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On the isotropic subspace theorems

by
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Abstract

In this paper we explore the Isotropic Subspace Theorems of Catanese and Bauer, establishing relations between isotropic subspaces in the 1-cohomology of a quasi-projective variety M and certain irrational pencils $f : M \to C$, from the point of view of the Tangent Cone Theorem due to Papadima, Suciu and the author.

In the proper case the picture is completely clear, and is described in section 3. For the quasi-projective case and the associated logarithmic pencils, the results are satisfactory only under the additional technical restriction that M is 1-formal, see section 4.

The example of the configuration space of n distinct labeled points on an elliptic curve, see Example 2.11, and that of the algebraic link of an isolated $\mathbb{C}^*$-surface singularity, see subsection (4.10), illustrate well the difficulties in the general case.

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1 Introduction

By a *fibration* we mean a surjective morphism $f : M \to N$ with connected fibers between two compact complex manifolds M and N. When M and N are quasi-projective varieties, a fibration is a surjective morphism with a connected general fiber (this is called an admissible morphism in [1]). Two fibrations $f : M \to C$ and $f' : M \to C'$ over quasi-projective curves C and C' are called *equivalent* if there is an isomorphism $g : C \to C'$ such that $f' = g \circ f$.

The isotropic subspace theorem, due to Catanese in the case M proper, see [4], Theorem 1.10 and [5], Isotropic Subspace Theorem 2.6., and to Bauer and Catanese in the general case, see Bauer [2], Theorem 2.1 and Catanese [5], Theorem 2.11,(Theorem of the logarithmic isotropic subspace), establishes roughly a bijection between the set $\mathcal{E}(M)$ of equivalence classes of fibrations $f : M \to C$ (where M is fixed and C is variable with $\chi(C) < 0$) and the set $\mathcal{I}(M)$ formed by certain isotropic linear subspaces in the cohomology group $H^1(M, \mathbb{C})$, for a precise statement refer to Theorems 3.1 and 4.1 below.

On the other hand, Arapura's work [1] establishes a bijection between the same set $\mathcal{E}(M)$ of equivalence classes of fibrations $f : M \to C$ and the set $\mathcal{I}C_1(M)$ of irreducible components of the first characteristic variety $\mathcal{V}_1(M)$ passing through the unit element 1 of the character group $\mathbb{T}(M)$ of M under the same assumption $\chi(C) < 0$, see Theorem 2.3 and Proposition 2.4 below.

Now taking for any irreducible component $W \in \mathcal{I}C_1(M)$ (which is a smooth subvariety in $\mathbb{T}(M)$ isomorphic to an affine torus) its tangent space $E = T_1W$ at the origin, yields a linear subspace in $H^1(M, \mathbb{C})$, which is e-isotropic, for $e \in \{0, 1\}$, see Definition 2.7.

Our first aim is to understand the relation between the Catanese-Bauer correspondence and the correspondence

$$\Phi : f \in \mathcal{E}(M) \mapsto W_f \in \mathcal{I}C_1(M) \mapsto T_1W_f \subset H^1(M, \mathbb{C})$$

induced by Arapura's results.

The missing link in general is the possibility to associate a component $W \in \mathcal{I}C_1(M)$ to a linear subspace (with some isotropy properties) in $H^1(M, \mathbb{C})$. Due to the Tangent Cone Theorem, one of the main results in [14], this construction works perfectly well in the case of 1-formal manifolds M, and then our results can be considered satisfactory. Indeed, in this case the image of the bijective correspondence Φ is the set of strongly maximal e-isotropic subspaces in $H^1(M, \mathbb{C})$, see Definition 2.8, Lemma 2.9, Corollary 3.4 and Propositions 4.2 and 4.5. The simpliest example of this instance is when M is compact Kähler, which is discussed in the third section.

On the other hand, for manifolds which are not 1-formal, the results are not so clear yet. Our discussion in subsection 4.3 seems to suggest that the statement of Theorem 4.1 needs some modification.

Using these techniques, one may also study fibrations $f : M \to C$ with $\chi(C) = 0$, but only when they have multiple fibers, more precisely when the group $T(f)$ defined in (2.7) is non-trivial. Then such fibrations produce translated components in $\mathcal{V}_1(M)$, which are invisible if we look only at the origin in $\mathbb{T}(M)$, see Propositions 3.6 and 4.7.

A second aim is to show the usefulness of characteristic varieties in the study of multiple fibers of a fibration $f : M \to C$, in particular for questions related to the existence of such multiple fibers, see Corollary 2.15, and to the topological invariance of the number and multiplicities of such fibers, see Theorem 5.1.

2 Rank one local systems, characteristic and resonance varieties

Let M be a connected complex manifold of the form $X \setminus D$, where X is compact Kähler and D is a normal crossing divisor. Clearly M is compact if and only if D is empty. We refer to this situation by saying that M is a quasi-Kähler manifold.

Let $\mathbb{T}(M) = \mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$ be the character variety of M. This is an algebraic group whose identity irreducible component is an algebraic torus $\mathbb{T}(M)_1 \simeq (\mathbb{C}^*)^{b_1(M)}$. Consider the exponential mapping

$$\exp : H^1(M, \mathbb{C}) \to H^1(M, \mathbb{C}^*) = \mathbb{T}(M) \tag{2.1}$$

induced by the usual exponential function $\exp : \mathbb{C} \to \mathbb{C}^*$.

Clearly $\exp(H^1(M, \mathbb{C})) = \mathbb{T}(M)_1$.

The *characteristic varieties* of M are the jumping loci for the cohomology of M, with coefficients in rank 1 local systems:

$$\mathcal{V}^i_k(M) = \{\rho \in \mathbb{T}(M) \mid \dim H^i(M, \mathcal{L}_\rho) \geq k\}. \tag{2.2}$$

When $i = 1$, we use the simpler notation $\mathcal{V}_k(M) = \mathcal{V}^1_k(M)$.

Remark 2.1. *It is clear that the first characteristic varieties $\mathcal{V}_k(M)$ depend only on the fundamental group $G = \pi_1(M)$. Actually, the varieties $\mathcal{V}_k(M)$ depend only on the maximal metabelian quotient $G_{meta} = G/G''$ of G, see Corollary 2.5 in [15] or [9].*

The *resonance varieties* of M are the jumping loci for the cohomology of the complex $H^*(H^*(M, \mathbb{C}), \alpha\wedge)$, namely:

$$\mathbb{R}^i_k(M) = \{\alpha \in H^1(M, \mathbb{C}) \mid \dim H^i(H^*(M, \mathbb{C}), \alpha\wedge) \geq k\}. \tag{2.3}$$

When $i = 1$, we use the simpler notation $\mathbb{R}_k(M) = \mathbb{R}^1_k(M)$.

Example 2.2. *Assume that* $\dim M = 1$ *and* $\chi(M) < 0$*. It is easy to see that that*

$$\mathcal{V}_1(M) = \mathbb{T}(M) \text{ and } \mathbb{R}_1(M) = H^1(M, \mathbb{C}).$$

In the sequel we concentrate ourselves on the strictly positive dimensional irreducible components of the first characteristic variety $\mathcal{V}_1(M)$. They have the following rather explicit description, given by Arapura [1], see also Theorem 3.6 in [11].

Theorem 2.3. *Let M be a quasi-Kähler manifold. Let W be a d-dimensional irreducible component of the first characteristic variety $\mathcal{V}_1(M)$, with $d > 0$. Then there is a regular morphism $f : M \to C$ onto a smooth curve $C = C_W$ with $b_1(C) = d$ such that the generic fiber F of f is connected, and a torsion character $\rho \in \mathbb{T}(M)$ such that the composition*

$$\pi_1(F) \xrightarrow{i_\sharp} \pi_1(M) \xrightarrow{\rho} \mathbb{C}^*,$$

where $i : F \to M$ is the inclusion, is trivial and

$$W = \rho \cdot f^*(\mathbb{T}(C)).$$

In addition, $\dim W = -\chi(C_W) + e$, *with* $e = 1$ *if* C_W *is affine and* $e = 2$ *if* C_W *is proper. If* $\mathcal{L} \in W$, *then* $\dim H^1(M, \mathcal{L}) \geq -\chi(C_W)$ *and equality holds for all such* $\mathcal{L}$ *with finitely many exceptions when* $1 \in W$.

If $1 \in W$, we say that W is a *non-translated component* and then one can take $\rho = 1$. If $1 \notin W$, we say that W is a *translated component.*

One has the following partial converse, see Arapura [1].

Proposition 2.4. *If* $f : M \to C$ *is a fibration with* $\chi(C) < 0$, *then* $W_f = f^*(\mathbb{T}(C))$ *is an irreducible component of the first characteristic variety* $\mathcal{V}_1(M)$. *Moreover, the correspondence* $\mathcal{E}(M) \to \mathcal{IC}_1(M)$ *given by* $[f] \mapsto W_f$ *is a bijection. In particular, the set* $\mathcal{E}(M)$ *is finite.*

Propositions 2.4 implies the following related result.

Corollary 2.5. *Let* $f_0, f_1 : M \to C$ *be two fibrations onto the curve* C *with* $\chi(C) < 0$. *If* f_0 *and* f_1 *are homotopic mappings, then* f_0 *and* f_1 *are equivalent fibrations. In particular, in this case* f_0 *and* f_1 *have the same number and multiplicities of multiple fibers.*

Proof: Since f_0 and f_1 are homotopic mappings, it follows that

$$W = f_0^*(H^1(C, \mathbb{C}^*)) = f_1^*(H^1(C, \mathbb{C}^*)).$$

The irreducible component $W \in \mathcal{IC}_1(M)$ determines uniquely an equivalence class of fibrations in $\mathcal{E}(M)$, so the result follows.

□

The precise relation between the resonance and characteristic varieties is clear only for 1-formal spaces and can be summarized as follows, see [14].

Theorem 2.6. *Assume that the quasi-Kähler manifold* M *is 1-formal. Then the irreducible components* E *of the resonance variety* $\mathbb{R}_1(M)$ *are linear subspaces in* $H^1(M, \mathbb{C})$ *and the exponential mapping* (2.1) *sends these irreducible components* E *onto the irreducible components* W *of* $\mathcal{V}_1(M)$ *with* $1 \in W$. *Moreover, if* E *and* E' *are distinct components of* $\mathbb{R}_1(M)$, *then* $E \cap E' = 0$.

The property of M to be 1-formal depends only on the fundamental group $\pi_1(M)$, see [14] for details. Note that the class of 1-formal varieties is large enough, as it contains all the projective smooth varieties and any hypersurface complement in $\mathbb{P}^n$, see [14]. In fact, if the Deligne mixed Hodge structure on $H^1(M, \mathbb{Q})$ is pure of weight 2, then the smooth quasi-projective variety M is 1-formal, see [18].

We recall the following definition from [14].

Definition 2.7. *A linear subspace $E \subset H^1(M, \mathbb{C})$ is called 0-isotropic (or simply isotropic), if it is isotropic with respect to the cup-product $H^1(M, \mathbb{C}) \times H^1(M, \mathbb{C}) \to H^2(M, \mathbb{C})$.*

A linear subspace $E \subset H^1(M, \mathbb{C})$ is called 1-isotropic if the restriction of the cup-product $H^1(M, \mathbb{C}) \times H^1(M, \mathbb{C}) \to H^2(M, \mathbb{C})$ to $E \times E$ has a 1-dimensional image, and the resulting skew-symmetric form is non-degenerate.

Note that a maximal isotropic subspace can be contained in a (maximal) 1-isotropic subspace. To avoid such situations, we introduce the following notion.

Definition 2.8. *A linear subspace $E \subset H^1(M, \mathbb{C})$ which is e-isotropic for $e = 0$ or $e = 1$ is called strongly maximal if given any e'-isotropic subspace $E' \subset H^1(M, \mathbb{C})$ for $e' = 0$ or $e' = 1$ such that $E \subset E'$, one has $E = E'$*

This notion is natural in this context due to the following.

Lemma 2.9. *Assume that the quasi-Kähler manifold M is 1-formal. Then the irreducible components E of the resonance variety $\mathcal{R}_1(M)$ are exactly the strongly maximal e-isotropic subspaces in $H^1(M, \mathbb{C})$ for $e = 0$ or $e = 1$ such that $\dim E \geq 2 + e$.*

Proof: It follows from Theorem 2.3 that the irreducible components E of the resonance variety coincide to subspaces of the form $f^*(H^1(M, \mathbb{C}))$, which are clearly e-isotropic. $\mathcal{R}_1(M)$

Assume that one has an e'-isotropic subspace $E' \subset H^1(M, \mathbb{C})$ for $e' = 0$ or $e' = 1$ such that $E \subset E'$ and $E \neq E'$. Since $\dim E' > \dim E \geq 2$, it follows that $E' \subset \mathcal{R}_1(M)$. This is a contradiction with the fact that E is an irreducible component of the resonance variety $\mathcal{R}_1(M)$. The second claim follows in a similar way by noting that $E \subset \mathcal{R}_1(M)$.

□

Remark 2.10. *Since $H^1(M, \mathbb{C}) = H^1(M, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, it follows that $H^1(M, \mathbb{C})$ has a natural complex conjugation involution, denoted simply by $a \mapsto \overline{a}$. We say that a subspace $E \subset H^1(M, \mathbb{C})$ is real if $\overline{E} = E$ or, equivalently, if E comes from a real subspace in $E_{\mathbb{R}} \subset H^1(M, \mathbb{R})$, i.e. $E = E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. One interesting consequence of the above results is that for M 1-formal, a strongly maximal e-isotropic subspace as in Lemma 2.9 is necessarily real.*

Example 2.11. *Let C be a smooth compact complex curve of genus $g = 1$. Consider the* configuration space *of n distinct labeled points in C,*

$$M_{1,n} = C^n \setminus \bigcup_{i<j} \Delta_{ij},$$

where Δ_{ij} is the diagonal $\{s \in C^n \mid s_i = s_j\}$. It is straightforward to check that

(i) the inclusion $j : M_{1,n} \to C^n$ *induces an isomorphism* $j^* : H^1(C^n, \mathbb{C}) \to H^1(M_{1,n}, \mathbb{C})$. *In particular* $W_1(H^1(M_{1,n}, \mathbb{C})) = H^1(M_{1,n}, \mathbb{C})$.
(ii) using the above isomorphism, the cup-product map

$$\bigwedge^2 H^1(M_{1,n}, \mathbb{C}) \to H^2(M_{1,n}, \mathbb{C})$$

is equivalent to the composite

$$\mu_{1,n}\colon \ \bigwedge^2 H^1(C^n, \mathbb{C}) \xrightarrow{\cup_{C^n}} H^2(C^n, \mathbb{C}) \twoheadrightarrow H^2(C^n, \mathbb{C})/\operatorname{span}\{[\Delta_{ij}]\}_{i<j} \ , \quad (2.4)$$

where $[\Delta_{ij}] \in H^2(C^n, \mathbb{C})$ *denotes the dual class of the diagonal* Δ_{ij}, *and the second arrow is the canonical projection. See Section 9 in [14] for more details.*

Let $\{a, b\}$ *be the standard basis of* $H^1(C, \mathbb{C}) = \mathbb{C}^2$. *Note that the cohomology algebra* $H^*(C^n, \mathbb{C})$ *is isomorphic to* $\bigwedge^*(a_1, b_1, \ldots, a_n, b_n)$. *Denote by* $(x_1, y_1, \ldots, x_n, y_n)$ *the coordinates of* $z \in H^1(M_{1,n}, \mathbb{C})$. *Using (2.4), it is readily seen that*

$$\mathcal{R}_1(M_{1,n}) = \left\{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \begin{array}{l} \sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0, \\ x_i y_j - x_j y_i = 0, \ \text{for } 1 \le i < j < n \end{array} \right\}.$$

Suppose $n \ge 3$. *Then* $\mathcal{R}_1(M_{1,n})$ *is the affine cone over a rational normal scroll in* $\mathbb{P}^{2n-3}$. *Indeed, one may use* $x_1, ..., x_{n-1}, y_1, ..., y_{n-1}$ *as coordinates on the vector space*

$$V = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n \ \mid \ \sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0\} \simeq \mathbb{C}^{2n-2}.$$

The resonance variety $\mathcal{R}_1(M_{1,n})$ *is contained in* V *and is given here by the equations*

$$x_i y_j - x_j y_i = 0, \ \text{for } 1 \le i < j < n.$$

These equations, regarded in $\mathbb{P}(V) \simeq \mathbb{P}^{2n-3}$, *define the image* $\Sigma_{n-2,1}$ *of the Segre mapping*

$$s : \mathbb{P}^{n-2} \times \mathbb{P}^1 \to \mathbb{P}(V), \quad (a, [u : v]) \mapsto (ua : va).$$

Now $\Sigma_{n-2,1}$ *is just the* $(n-1)$*-fold scroll* $S_{1,\ldots,1}$, *with* 1 *repeated* $(n-1)$*-times, see [17], Exercise 8.27. In particular,* $\mathcal{R}_1(M_{1,n})$ *is an irreducible, non-linear variety. From Theorem 2.6, we conclude that* $M_{1,n}$ *is not* 1*-formal.*

The maximal isotropic subspaces $E \subset H^1(M_{1,n}, \mathbb{C})$ *are (infinitely many and) all of dimension 2 since they have the form*

$$E_x = \{(\alpha x, \beta x) \ \mid \ (\alpha, \beta) \in \mathbb{C}^2\} = < (x, 0), (0, x) >$$

for a fixed $x \in \mathbb{C}^n$, $x \neq 0$ *with* $\sum_{i=1}^n x_i = 0$. *In other words these are the 2-planes corresponding to the lines* $s(a \times \mathbb{P}^1)$ *for* $a \in \mathbb{P}^{n-2}$. *Note that* E_x *is real if and only if* $x \in \mathbb{R}^n$.

The subspaces E_x come from non-isotropic subspaces $E'_x \subset H^1(C^n, \mathbb{C})$ and therefore, Theorem 4.1 says nothing about them. In fact one knows exactly which one of them are associated to an irrational pencil, see [13], Proposition 4.5. More precisely, they correspond to the vectors $x^{ij} \in \mathbb{C}^n$ having 1 on the i-th coordinate, -1 on the j-th coordinate and all the other coordinates zero for some pair $1 \leq i < j \leq n$.

Note also that besides these isotropic subspaces E_x, the resonance variety $\mathbb{R}_1(M_{1,n})$ contains some higher dimensional linear subspaces (coming from $s(\mathbb{P}^{n-2} \times b)$ for $b \in \mathbb{P}^1$, which are neither 0- nor 1- isotropic.

Let $f : M \to C$ be a surjective morphism with a generic connected fiber F. Let $C(f) \subset C$ be a finite, minimal subset such that if we put $C' = C \setminus C(f)$, $M' = f^{-1}(C')$, then the induced mapping $f : M' \to C'$ is a topologically locally trivial fibration. Then, we have an exact sequence

$$H_1(F) \xrightarrow{i'_*} H_1(M') \xrightarrow{f'_*} H_1(C') \to 0 \tag{2.5}$$

as well as a sequence

$$H_1(F) \xrightarrow{i_*} H_1(M) \xrightarrow{f_*} H_1(C) \to 0 \tag{2.6}$$

which is not necessarily exact in the middle, i.e. the group

$$T(f) = \frac{\ker f_*}{\hat{\imath} i_*} \tag{2.7}$$

is non-trivial in general. Here $i : F \to M$ and $i' : F \to M'$ denote the inclusions, and homology is taken with $\mathbb{Z}$-coefficients if not stated otherwise.

This group was studied for $f : M \to C$ proper by Serrano, see [19], but his results are correct only when M and S are compact. On the other hand, the situation when M is compact was studied by A. Beauville in [3].

For $c \in C(f)$ we denote by m_c the multiplicity of the divisor $F_c = f^{-1}(c)$. We have the following result, where the first claim is already in [3], see the remarks after Proposition 1.19, and in Serrano, see [19]. However, this second author wrongly claims that the isomorphism in (i) holds for the case (ii) as well, see [11] for details.

Theorem 2.12.
(i) If the curve C is proper, then

$$T(f) = \left(\oplus_{c \in C(f)} \mathbb{Z}/m_c\mathbb{Z}\right) / (\hat{1}, ..., \hat{1}).$$

(ii) If the curve C is not proper, then

$$T(f) = \oplus_{c \in C(f)} \mathbb{Z}/m_c\mathbb{Z}.$$

The main interest in this group comes from the fact that it parametrizes the translated components in $\mathcal{V}_1(M)$ parallel to the subtorus $f^*\mathbb{T}(C)$ in $\mathbb{T}(M)$, see Beauville [3] in the compact case and Corollary 5.8 in [11] in the general case.

Remark 2.13. *Note that the group $T(f)$ is trivial either if there are no multiple fibers or, in the case C proper, if the multiplicities $m_1, ..., m_s$ are pairwise coprime. Given M, we do not know which sets of pairwise coprime multiplicities $m_1, ..., m_s$ can actually occur.*

Now we can play the same game using fundamental groups instead of H_1-groups. Consider the sequence

$$\pi_1(F) \xrightarrow{i_\sharp} \pi_1(M) \xrightarrow{f_\sharp} \pi_1(C) \to 1 \tag{2.8}$$

and define the group

$$P(f) = \frac{\ker f_\sharp}{\hat{i}i_\sharp}. \tag{2.9}$$

This is possible since $\hat{i}i_\sharp$ is a normal subgroup in $\pi_1(M)$, being the epimorphic image of the normal subgroup $\ker f'_\sharp = \hat{i}i'_\sharp$ in $\pi_1(M')$. Unlike the group $T(f)$, the group $P(f)$ is not necessarily finitely generated. This is clarified by Catanese in [6], see especially the proofs of Lemmas 4.2 and 5.3.

Theorem 2.14. *Let M be a quasi-Kähler manifold and $f : M \to C$ be a fibration with $\chi(C) \leq 0$. Then the following conditions are equivalent.*

(i) the fibration f has no multiple fibers;

(ii) the group $\ker\{f_\sharp : \pi_1(M) \to \pi_1(C)\}$ is finitely generated;

(iii) the group $P(f)$ is finitely generated;

(iv) the group $P(f)$ is trivial, i.e. the sequence (2.8) is exact.

Combining Theorem 2.12 (ii) and Theorem 2.14 we get the following.

Corollary 2.15. *Let M be a quasi-Kähler manifold and $f : M \to C$ be a fibration with $\chi(C) \leq 0$. Then, if C is non compact, the following conditions are equivalent.*

(i) the group $P(f)$ is trivial, i.e. the sequence (2.8) is exact.

(ii) the group $T(f)$ is trivial, i.e. the sequence (2.6) is exact.

3 The isotropic subspace theorem: the proper case

The following fundamental result is due to Catanese, see [4], Theorem 1.10 and [5], Isotropic Subspace Theorem 2.6.

Theorem 3.1. *Let M be a compact Kähler manifold. Then the correspondence associating to a fibration $f : M \to C$ where C is a projective curve of genus $g \geq 2$ the subspace $E = f^*(H^1(C, \mathbb{C})) \subset H^1(M, \mathbb{C})$ induces a bijection between:*

(i) equivalence classes of fibrations $f : M \to C$ where C is a curve of genus $g \geq 2$, and

(ii) $2g$-dimensional subspaces $E \subset H^1(M, \mathbb{C})$ which can be written as $E = U \oplus \overline{U}$, with U a maximal isotropic subspace for the cup-product $H^1(M, \mathbb{C}) \times H^1(M, \mathbb{C}) \to H^2(M, \mathbb{C})$.

Remark 3.2. *(i) For any maximal isotropic subspace $V \subset H^1(M, \mathbb{C})$ of dimension $g \geq 2$ there is a fibration $f \colon M \to C$ onto a smooth curve of genus g and a maximal isotropic subspace $V' \subset H^1(C, \mathbb{C})$ such that $V = f^*V'$, see Theorem 1.10 in [4]. It follows from Theorem 2.6 that the fibration f is uniquely determined by V. On the other hand, given f, the set of maximal isotropic subspaces in $E = f^*(H^1(C, \mathbb{C}))$ is exactly the complex Lagrangian Grassmannian $LG(E)$, a complex manifold of dimension $g(g + 1)/2$ with g the genus of C, see for instance [16], Chapter 3. Moreover, the set of real maximal isotropic subspaces in $E = f^*(H^1(C, \mathbb{C}))$ is exactly the real Lagrangian Grassmannian $LG(E_{\mathbb{R}})$, a real manifold of dimension $g(g + 1)/2$. So in fact any fibration $f \colon M \to C$ as above is associated to some real maximal isotropic subspace $V_{\mathbb{R}} \subset H^1(M, \mathbb{R})$. This fact is useful in understanding the relation of Theorem 3.1 to Theorem 4.1.*

(ii) It is not true that any maximal isotropic subspace $U \subset H^1(M, \mathbb{C})$ with $\dim U = g \geq 2$ satisfies $U \cap \overline{U} = 0$, as claimed in Remark 2.7. (b) in [5]. Indeed, if one takes V' to be a real subspace in $H^1(C, \mathbb{C})$ (coming from a maximal isotropic subspace in $H^1(C, \mathbb{R})$) , then V will be a real maximal subspace in $H^1(M, \mathbb{C})$ and hence $V = \overline{V}$.

On the other hand, we can consider only maximal isotropic subspaces $U \subset F^1H^1(M, \mathbb{C}) = H^0(M, \Omega^1_M)$ and in this case the choice for U is unique and satisfies $U \cap \overline{U} = 0$. Note also that such isotropic subspaces V are maximal but neither strongly maximal nor real in general.

(iii) Any $2g$-dimensional subspace $E \subset H^1(M, \mathbb{C})$ as in Theorem 3.1, (ii) is 1-isotropic and real. Moreover E is strongly maximal as in Lemma 2.9, see the next Corollary.

Since a compact Kähler manifold is 1-formal, see [8], the Tangent Cone Theorem in [14] implies the following.

Corollary 3.3. *Let M be a compact Kähler manifold. Then, for any $k \geq 1$, the tangent cone at the origin $TC_1(\mathcal{V}_k(M))$ to the k-th characteristic variety $\mathcal{V}_k(M)$ is the union of all the $2g$-dimensional strongly maximal 1-isotropic subspaces $E \subset H^1(M, \mathbb{C})$ with $2g - 2 \geq k$.*

Corollary 3.4. *Let M be a compact Kähler manifold. Then the irreducible components of the resonance variety $\mathbb{R}_1(M)$ are linear subspaces and, for $g \geq 2$, the following three finite sets of subspaces of $H^1(M, \mathbb{C})$ coincide.*

(i) $2g$*-dimensional (strongly) maximal 1-isotropic (real) subspaces* $E \subset H^1(M, \mathbb{C})$*;*

(ii) $2g$*-dimensional irreducible components* E *of the resonance variety* $\mathcal{R}_1(M)$*;*

(iii) $2g$*-dimensional subspaces* $E \subset H^1(M, \mathbb{C})$ *which can be written as* $E = U \oplus \overline{U}$*, with* U *a maximal isotropic subspace for the cup-product* $H^1(M, \mathbb{C}) \times H^1(M, \mathbb{C}) \to H^2(M, \mathbb{C})$*.*

It is clear that this set of subspaces E, and hence, in view of Theorem 3.1, the corresponding fibrations $f_E : M \to C$, with $g(C) = g \geq 2$ are completely determined by the cup-product $H^1(M, \mathbb{C}) \times H^1(M, \mathbb{C}) \to H^2(M, \mathbb{C})$.

We can ask the following natural question: can we decide from the subspace E and the cup-product (or some other (co)homological data not involving fundamental groups) whether the associated fibration f_E has multiple fibers?

We have only a very partial answer in the compact case coming from Theorem 2.12 (i).

Corollary 3.5. *Let* M *be a compact Kähler manifold and* $f : M \to C$ *be a fibration. Then the associated group* $T(f)$ *is trivial if and only if either* $f : M \to C$ *has no multiple fibers, or the multiplicities* $m_1, ..., m_s$ *are pairwise coprime.*

The existence of fibrations $f : M \to C$ in the case $g(C) = 1$ is partially settled by the following.

Proposition 3.6. *Let* M *be a compact Kähler manifold. Then there exists a fibration* $f : M \to C$ *with* $g(C) = 1$ *and* $T(f)$ *non-trivial if and only if there is at least one 2-dimensional (necessarily translated) component* W *in the characteristic variety* $\mathcal{V}_1(M)$*. In such a case there is a component* W *as above which in addition is parallel to the subtorus* $W_0 = f^*\mathbb{T}(C)$ *in* $\mathbb{T}(M)$*.*

Proof: This claim follows directly from the description of the translated components in [3], see also [11]. Note that in this case the corresponding non-translated component W_0 is missing, and this is the key difference with the case $g \geq 2$. □

4 The isotropic subspace theorem: the logarithmic case

Suppose now that M is a non-compact quasi-projective manifold. Then the cohomology group $H^1(M, \mathbb{Q})$ carries a weight filtration $0 = W_0 \subset W_1 \subset W_2 = H^1(M, \mathbb{Q})$, such that for any smooth compactification $j : M \subset X$, with X projective, the morphism $j^* : H^1(X, \mathbb{Q}) \to H^1(M, \mathbb{Q})$ is injective (hence one can regard $H^1(X, \mathbb{Q})$ as a subspace of $H^1(M, \mathbb{Q})$) and $W_1 = j^*H^1(X, \mathbb{Q})$.

The correspondence between irrational pencils and isotropic subspaces is much more subtle in this case, see Bauer [2], Theorem 2.1 and Catanese [5], Theorem 2.11, Theorem of the logarithmic isotropic subspace, which we reproduce below.

Theorem 4.1. *Let M be a quasi-projective manifold, $M = X \setminus D$, with X smooth and projective and D a normal crossing divisor. Then every real maximal isotropic subspace V of $H^1(M, \mathbb{R})$ either of dimension ≥ 3 or of dimension 2 but not coming from a non-isotropic subspace V' of $H^1(X, \mathbb{R})$ (this case is not covered by the theorem) determines a unique logarithmic irrational pencil $f : M \to C$ onto a curve C with logarithmic genus $g^* \geq 2$.*

The curve C is projective if and only if $V \subset H^1(X, \mathbb{R})$, and is isotropic there, otherwise $V = f^*(H^1(C, \mathbb{R}))$, and one says that the pencil is strictly logarithmic.

Here the logarithmic genus g^* of the curve C is defined by the equality $b_1(C) = g + g^*$, where g is the genus of (a compactification of) C. Note that $g^* \geq 2$ implies (but it is not equivalent to) $\chi(C) < 0$. A key point here is that there is no bijection result similar to the compact case covered by Theorem 3.1, i.e. there might exist some logarithmic irrational pencils not coming from a real maximal isotropic subspace as in Theorem 4.1. There are some bijection claims in Theorem 2.1 and Theorem 2.4 in [2], but they do not hold as stated, see for instance Example 4.6.

The difficulties of the case is not covered by the theorem are highlighted by our Example 2.11, in which there are infinitely many real maximal isotropic subspace V of $H^1(M, \mathbb{R})$ of dimension 2 coming from non-isotropic subspaces V' of $H^1(X, \mathbb{R})$, and some of them (finitely many) are associated to logarithmic irrational pencil $f_{ij} : M \to C_1$ onto a curve C_1 with logarithmic genus $g^* = 1$ and $\chi(C_1) = -1$.

We intend to look at this non-proper situation closer, by imposing at some points the condition that M is 1-formal, in order to better grasp the correspondence between irrational pencils and isotropic subspaces. With this additional hypothesis, we recover the bijection between equivalence classes of pencils and certain e-isotropic subspaces, see Propositions 4.2 and 4.5.

In this situation there are the following two cases to discuss.

4.1 The case C proper, of genus $g \geq 2$

Assume $f : M \to C$ is a fibration. Then M admits a compactification X such that f extends to a fibration $\tilde{f} : X \to C$. According to Theorem 3.1, $\tilde{f}$ is uniquely determined by a $2g$-dimensional strongly maximal 1-isotropic subspace $\tilde{E} = \tilde{f}^*(H^1(C, \mathbb{C}))$ in $H^1(X, \mathbb{C})$ or, equivalently, by the g-dimensional maximal isotropic subspace $\tilde{U} = \tilde{f}^*(H^0(C, \Omega^1_C))$.

Using the injection $j^* : H^1(X, \mathbb{C}) \to H^1(M, \mathbb{C})$, it follows that $E = j^*(\tilde{E})$ is an e-isotropic subspace, where $e = 1$ if the morphism

$$f^* : H^2(C, \mathbb{C}) \to H^2(M, \mathbb{C}) \tag{4.1}$$

is non-trivial, and $e = 0$ otherwise. Both cases are possible: indeed, if M is 1-formal then $e = 1$, see Prop. 5.10 (3) in [14], while an example with $e = 0$ is given in Example 5.11 in [14]. We have the following.

Proposition 4.2. *Assume M is 1-formal and $f : M \to C$ is a fibration on the proper curve C of genus $g \geq 2$. Then:*
(i) $E = f^*(H^1(C, \mathbb{C}))$ is a $2g$-dimensional strongly maximal 1-isotropic real subspace of $H^1(M, \mathbb{C})$. Moreover, E is contained in $W_1H^1(M, \mathbb{C}) = H^1(X, \mathbb{C})$ and is 1-isotropic there. Conversely, for any $2g$-dimensional strongly maximal 1-isotropic real subspace of $H^1(M, \mathbb{C})$ contained in $W_1H^1(M, \mathbb{C}) = H^1(X, \mathbb{C})$, there is a fibration $f : M \to C$ on the proper curve C of genus $g \geq 2$ such that $E = f^*(H^1(C, \mathbb{C}))$.
(ii) $U = f^*(H^1(C, \Omega^1_C))$ is a g-dimensional maximal isotropic subspace of $H^1(M, \mathbb{C})$ such that $E = U \oplus \overline{U}$. Moreover, U is contained in $W_1H^1(M, \mathbb{C}) = H^1(X, \mathbb{C})$ and is isotropic there.
(iii) the equivalence class of the fibration f is determined by the subspace E. Conversely, any $2g$-dimensional strongly maximal 1-isotropic subspace of $H^1(M, \mathbb{C})$ determines such an equivalence class of fibrations.

Proof: The first claim in (i) is clear. For the converse claim, use Theorem 3.1 and get a map $\tilde{f} : X \to C$ and then set $f = \tilde{F}|M$.

The claim (ii) is obvious.

In (iii), the subspace E has dimension at least 4, hence produces an irreducible component of the resonance variety $\mathbb{R}_1(M)$ in view of Lemma 2.9. Using 1-formality, we know that $W = \exp(E)$ is an irreducible component of the characteristic variety. By Arapura's results, this comes from a fibration onto a projective curve of genus g, whose equivalence class is determined by W (this follows for instance from Lemmas 6.2, 6.3 and the proof of Part 3 in Theorem 6.4 in [14].) □

Corollary 4.3. *Let M be a quasi-projective manifold which is 1-formal. Then any strongly maximal 1-isotropic subspace of $H^1(M, \mathbb{C})$ of dimension at least 4 is contained in $W_1H^1(M, \mathbb{C})$.*

Remark 4.4. *When M is not 1-formal, Example 5.11 in [14] shows that $E = f^*(H^1(C, \mathbb{C}))$ can be isotropic in $H^1(M, \mathbb{C})$ but 1-isotropic in $H^1(X, \mathbb{C})$. See also subsection 4.3 below for other examples. In such cases, one should decide which of the isotropic subspaces E and U should be associated to f, which explains the rather complicated statement in Theorem 4.1. It is not clear whether E is strongly maximal in this situation.*

One advantage of using the subspaces U is that they stay isotropic both in $H^1(M, \mathbb{C})$ and in $H^1(X, \mathbb{C})$.

4.2 The case C non-proper, with $\chi(C) = 1 - g - g^* < 0$

In this case $E = f^*(H^1(C, \mathbb{C}))$ is an isotropic subspace in $H^1(M, \mathbb{C})$. Moreover $E \subset W_1H^1(M, \mathbb{C})$ exactly when C is obtained from a projective curve $\tilde{C}$ by deleting one point p. In this special case, the inclusion $C \subset \tilde{C}$ induces an isomorphism of Hodge structures

$$H^1(C, \mathbb{C}) \simeq H^1(\tilde{C}, \mathbb{C})$$

which implies that E is pure of weight 1. Moreover, there is a compactification X of M and an extension $\tilde{f} : X \to \tilde{C}$ of f. It follows that E, regarded as a subspace of $W_1H^1(M, \mathbb{C}) = H^1(X, \mathbb{C})$ is 1-isotropic as in the previous subsection.

This proves the following.

Proposition 4.5. *Assume M is 1-formal and $f : M \to C$ is a fibration on the non-proper curve C with $\chi(C) = 1 - g - g^* < 0$. Then:*
(i) $E = f^*(H^1(C, \mathbb{C}))$ is a $(g + g^*)$-dimensional strongly maximal isotropic subspace of $H^1(M, \mathbb{C})$. The equivalence class of the fibration f is determined by the subspace E. Conversely, any strongly maximal isotropic subspace of $H^1(M, \mathbb{C})$ of dimension at least 2 determines such an equivalence class of fibrations.
(ii) $E \subset W_1H^1(M, \mathbb{C})$ exactly when C is obtained from a projective curve $\tilde{C}$ by deleting one point p. In this case one has the following

(a) E, regarded as a subspace of $W_1H^1(M, \mathbb{C}) = H^1(X, \mathbb{C})$ is 1-isotropic;

(b) $U = f^*(H^1(C, \Omega^1_{\tilde{C}}))$ is a g-dimensional maximal isotropic subspace of $H^1(M, \mathbb{C})$ such that $E = U \oplus \overline{U}$, U is contained in $W_1H^1(M, \mathbb{C}) = H^1(X, \mathbb{C})$ and is isotropic there.

Note that the last case (ii) when in addition $g = 1$ is excluded in Theorem 4.1, but is covered by our Proposition 4.5.

Example 4.6. *Let C_1 and C_2 be two smooth projective curves of genus $g_1 \geq 2$ and respectively $g_2 \geq 2$. Pick points $p_1 \in C_1$ and $p_2 \in C_2$ and set*

$$M = (C_1 \setminus \{p_1\}) \times (C_2 \setminus \{p_2\}).$$

The surface M is 1-formal by Proposition 7.2 in [14] and the cohomology algebra $H^*(M, \mathbb{K})$ is easy to determine for any field $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Let $p_i : M \to C_i \setminus \{p_i\}$ be the two projections, for $i = 1, 2$. The only (strongly) maximal isotropic subspaces in the cohomology group $H^1(M, \mathbb{K})$ are $E_i = p_i^*(H^1(C_i \setminus \{p_i\}, \mathbb{K}))$ for $i = 1, 2$ and both of them are subspaces of $W_1H^1(M, \mathbb{K}) = H^1(X, \mathbb{K})$ and are 1-isotropic there. It follows that the only irrational pencils $f : M \to C$ are those equivalent to p_1 and p_2.

This example shows in particular that Theorem 2.1 and Theorem 2.4 in [2] do not hold as stated (indeed, the sets of isotropic subspaces considered in the final statement in Theorem 2.1 and in Theorem 2.4 are empty for our example (in the latter case there are no strongly isotropic subspaces), but we have the two irrational pencils p_1 and p_2).

The existence of fibrations $f : M \to \mathbb{C}^*$ is partially settled by the following analog of Proposition 3.6

Proposition 4.7. *Let M be a quasi-projective manifold. Then there exists a fibration $f : M \to \mathbb{C}^*$ with $T(f)$ non-trivial if and only if there is at least one*

1-dimensional (necessarily translated) component W in the characteristic variety $\mathcal{V}_1(M)$. In such a case there is a component W as above which in addition is parallel to the subtorus $W_0 = f^*\mathbb{T}(\mathbb{C}^*)$ in $\mathbb{T}(M)$.

Proof: This claim follows directly from the description of the translated components in [11]. Note that in this case, exacly as in the compact case, the corresponding non-translated component W_0. □

4.3 A non-formal example: isolated surface singularities with $\mathbb{C}^*$-action

Let $(Y, 0)$ be an isolated surface singularity with a good $\mathbb{C}^*$-action. Represent the singularity $(Y, 0)$ by an affine surface Y with a good $\mathbb{C}^*$-action. If $M := X \setminus \{0\}$, then M is homotopy equivalent to K, the link of the singularity $(Y, 0)$ and the quotient $C := M/\mathbb{C}^*$ is a smooth projective curve of genus g such that $b_1(M) = 2g$, see [10], p. 52 and p. 66. Assume in the sequel that $g \geq 1$.

By one of Sullivan's results in [20], we know that the cup-product is trivial on $H^1(M, \mathbb{K})$, for $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ as above. The projection $p : M \to C$ is surjective and the fibers are all connected, since isomorphic to $\mathbb{C}^*$. It follows that $p_* : H_1(M, \mathbb{Z}) \to H_1(C, \mathbb{Z}) = \mathbb{Z}^{2g}$ is an epimorphism inducing an isomorphism

$$p^* : H^1(C, \mathbb{Q}) \to H^1(M, \mathbb{Q}).$$

It follows that the mixed Hodge structure on $H^1(M, \mathbb{Q})$ is pure of weight 1. This shows that for any compactification $j : M \to X$, with X smooth projective, the induced morphism

$$j^* : H^1(X, \mathbb{Q}) \to H^1(M, \mathbb{Q})$$

is an isomorphism.

If $G = \pi_1(M)$ were 1-formal, then Proposition 5.10 in [14] would imply that

$$p^* : H^2(C, \mathbb{C}) \to H^2(M, \mathbb{C})$$

is injective, in contradiction to the triviality of the cup-product on $H^1(M, \mathbb{C})$. Therefore M is not 1-formal.

Now let's try to apply Theorem 4.1 to the quasi-projective manifold M and the real maximal isotropic subspace $V = H^1(M, \mathbb{R})$ in the case $g \geq 2$, i.e. $\dim V \geq 4$. Note that V is coming from a non-isotropic subspace of $H^1(X, \mathbb{R}) = H^1(M, \mathbb{R})$ (due to the Hodge-Riemann bilinear relations for X it follows that the cup-product is not trivial on $H^1(X, \mathbb{C})$, and hence non-trivial on $H^1(X, \mathbb{R})$), but it is not excluded since $\dim V \geq 4$.

Let $f : M \to C'$ be the fibration associated to V by Theorem 4.1. Since V is not isotropic in $H^1(X, \mathbb{R})$, it follows that C' is a non-compact curve with $b_1(C') = 2g \geq 4$. Since the only morphisms $\mathbb{C}^* \to C'$ in this setting are the constant ones, it follows that f is constant on the fibers of p.

This remark produces an induced mapping $\phi : C \to C'$ such that $\phi \circ p = f$. Since C is compact and C' is not compact, the only posibility is that ϕ is constant, but this contradicts the surjectivity of f. In conclusion, the last claim in Theorem 4.1 does not hold as stated in [5].

It may seem that one can avoid this problem by discarding all isotropic subspaces $V \subset H^1(M, \mathbb{R})$ coming from a non-isotropic subspace of $H^1(X, \mathbb{R})$, not just those of dimension 2. However, Proposition 4.5 (ii) and Example 4.6 show that these subspaces cannot be discarded without losing certain associated irrational fibrations.

5 On the multiple fibers of a strictly logarithmic irrational pencil

Let $f : M \to C$ be a strictly logarithmic irrational pencil, i.e. C is a non-compact curve with $\chi(C) < 0$. Let M' be another quasi-projective manifold and assume $h : M \to M'$ is a homeomorphism. Then h induces an isomomorphism of algebraic groups $h^* : \mathbb{T}(M') \to \mathbb{T}(M)$ given by $\mathcal{L} \mapsto h^{-1}\mathcal{L}$, the sheaf theoretic inverse image, such that

$$(h^*)^{-1}(\mathcal{V}_1(M)) = \mathcal{V}_1(M').$$

Indeed, one obviously has

$$\dim H^1(M, h^{-1}\mathcal{L}') = \dim H^1(M', \mathcal{L}'). \tag{5.1}$$

According to Proposition 2.4, the mapping f produces a non-translated irreducible component W_f of the characteristic variety $\mathcal{V}_1(M)$. The set $(h^*)^{-1}(W_f)$ is then a non-translated irreducible component $W_{f'}$ of the characteristic variety $\mathcal{V}_1(M')$, corresponding to a pencil $f' : M' \to C'$. Since the generic dimension of $H^1(M, \mathcal{L})$ along the non-translated component W_f is exactly $-\chi(C)$, see Theorem 2.3, it follows that $\chi(C') = \chi(C)$. Moreover, the equality $\dim W_f = \dim W_{f'}$ (topological invariance of dimension), combined with the formula given in Theorem 2.3 for these dimension, shows that C' is also a non-compact curve.

We have the following result.

Theorem 5.1. *Let $f : M \to C$ be a strictly logarithmic irrational pencil. Let M' be another quasi-projective manifold and assume $h : M \to M'$ is a homeomorphism. Then there is an associated strictly logarithmic irrational pencil $f' : M' \to C'$ onto a curve C' with $\chi(C') = \chi(C)$. Moreover, the two strictly logarithmic irrational pencils $f : M \to C$ and $f' : M' \to C'$ have the same number and multiplicies of multiple fibers.*

Proof: Suppose that f has s multiple fibers, with respective multiplicities $m_1, ..., m_s$. It follows from Theorem 2.12 that

$$T(f) = \oplus_{i=1,s}\mathbb{Z}/m_s\mathbb{Z}.$$

Let $s', m'_1, ...m'_{s'}$ be the corresponding data for f'. It follows from [11] that for each element $k = (\hat{k}_1, ..., \hat{k}_s) \in T(f)$, $k \neq 0$, there is exactly one translated component W_k in $\mathcal{V}_1(M)$ parallel to the component $W_f = W_0$. Moreover the generic dimension of $H^1(M, \mathcal{L})$ along the translated component W_k is exactly $-\chi(C) + n(k)$, where $n(k)$ is the number of non-trivial components in k, see [11]. It follows that the number of such translated components along which the generic dimension of $H^1(M, \mathcal{L})$ is $-\chi(C) + r$ is exactly $\sigma_r(m_1 - 1, ..., m_s - 1)$, where σ_r is the r-th elementary symmetric function, i.e. $\sigma_1(m_1 - 1, ..., m_s - 1) = \sum(m_i - 1)$, $\sigma_2(m_1 - 1, ..., m_s - 1) = \sum_{i<j}(m_i - 1)(m_j - 1)$ and so on. The maximal value of r is s, the total number of multiple fibers of f.

It is clear that the number of the translated components parallel to W_f along which the generic dimension of $H^1(M, \mathcal{L})$ has a given value, say $-\chi(C) + r$, is invariant by the homeomorphism h, in view of the formula (5.1). Hence we get $s = s'$ and

$$\sigma_r(m_1 - 1, ..., m_s - 1) = \sigma_r(m'_1 - 1, ..., m'_s - 1)$$

for $r = 1, ..., s$, which completes the proof of the claim. □

Remark 5.2. *The analogous result to Theorem 5.1, for M proper and the genus g of C at least 1, is also true, and it follows immediately from Theorem 4.14 in Catanese's recent paper [7]. The condition imposed there that $(g, m_1, ..., m_s)$ is a hyperbolic type is no restriction, since it is equivalent to*

$$\chi^{orb}(C) = \chi(C) - \sum_{i=1,s}(1 - \frac{1}{m_i}) \leq 0,$$

see Delzant [9]. In fact a similar approach, using the results in [6], may provide an alternative proof for Theorem 5.1 and also of the corresponding result in the case M non proper but C proper.

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