0,\lambda_1)$ the operator b_λ verifies the following properties:*
(i) b_λ is bilinear and continuous on $H_0^1(\Omega)\times H_0^1(\Omega)$;
(ii) $b_\lambda(u,u)\geq 0$ for all $u\in H_0^1(\Omega)$;
(iii) there exists $M>0$ such that

$$|b_\lambda(u,w)-b_\lambda(v,w)|\leq M\cdot\|u-v\|\cdot\|w\|,\quad \forall\, u,v,w\in H_0^1(\Omega).$$

Proof: (i) It is clear that b_λ is a bilinear operator on $H_0^1(\Omega)\times H_0^1(\Omega)$. On the other hand, Hölder's inequality and the Sobolev continuous embedding theorem of $H_0^1(\Omega)$ in $L^2(\Omega)$ imply

$$|b_\lambda(u,v)|\leq\lambda\cdot\|u\|_{L^2}\cdot\|v\|_{L^2}\leq c\cdot\|u\|\cdot\|v\|,\quad\forall\, u,v\in H_0^1(\Omega)$$

where c is a positive constant. That fact shows that b_λ is continuous.
(ii) For any $u\in H_0^1(\Omega)$ we have $b_\lambda(u,u)=\lambda\|u\|_{L^2}^2\geq 0$.
(iii) Using again Hölder's inequality and the Sobolev continuous embedding theorem of $H_0^1(\Omega)$ in $L^2(\Omega)$ we obtain

$$\begin{aligned}
|b_\lambda(u,w)-b_\lambda(v,w)| &= \lambda\left|\int_\Omega (u-v)w\,dx\right| \\
&\leq \lambda\cdot\|u-v\|_{L^2}\cdot\|w\|_{L^2} \\
&\leq M\cdot\|u-v\|\cdot\|w\|,\quad\forall\, u,v,w\in H_0^1(\Omega)
\end{aligned}$$

where M is a positive constant. The proof of Proposition 2 is complete. □

Proof of Theorem 1. Let $\lambda\in(0,\lambda_1)$ be arbitrary but fixed. By Proposition 1 (i) and the Riesz theorem we deduce that for each $u\in H_0^1(\Omega)$ there is an element called $Au \in H_0^1(\Omega)$ such that

$$a(u,v) = \langle Au, v\rangle = \int_\Omega \nabla Au \nabla v \; dx, \quad \forall \; v \in H_0^1(\Omega).$$

Thus we can define an operator $A : H_0^1(\Omega) \to H_0^1(\Omega)$. By Proposition 1 (ii) and (iii) it follows that A verifies the properties

$$\|u-v\|^2 \leq \langle Au, u-v\rangle - \langle Av, u-v\rangle, \quad \forall \; u,v \in H_0^1(\Omega) \tag{3}$$

i.e. A is strongly monotone, and

$$|\langle Au, w\rangle - \langle Av, w\rangle| \leq 3 \cdot \|u-v\| \cdot \|w\|, \quad \forall \; u,v,w \in H_0^1(\Omega). \tag{4}$$

Relation (4) implies

$$\|Au - Av\| = \sup_{\|w\|\leq 1} |\langle Au - Av, w\rangle| \leq 3 \cdot \|u-v\|, \quad \forall \; u,v \in H_0^1(\Omega). \tag{5}$$

On the other hand, by Proposition 2 (i) and the Riesz theorem we deduce that for each $u \in H_0^1(\Omega)$ there is an element called $B_\lambda u \in H_0^1(\Omega)$ such that

$$b_\lambda(u,v) = \langle B_\lambda u, v\rangle = \int_\Omega \nabla B_\lambda u \nabla v \; dx, \quad \forall \; v \in H_0^1(\Omega).$$

In this way we can define an operator $B_\lambda : H_0^1(\Omega) \to H_0^1(\Omega)$. By Proposition 2 and the variational characterization of λ_1 it follows that B_λ is a linear operator which satisfies the properties:

$$\langle B_\lambda u, u-v\rangle - \langle B_\lambda v, u-v\rangle = b_\lambda(u-v, u-v) \leq \frac{\lambda}{\lambda_1} \cdot \|u-v\|^2, \quad \forall \; u,v \in H_0^1(\Omega) \tag{6}$$

and

$$\|B_\lambda u - B_\lambda v\| = \sup_{\|w\|\leq 1} |\langle B_\lambda u - B_\lambda v, w\rangle| =$$

$$= \sup_{\|w\|\leq 1} |b_\lambda(u-v, w)| \leq M \cdot \|u-v\|, \quad \forall \; u,v \in H_0^1(\Omega). \tag{7}$$

We define the operator $S : H_0^1(\Omega) \to H_0^1(\Omega)$ by

$$Su = u - t(Au - B_\lambda u)$$

where $t \in (0, \frac{2(1-\lambda/\lambda_1)}{(3+M)^2})$. The relations (3), (5), (6) and (7) show that for each

v_1, $v_2 \in H_0^1(\Omega)$ the following inequalities hold true

$$\begin{aligned}
& \|Sv_1 - Sv_2\|^2 \\
= \;& \langle Sv_1 - Sv_2, Sv_1 - Sv_2 \rangle \\
= \;& \langle (v_1 - v_2) - t(Av_1 - Av_2) + t(B_\lambda v_1 - B_\lambda v_2), (v_1 - v_2) - t(Av_1 - Av_2) + \\
& t(B_\lambda v_1 - B_\lambda v_2) \rangle \\
= \;& \|v_1 - v_2\|^2 - 2t\langle Av_1 - Av_2, v_1 - v_2 \rangle + 2t\langle B_\lambda v_1 - B_\lambda v_2, v_1 - v_2 \rangle - \\
& 2t^2 \langle Av_1 - Av_2, B_\lambda v_1 - B_\lambda v_2 \rangle + t^2\|Av_1 - Av_2\|^2 + t^2 \|B_\lambda v_1 - B_\lambda v_2\|^2 \\
\leq \;& \|v_1 - v_2\|^2 - 2t\|v_1 - v_2\|^2 + \\
& 2t\frac{\lambda}{\lambda_1}\|v_1 - v_2\|^2 + 2t^2\|Av_1 - Av_2\| \cdot \|B_\lambda v_1 - B_\lambda v_2\| + \\
& t^2\|Av_1 - Av_2\|^2 + t^2\|B_\lambda v_1 - B_\lambda v_2\| \\
\leq \;& \left(1 - 2t\left(1 - \frac{\lambda}{\lambda_1}\right) + 6Mt^2 + 9t^2 + M^2t^2\right) \cdot \|v_1 - v_2\|^2 \\
= \;& \alpha \cdot \|v_1 - v_2\|^2
\end{aligned}$$

where $\alpha = 1 - 2(1 - \frac{\lambda}{\lambda_1})t + (3+M)^2t^2 \geq 0$. If $t = 0$ or $t = \frac{2(1-\lambda/\lambda_1)}{(3+M)^2}$ then $\alpha = 1$. This implies that $\sqrt{\alpha} < 1$ for all $t \in (0, \frac{2(1-\lambda/\lambda_1)}{(3+M)^2})$.

Therefore,

$$\|Sv_1 - Sv_2\| \leq \sqrt{\alpha} \cdot \|u - v\|, \quad \forall\, u, v \in H_0^1(\Omega)$$

i.e. S is $\sqrt{\alpha}$-contractive with $\sqrt{\alpha} < 1$. By the Banach fixed point theorem (see Zeidler [18], Section 1.6) it follows that the problem

$$u = Su$$

has a unique solution $u \in H_0^1(\Omega)$, i.e. the problem

$$Au = B_\lambda u$$

has a unique solution $u \in H_0^1(\Omega)$. It follows that

$$\langle Au, \varphi \rangle = \langle B_\lambda u, \varphi \rangle, \quad \forall\, \varphi \in H_0^1(\Omega)$$

i.e.

$$\int_\Omega \mathcal{A}(\nabla u)\nabla\varphi\, dx = \lambda \int_\Omega u\varphi\, dx, \quad \forall\, \varphi \in H_0^1(\Omega).$$

Finally we remark that $u \neq 0$ since B_λ vanishes in the origin while A does not vanish in the origin. Thus we have proved that any $\lambda \in (0, \lambda_1)$ is an eigenvalue of problem (1). The proof of Theorem 1 is complete. □

3 Proof of Theorem 2

First, we point out the fact that under the hypotheses of Theorem 2 the conclusion of Theorem 1 does not hold. Indeed, in that case we have $\mathcal{A}(0) = 0$ and thus the non-triviality of the solution obtained by applying the Banach fixed point theorem can not be stated. However, we can prove the existence of a positive eigenvalue of problem (1) under the hypotheses of Theorem 2 using a minimization technique. Such techniques are usually used in finding principal eigenvalues (see e.g. Szulkin-Willem [14]). We remark that the minimization procedure can be also used in order to prove Theorem 3.

We define the functional $I : H_0^1(\Omega) \to \mathbf{R}$,

$$I(u) = \int_\Omega F(\nabla u)\ dx, \quad \forall\ u \in H_0^1(\Omega)$$

where $F : \mathbf{R}^N \to \mathbf{R}$ is the function $F(\xi) = \sum_{i=1}^N (-\cos(\xi_i) + \xi_i^2)$, for all $\xi = (\xi_1, ..., \xi_N) \in \mathbf{R}^N$. It is clear that

$$\frac{\partial F}{\partial x_i}(\xi) = \sin(\xi_i) + 2\xi_i = \mathcal{A}_i(\xi), \quad \forall\ i \in \{1, ..., N\} \text{ and } \forall\ \xi \in \mathbf{R}^N,$$

i.e. $\nabla F(\xi) = \mathcal{A}(\xi)$ for all $\xi \in \mathbf{R}^N$. Thus it is easy to remark that I is of class C^1 on $H_0^1(\Omega)$ with the derivative given by

$$\langle I^{'}(u), v\rangle = \int_\Omega \mathcal{A}(\nabla u)\nabla v\ dx, \quad \forall\ u, v \in H_0^1(\Omega).$$

We consider the minimization problem
(P) minimize $I(u)$ under conditions $u \in H_0^1(\Omega)$ and $\int_\Omega u^2\ dx = 1$.

We point out the fact that problem (P) is well defined. Indeed, for all $u \in H_0^1(\Omega)$ with $\int_\Omega u^2\ dx = 1$ we have

$$I(u) = \sum_{i=1}^N \int_\Omega -\cos\left(\frac{\partial u}{\partial x_i}\right)\ dx + \|u\|^2 \geq -N \cdot |\Omega| + \lambda_1 > -\infty \tag{8}$$

where λ_1 is the first eigenvalue of the Laplace operator.

Proposition 3. *The functional I is weakly lower semicontinuous on $H_0^1(\Omega)$.*

Proof: Let (u_n) be a sequence in $H_0^1(\Omega)$ such that u_n converges weakly to u in $H_0^1(\Omega)$. We show that

$$\liminf_{n\to\infty} I(u_n) \geq I(u).$$

First, we remark that

$$I(u_n) - I(u) = \sum_{i=1}^N \int_\Omega \left(-\cos\left(\frac{\partial u_n}{\partial x_i}\right) - \left(-\cos\left(\frac{\partial u}{\partial x_i}\right)\right)\right)\ dx + \|u_n\|^2 - \|u\|^2.$$

Applying the mean-value theorem it follows that

$$I(u_n) - I(u) = \sum_{i=1}^{N} \int_\Omega \sin(w_n^{(i)}) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) \, dx + \|u_n\|^2 - \|u\|^2 \tag{9}$$

where $w_n^{(i)}(x) = \mu_n^{(i)}(x) \frac{\partial u_n}{\partial x_i}(x) + (1 - \mu_n^{(i)}(x)) \frac{\partial u}{\partial x_i}(x)$ for all n, all $i \in \{1, ..., N\}$ and all $x \in \Omega$ with $\mu_n^{(i)}(x) \in [0,1]$ for all n, all $i \in \{1, ..., N\}$ and all $x \in \Omega$.

On the other hand, we have

$$\begin{aligned}
& \sum_{i=1}^{N} \int_\Omega \sin(w_n^{(i)}) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) \, dx \\
= & \sum_{i=1}^{N} \int_\Omega \sin \left(\mu_n^{(i)}(x) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) + \frac{\partial u}{\partial x_i} \right) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) \, dx \\
= & \sum_{i=1}^{N} \int_\Omega \left[\sin \left(\mu_n^{(i)}(x) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) + \frac{\partial u}{\partial x_i} \right) - \sin \left(\frac{\partial u}{\partial x_i} \right) \right] \cdot \\
& \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) \, dx + \sum_{i=1}^{N} \int_\Omega \sin \left(\frac{\partial u}{\partial x_i} \right) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) \, dx.
\end{aligned}$$

Applying again the mean value theorem we get

$$\begin{aligned}
& \left| \sum_{i=1}^{N} \int_\Omega \sin(w_n^{(i)}) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) \, dx \right| \\
= & \left| \sum_{i=1}^{N} \int_\Omega \mu_n^{(i)}(x) \cos(\xi_n^{(i)}) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right)^2 \, dx + \right. \\
& \left. \sum_{i=1}^{N} \int_\Omega \sin \left(\frac{\partial u}{\partial x_i} \right) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) \, dx \right| \\
\leq & \|u_n - u\|^2 + \left| \sum_{i=1}^{N} \int_\Omega \sin \left(\frac{\partial u}{\partial x_i} \right) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) \, dx \right|.
\end{aligned} \tag{10}$$

Relations (9) and (10) imply

$$\begin{aligned}
& I(u_n) - I(u) \\
\geq & \|u_n\|^2 - \|u_n - u\|^2 - \|u\|^2 - \left| \sum_{i=1}^{N} \int_\Omega \sin \left(\frac{\partial u}{\partial x_i} \right) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) \, dx \right|.
\end{aligned} \tag{11}$$

Since u_n converges weakly to u in the Hilbert space $H_0^1(\Omega)$ using Remark 1.33 on p. 22 in [17] we deduce

$$\lim_{n \to \infty} (\|u_n\|^2 - \|u_n - u\|^2 - \|u\|^2) = 0 \tag{12}$$

On the other hand, we define the functional $T : H_0^1(\Omega) \to \mathbf{R}$ by

$$\langle T, \varphi \rangle = \sum_{i=1}^{N} \int_\Omega \sin\left(\frac{\partial u}{\partial x_i}\right) \frac{\partial \varphi}{\partial x_i}\ dx, \quad \forall\, \varphi \in H_0^1(\Omega).$$

It is clear that T is linear on $H_0^1(\Omega)$. Using Hölder's inequality we deduce

$$\begin{aligned} |\langle T, \varphi \rangle| &\leq \sum_{i=1}^{N} \int_\Omega \left|\sin\left(\frac{\partial u}{\partial x_i}\right)\right| \left|\frac{\partial \varphi}{\partial x_i}\right|\ dx \\ &\leq \sum_{i=1}^{N} \left\|\sin\left(\frac{\partial u}{\partial x_i}\right)\right\|_{L^2} \cdot \left\|\frac{\partial \varphi}{\partial x_i}\right\|_{L^2} \\ &\leq \left(\sum_{i=1}^{N} \left\|\sin\left(\frac{\partial u}{\partial x_i}\right)\right\|_{L^2}^2\right)^{1/2} \cdot \left(\sum_{i=1}^{N} \left\|\frac{\partial \varphi}{\partial x_i}\right\|_{L^2}^2\right)^{1/2} \\ &\leq \left(\sum_{i=1}^{N} \left\|\sin\left(\frac{\partial u}{\partial x_i}\right)\right\|_{L^2}^2\right)^{1/2} \cdot \|\varphi\|, \quad \forall\, \varphi \in H_0^1(\Omega). \end{aligned}$$

Thus, T is linear and continuous on $H_0^1(\Omega)$. Since u_n converges weakly to u in $H_0^1(\Omega)$ we obtain

$$\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega \sin\left(\frac{\partial u}{\partial x_i}\right) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)\ dx = 0. \tag{13}$$

Relations (11), (12) and (13) imply

$$\liminf_{n\to\infty} I(u_n) \geq I(u).$$

We conclude that I is weakly lower semicontinuous on $H_0^1(\Omega)$. The proof of Proposition 3 is complete. □

Proof of Theorem 2. By relation (8) there exists $\Lambda_1 \in \mathbf{R}$ such that

$$\Lambda_1 = \inf_{u \in H_0^1(\Omega),\ \int_\Omega u^2\ dx = 1} I(u).$$

There exists (u_n), a minimizing sequence in $H_0^1(\Omega)$, i.e.

$$I(u_n) \to \Lambda_1$$

and $\int_\Omega u_n^2\ dx = 1$ for all n. We point out the fact that (u_n) is bounded in $H_0^1(\Omega)$. Indeed, the above information shows that

$$\begin{aligned} \|u_n\|^2 &= I(u_n) + \sum_{i=1}^{n} \int_\Omega \cos\left(\frac{\partial u_n}{\partial x_i}\right)\ dx \\ &\leq I(u_n) + N|\Omega| \\ &\leq \Lambda_1 + N|\Omega| + c, \quad \forall\, n \end{aligned}$$

where c is a positive constant.

The fact that (u_n) is bounded in $H_0^1(\Omega)$ implies that there exists $u \in H_0^1(\Omega)$ such that u_n converges weakly to u in $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ we deduce that $\int_\Omega u^2\ dx = 1$. On the other hand, by Proposition 3 we have

$$\Lambda_1 = \liminf_{n\to\infty} I(u_n) \geq I(u).$$

Thus we obtain $I(u) = \Lambda_1$, i.e. u is a solution of problem (P).

Let $v \in H_0^1(\Omega)$ be arbitrary but fixed. Then for all ϵ in a suitable neighborhood of the origin the function

$$g(\epsilon) = I\left(\frac{u+\epsilon v}{\|u+\epsilon v\|_{L^2}}\right) = \int_\Omega F\left(\frac{\nabla u + \epsilon \nabla v}{\|u+\epsilon v\|_{L^2}}\right)\ dx$$

is well defined and possesses a minimum in $\epsilon = 0$. Then it is clear that $g^{'}(0) = 0$. A simple computation shows that

$$g^{'}(\epsilon) =$$

$$\int_\Omega \mathcal{A}\left(\frac{\nabla u + \epsilon\nabla v}{\|u+\epsilon v\|_{L^2}}\right) \cdot \frac{\nabla v \cdot \|u+\epsilon v\|_{L^2}^2 - (\nabla u + \epsilon \nabla v)\cdot\left(\int_\Omega uv\ dx + \epsilon\int_\Omega v^2\ dx\right)}{\|u+\epsilon v\|_{L^2}^3}\ dx$$

Since $\int_\Omega u^2\ dx = 1$ we get

$$g^{'}(0) = \int_\Omega \mathcal{A}(\nabla u)\left(\nabla v - \nabla u \int_\Omega uv\ dx\right)\ dx$$

and thus

$$\int_\Omega \mathcal{A}(\nabla u)\nabla v\ dx = \lambda \int_\Omega uv\ dx$$

where $\lambda = \int_\Omega \mathcal{A}(\nabla u)\nabla u\ dx \geq \|u\|^2 \geq \lambda_1 \int_\Omega u^2\ dx = \lambda_1 > 0$. We conclude that $\lambda \geq \lambda_1$ is an eigenvalue for problem (1). The proof of Theorem 2 is complete. □

4 Final remarks

In this section we point out the fact that our study can be extended to the operators $\mathcal{A} : \mathbf{R}^N \to \mathbf{R}^N$ of the type

$$\mathcal{A}(\xi) = (\mathcal{A}_1(\xi), ..., \mathcal{A}_N(\xi)), \quad \forall\ \xi = (\xi_1, ..., \xi_N) \in \mathbf{R}^N$$

with $\mathcal{A}_1, ..., \mathcal{A}_N : \mathbf{R}^N \to \mathbf{R}$ and $\mathcal{A}_i(\xi) = h_i(\xi_i) + (k+1)\xi_i$ for all $\xi = (\xi_1, ..., \xi_N) \in \mathbf{R}^N$ and all $i \in \{1, ..., N\}$, where k is a positive constant and for all $i \in \{1, ..., N\}$, $h_i : \mathbf{R} \to \mathbf{R}$ are given functions. Assume that for any $i \in \{1, ..., N\}$ the function h_i is of class C^1 on $\mathbf{R}$ and admits a bounded primitive $H_i : \mathbf{R} \to \mathbf{R}$. Moreover, we assume that

$$|h_i(\xi)| \leq k \quad \text{and} \quad |h_i^{'}(\xi)| \leq \min\left\{k, \frac{k+1}{2}\right\}, \quad \forall\ \xi \in \mathbf{R},\ i \in \{1, ..., N\}.$$

Remark. In the case when $h_i(\xi) = \cos(\xi)$ or $h_i(\xi) = \sin(\xi)$ and $k = 1$ we obtain the operators studied in the above sections. We remark that there exists also other functions h_i which satisfy the above conditions. An example can be $h_i(\xi) = \frac{1}{\alpha+1} \cdot \exp(-|\xi|) \cdot \sin(\alpha \cdot \xi)$, for all $\xi \in \mathbf{R}$, where $\alpha > 0$ and $k = 1$.

In the following we will say that an operator $\mathcal{A} : \mathbf{R}^N \to \mathbf{R}^N$ is of the type (T) if it verifies the above conditions. In the case of such an operator the same arguments used in the proof of Theorems 1-3 enable us to state the following result:

Theorem 4. *(i) Assume $\mathcal{A}$ is an operator of type (T). Then there exists at least a positive eigenvalue λ of problem (1), such that $\lambda \geq \lambda_1$ where λ_1 is the first eigenvalue of the Laplace operator.*
(ii) Assume that $\mathcal{A}$ is an operator of type (T) and there exists $i_0 \in \{1, ..., N\}$ such that h_{i_0} does not vanish in the origin. Then any $\lambda \in (0, \lambda_1)$ is an eigenvalue of problem (1), where λ_1 is the first eigenvalue of the Laplace operator.

References

[1] H. Brezis, *Analyse fonctionnelle: théorie et applications*, Masson, Paris, 1992.

[2] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer Verlag, Berlin-Heidelberg, 1983.

[3] E. Lieb, On the lowest eigenvalue of the Laplacian for the intersection of two domains, *Inventiones Mathematicae* **74** (1983), 441-448.

[4] P. Lindqvist, On the equation $\text{div}(|\nabla u|^{p-2}\nabla u)+\lambda|u|^{p-2}u = 0$, *Proc. Amer. Math. Soc.* **109** (1990) 157-164.

[5] M. Mihăilescu and G. Moroşanu, On an eigenvalue problem for an anisotropic elliptic equation involving variable exponents, submitted.

[6] M. Mihăilescu, P. Pucci and V. Rădulescu, Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent, *J. Math. Anal. Appl.* **340** (2008), 687-698.

[7] M. Mihăilescu and V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proc. Amer. Math. Soc.* **135** (2007), 2929-2937.

[8] M. Mihăilescu and V. Rădulescu, Continuous spectrum for a class of nonhomogeneous differential operators, *Manuscripta Mathematica* **125** (2008) 157-167.

[9] M. Mihăilescu and V. Rădulescu, Eigenvalue problems associated to nonhomogeneous differential operators in Orlicz-Sobolev spaces, *Analysis and Applications* **6** (2008), No. 1, 1-16.

[10] M. Mihăilescu and V. Rădulescu, A continuous spectrum for nonhomogeneous differential operators in Orlicz-Sobolev spaces, *Mathematica Scandinavica*, in press.

[11] M. Mihăilescu and V. Rădulescu, Spectrum consisting in an unbounded interval for a class of nonhomogeneous differential operators, *Bulletin of the London Mathematical Society*, in press.

[12] M. Mihăilescu, V. Rădulescu and S. Tersian, Eigenvalue Problems for Anisotropic Discrete Boundary Value Problems, *Journal of Difference Equations and Applications*, in press.

[13] G. Polya and G. Szego, *Isoperimetric inequalities in mathematical physics*, Princeton Univ. Press, Princeton, NJ, 1951.

[14] A. Szulkin and M. Willem, Eigenvalue problems with indefinite weight, *Studia Mathemetica*, 135**(2)** (1999), 191-201.

[15] F. de Thélin, Quelques résultats d'existence et de non-existence pour une E.D.P. elliptique non linèaire, *C.R. Acad. Sci. Paris Sér. I Math* **299** (1984), 911-914.

[16] F. de Thélin, Sur l'espace propre associé à la première valeur propre du pseudo-laplacien, *C.R. Acad. Sci. Paris Sér. I Math* **303** (1986), 355-358.

[17] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

[18] E. Zeidler, *Applied Functional Analysis: Applications to Mathematical Physics*, Springer Verlag, New York, 1995.

Received: 28.08.2008.

Department of Mathematics, University of Craiova,
A.I. Cuza 13, 200585 Craiova, Romania
Department of Mathematics and Its Applications,
Central European University,
1051 Budapest, Hungary
E-mail: `mmihailes@yahoo.com`