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On holomorphic curvature of η - Einstein complex Finsler spaces

by
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Abstract

This paper comprises a class of complex Finsler metrics, namely $\eta-$ Einstein, which satisfies some special conditions on the curvature. By means of Chern complex linear connection on the pull-back tangent bundle, a special approach is devoted to obtain the equivalence conditions that a complex Finsler space should be $\eta-$ Einstein, (§3). A Schur type theorem for a $\eta-$ Einstein complex Finsler space, weakly Kähler, and other characterizations of the holomorphic curvature of this space are given in §4.

Key Words: Chern (*c.l.c*), η - Einstein space, constant curvature.
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1 Introduction

The study of the complex Finsler metrics of constant holomorphic curvature is an interesting problem in complex Finsler geometry. M. Abate and G. Patrizio [1] gave a characterization of the constant holomorphic curvature through complex geodesics, with the main result that any complex Finsler metric of holomorphic curvature $K_F = -4$ and which satisfies some regularity conditions is the Kobayashi metric. The first proof is due to J. Faran [8], who used the method of equivalence problem in his work. Another result, due to M. Abate and G. Patrizio [2], asserts that if the complex Finsler spaces satisfy the notion of Kähler, a symmetry condition on the curvature and with positive constant holomorphic curvature, then they are purely Hermitian.

In a previous paper, [4], we started the study of the curvature of complex Finsler spaces, with respect to the Chern complex linear connection, briefly Chern (*c.l.c*), on the pull-back tangent bundle. Our goal was to determine the conditions in which a complex Finsler metric has constant holomorphic curvature. We solved this problem for a special class of complex Finsler spaces, called generalized

Einstein, briefly $(g.E.)$. In the present paper we shall introduce a new class of complex Finsler metrics, called η - Einstein, briefly $(\eta - E)$, which generalize the class of $(g.E.)$ complex Finsler metrics. We shall obtain necessary and sufficient conditions that a complex Finsler metric should be $(\eta - E)$, (Theorem 3.1). These results permit us to find the conditions in which a $(\eta - E)$ complex Finsler space is $(g.E.)$, (Corollary 3.2). With the additional condition of Kähler, we prove that the $(\eta - E)$ complex Finsler spaces of nonzero holomorphic curvature are purely Hermitian (Corollary 3.3). We prove a Schur type theorem for $(\eta - E)$ complex Finsler spaces (Theorem 4.1). Another result is that the $(\eta - E)$ complex Finsler spaces of nonzero constant holomorphic curvature are weakly Kähler (Proposition 4.1). Moreover, a $(\eta - E)$ complex Finsler metric with holomorphic curvature $K_F = -4$ is the Kobayashi metric, (Proposition 4.4).

2 Notation and definitions

In the present section we recall only the basic notions which are needed; for more information see [1], [12], [5]. For the beginning, we shall make an introduction to the geometry of the pull-back tangent bundle with the Chern $(c.l.c)$, [5]. Let M be a complex manifold, $\dim_C M = n$, and $T'M$ the holomorphic tangent bundle in which as a complex manifold the local coordinates will be denoted by (z^k, η^k). The complexified tangent bundle of $T'M$ is decomposed in $T_C(T'M) = T'(T'M) \oplus T''(T'M)$.

Considering the restriction of the projection to $\widetilde{T'M} = T'M \setminus \{0\}$, for pulling the holomorphic tangent bundle $T'M$ back, we obtain a holomorphic tangent bundle $\pi' : \pi^*(T'M) \longrightarrow \widetilde{T'M}$, called *the pull-back tangent bundle* over the slit $\widetilde{T'M}$. We denote by $\left\{\frac{\partial}{\partial z^k}^*, \frac{\partial}{\partial \bar{z}^k}^*\right\}$, and by $\left\{dz^{*k}, d\bar{z}^{*k}\right\}$, the local frame and its dual.

Let $V(T'M) = \ker \pi_* \subset T'(T'M)$ be the vertical bundle, spanned locally by $\{\frac{\partial}{\partial \eta^k}\}$. A complex nonlinear connection, briefly $(c.n.c.)$, determines a supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. The adapted frames of the $(c.n.c.)$ is $\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}$, where $N_k^j(z, \eta)$ are the coefficients of the $(c.n.c.)$. Further on we shall use the abbreviations $\delta_i = \frac{\delta}{\delta z^i}$, $\dot{\partial}_i = \frac{\partial}{\partial \eta^i}$, $\delta_{\bar{i}} = \frac{\delta}{\delta \bar{z}^i}$, $\dot{\partial}_{\bar{i}} = \frac{\partial}{\partial \bar{\eta}^i}$, and theirs conjugates ([1], [3], [12]). On $T'M$ let $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ be the fundamental metric tensor of a complex Finsler space $(M, F^2 = L)$. The isomorphism between $\pi^*(T'M)$ and $T'M$ induces an isomorphism of $\pi^*(T_C M)$ and $T_C M$. Thus, $g_{i\bar{j}}$ defines an Hermitian metric structure $\mathcal{G}(z, \eta) := g_{j\bar{k}} dz^{*j} \otimes d\bar{z}^{*k}$on $\pi^*(T_C M)$, with respect to the natural complex structure. On the other hand, $H(T'M)$ and $\pi^*(T'M)$ are isomorphic. Therefore the structures on $\pi^*(T_C M)$ can be pulled-back to $H(T'M) \oplus \overline{H(T'M)}$. By this isomorphism the natural cobasis dz^{*j} is identified with dz^j.

In view of this construction the pull-back tangent bundle $\pi^*(T'M)$ admits a unique complex linear connection ∇, called the Chern $(c.l.c.)$, which is metric with respect to $\mathcal{G}$ and of $(1,0)-$ type. Its connection form is $\omega^i_j(z,\eta) = L^i_{jk}(z,\eta)dz^k + C^i_{jk}(z,\eta)\delta\eta^k$, where $L^i_{jk} = g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta z^k}$, $C^i_{jk} = g^{\overline{m}i}\frac{\partial g_{j\overline{m}}}{\partial \eta^k}$, [5]. The covariant derivative of $X := X^j(z,\eta)\frac{\partial^*}{\partial z^j}$, associated to the Chern (*c.l.c*) is

$$\nabla X = \left(X^i_{|k}dz^k + X^i|_k\delta\eta^k + X^i_{|\overline{k}}d\overline{z}^k + X^i|_{\overline{k}}\delta\overline{\eta}^k \right)\frac{\partial^*}{\partial z^i},$$

with $X^i_{|k} := \delta_k X^i + X^l L^i_{lk}$; $X^i|_k := \dot{\partial}_k X^i + X^l C^i_{lk}$; $X^i_{|\overline{k}} := \delta_{\overline{k}} X^i$; $X^i|_{\overline{k}} := \dot{\partial}_{\overline{k}} X^i$. The Chern (*c.l.c.*) on $\pi^*(T'M)$ determines the Chern-Finsler (*c.n.c.*) on $T'M$, with the coefficients $\overset{CF}{N^i_k} = g^{\overline{m}i}\frac{\partial g_{j\overline{m}}}{\partial z^k}\eta^j$, and its local coefficients of torsion and curvature are

$$\begin{aligned} T^i_{jk} \ &: = L^i_{jk} - L^i_{kj}\ ; \\ R^i_{j\overline{h}k} \ &: = -\delta_{\overline{h}}L^i_{jk} - \delta_{\overline{h}}(\overset{CF}{N^l_k})C^i_{jl}\ ;\ \Xi^i_{j\overline{h}k} := -\delta_{\overline{h}}C^i_{jk} = \Xi^i_{k\overline{h}j}; \\ P^i_{j\overline{h}k} \ &: = -\dot{\partial}_{\overline{h}}L^i_{jk} - \dot{\partial}_{\overline{h}}(\overset{CF}{N^l_k})C^i_{jl}\ ;\ S^i_{j\overline{h}k} := -\dot{\partial}_{\overline{h}}C^i_{jk} = S^i_{k\overline{h}j}. \end{aligned} \tag{2.1}$$

The Riemann type tensor $\mathbf{R}(W,\overline{Z},X,\overline{Y}) := \mathcal{G}(R(X,\overline{Y})W,\overline{Z})$ has the properties:

$$\begin{aligned} \mathbf{R}(W,\overline{Z},X,\overline{Y}) &= W^i\overline{Z}^j X^k\overline{Y}^h R_{i\overline{j}k\overline{h}};\ \ R_{\overline{j}i\overline{h}k} := R^l_{i\overline{h}k}g_{l\overline{j}}; \\ R_{i\overline{j}k\overline{h}} &= -R_{i\overline{j}\overline{h}k} = \overline{R_{\overline{j}i\overline{h}\overline{k}}} = R_{\overline{j}i\overline{h}k}; \\ \text{If } R^i_{j\overline{h}k} &= R^i_{k\overline{h}j} \text{ then } R_{i\overline{j}k\overline{h}} = R_{k\overline{j}i\overline{h}} = R_{k\overline{h}i\overline{j}}. \end{aligned} \tag{2.2}$$

By setting $R_{\overline{j}k} := R_{i\overline{j}k\overline{h}}\eta^i\overline{\eta}^h = -g_{l\overline{j}}\delta_{\overline{h}}(\overset{CF}{N^l_k})\overline{\eta}^h$, the *Ricci scalar* and the *Ricci tensor* associated to the Chern (*c.l.c.*) on $\pi^*(T'M)$ are defined by $Ric := g^{\overline{j}k}R_{\overline{j}k} = R^k_{i\overline{h}k}\eta^i\overline{\eta}^h$; $Ric_{i\overline{j}} := \frac{\partial^2 Ric}{\partial\eta^i\partial\overline{\eta}^j}$. An easy computation shows that the functions $R_{\overline{j}k}$ are $1-$ homogeneous with respect to η, i.e. $\frac{\partial R_{\overline{j}k}}{\partial\eta^i}\eta^i = R_{\overline{j}k}$.

According to [1] the complex Finsler space (M,F) is *strongly Kähler* iff $T^i_{jk} = 0$, *Kähler* iff $T^i_{jk}\eta^j = 0$ and *weakly Kähler* iff $g_{i\overline{l}}T^i_{jk}\eta^j\overline{\eta}^l = 0$. Note that for a complex Finsler metric which comes from a Hermitian metric on M, so-called *purely Hermitian metric* in [12], i.e. $g_{i\overline{j}} = g_{i\overline{j}}(z)$, the three nuances of Kähler spaces coincide, [14]. In [1], the holomorphic curvature of F in direction η, with respect to the Chern (*c.l.c.*), is

$$\mathcal{K}_F(z,\eta) := \frac{2R(\eta,\overline{\eta},\eta,\overline{\eta})}{\mathcal{G}^2(\eta,\overline{\eta})} = \frac{2\overline{\eta}^j\eta^k R_{\overline{j}k}}{L^2(z,\eta)}, \tag{2.3}$$

where η is viewed as local section of $\pi^*(T'M)$, *i.e.* $\eta := \eta^i\frac{\partial}{\partial z^i}^*$. Further on, we shall simply call it holomorphic curvature. It depends both on the position $z \in M$ and the direction η. Moreover, it is $0-$ homogeneous with respect to η.

In this context, we introduced in [4] the following concept:

Definition 2.1. *The complex Finsler space (M, F) is called generalized Einstein if $R_{\bar{j}k}$ is proportional to $t_{k\bar{j}}$, i.e. if there exists a real valuated function $K(z, \eta)$, such that*

$$R_{\bar{j}k} = K(z, \eta)t_{k\bar{j}}, \tag{2.4}$$

where $t_{k\bar{j}} := L(z, \eta)g_{k\bar{j}} + \eta_k \bar{\eta}_j$, $\eta_k := \frac{\partial L}{\partial \eta^k}$, $\bar{\eta}_j := \frac{\partial L}{\partial \bar{\eta}^j}$.

The main properties of the $(g.E.)$ complex Finsler spaces are collected in:

Theorem 2.1. *Let (M, F) be a $(g.E.)$ complex Finsler space. Then*

i) $K(z, \eta) = \frac{1}{4}\mathcal{K}_F(z, \eta)$ and it depends on z alone.

ii) If (M, F) is connected and weakly Kähler, of complex dimension $n \geq 2$, then it is a space with constant holomorphic curvature.

iii) If the space is of nonzero constant holomorphic curvature, then F is weakly Kähler.

iv) If the space is Kähler of nonzero constant holomorphic curvature, then F is purely Hermitian. Conversely, a purely Hermitian complex Finsler space, which is Kähler of constant holomorphic curvature, is $(g.E.)$.

Note that for the particular case of the complex Finsler spaces which are Kähler of nonzero constant holomorphic curvature, the notions of $(g.E.)$ and purely Hermitian spaces coincide.

3 $\eta -$ Einstein complex Finsler metrics

Definition 3.1. *The complex Finsler space (M, F) is called $\eta-$ Einstein, briefly $(\eta - E)$, if there exists two smooth functions $K_i(z, \eta) : T'M \to \mathbf{R}$, $i = 1, 2$, such that*

$$R_{\bar{j}k} = K_1(z, \eta)Lg_{k\bar{j}} + K_2(z, \eta)\eta_k \bar{\eta}_j. \tag{3.1}$$

Under the changes rule of complex coordinates on $T'M$, the functions $K_i(z, \eta)$ are well defined on $T'M$. The main examples of $(\eta - E)$ - spaces are $(g.E)$ - spaces. From formula (3.1) we deduce:

Proposition 3.1. *Let (M, F) be a $(\eta - E)$ complex Finsler space of complex dimension n. Then*

i) $K_1(z, \eta) + K_2(z, \eta) = \frac{1}{2}\mathcal{K}_F(z, \eta)$;

ii) $(\dot{\partial}_l K_1)\eta^l Lg_{k\bar{j}} + (\dot{\partial}_l K_2)\eta^l \eta_k \bar{\eta}_j = 0$;

iii) $(K_1(z, \eta) + K_2(z, \eta))\,|_k \eta^k = (K_1(z, \eta) + K_2(z, \eta))\,|_{\bar{j}}|_k \eta^k = 0$ and its conjugates.

iv) $\overline{R_{\bar{j}k}} = R_{\bar{k}j}$;

v) the functions $K_i(z, \eta)$, $i = 1, 2$, are $0-$ homogenous with respect to η, if $n \geq 2$.

Proof: Contracting the relation (3.1) with $\eta^k \overline{\eta}^j$ and taking into account (2.3), we obtain i).

For ii) we have

$$\frac{\partial R_{\bar{j}k}}{\partial \eta^l}\eta^l = (\dot{\partial}_l K_1)\eta^l L g_{k\bar{j}} + K_1 L g_{k\bar{j}} + (\dot{\partial}_l K_2)\eta^l \eta_k \overline{\eta}_j + K_2 \eta_k \overline{\eta}_j.$$

Because the functions $R_{\bar{j}k}$ are $1-$ homogeneous with respect to η, it follows that

$R_{\bar{j}k} = (\dot{\partial}_l K_1)\eta^l L g_{k\bar{j}} + (\dot{\partial}_l K_2)\eta^l \eta_k \overline{\eta}_j + R_{\bar{j}k}$ and so, ii).

Using i) and the fact that $\mathcal{K}_F(z,\eta)$ is $0-$ homogeneous with respect to η, i.e. $\mathcal{K}_F(z,\lambda\eta) = \mathcal{K}_F(z,\eta)$, for any $\lambda \in \mathbf{C}$, we have

$$K_1(z,\lambda\eta) + K_2(z,\lambda\eta) = \frac{1}{2}\mathcal{K}_F(z,\eta). \tag{3.2}$$

Thus, differentiating in (3.2) with respect to λ and setting $\lambda = 1$ we get $\dot{\partial}_k \left(K_1(z,\eta) + K_2(z,\eta)\right)\eta^k = 0$. Therefore,

$\left(K_1(z,\eta) + K_2(z,\eta)\right)|_k \eta^k = \dot{\partial}_k \left(K_1(z,\eta) + K_2(z,\eta)\right)\eta^k = 0.$

$\left(K_1(z,\eta) + K_2(z,\eta)\right)|_{\bar{j}}|_k \eta^k = \dot{\partial}_k \left(\left(K_1(z,\eta) + K_2(z,\eta)\right)|_{\bar{j}}\right)\eta^k$

$= \dot{\partial}_k \left(\dot{\partial}_{\bar{j}}(K_1(z,\eta) + K_2(z,\eta))\right)\eta^k = \dot{\partial}_{\bar{j}}\left(\dot{\partial}_k (K_1(z,\eta) + K_2(z,\eta))\eta^k\right) = 0$. So, iii) is proved.

By conjugation in (3.1), it results

$$\overline{R_{\bar{j}k}} = \overline{K_1(z,\eta) L g_{k\bar{j}} + K_2(z,\eta)\eta_k \overline{\eta}_j} = K_1(z,\eta) L g_{j\bar{k}} + K_2(z,\eta)\eta_j \overline{\eta}_k = R_{\bar{k}j},$$

i.e. iv).

In order to prove v), we write ii) as $K_1(z,\eta)|_l \eta^l L g_{k\bar{j}} + K_2(z,\eta)|_l \eta^l \eta_k \overline{\eta}_j = 0$. Because $K_1(z,\eta)|_l \eta^l = -K_2(z,\eta)|_l \eta^l$, the last relation can be written in the form

$$L h_{k\bar{j}} K_2(z,\eta)|_l \eta^l = 0,$$

where

$$h_{k\bar{j}} := g_{k\bar{j}} - \frac{1}{L(z,\eta)}\eta_k \overline{\eta}_j.$$

But, $h_{k\bar{j}} g^{\bar{j}k} = n-1$ and $n \geq 2$, therefore $L(n-1)K_2(z,\eta)|_l \eta^l = 0$, and from here results

$$K_2(z,\eta)|_{\bar{h}}\overline{\eta}^h = (\dot{\partial}_l K_2(z,\eta))\eta^l = 0,$$

i.e. $K_2(z,\eta)$ is $0-$ homogeneous with respect to η. Using again iii) we get that $K_1(z,\eta)$ is $0-$ homogeneous with respect to η. $\square$

Theorem 3.1. *Let (M,F) be a complex Finsler space, of complex dimension ≥ 2. The following statements are equivalent:*

i) (M,F) *is* $(\eta - E)$;

ii) There exists two smooth functions $K_i(z,\eta) : T'M \to \mathbf{R}$, $i=1,2$, which are $0-$ homogeneous with respect to η and such that

$$\begin{aligned} R_{\bar{j}\bar{h}k} \quad : &= R^l_{\bar{h}k} g_{l\bar{j}} = K_1(z,\eta) g_{k\bar{j}} \bar{\eta}_h + K_2(z,\eta) g_{k\bar{h}} \bar{\eta}_j + \\ & K_1(z,\eta)|_{\bar{h}} L g_{k\bar{j}} + K_2(z,\eta)|_{\bar{h}} \eta_k \bar{\eta}_j + C_{\bar{j}\bar{h}|k|\bar{m}} \bar{\eta}^m, \end{aligned} \tag{3.3}$$

where $R^l_{\bar{h}k} := R^l_{m\bar{h}k}\eta^m$.

iii) There exists two smooth functions $K_i(z,\eta) : T'M \to \mathbf{R}$, $i=1,2$, which are $0-$ homogeneous with respect to η and such that

$$\begin{aligned} R_{\bar{j}l\bar{h}k} \quad = \quad & K_1(z,\eta)\left(C_{k\bar{j}l}\bar{\eta}_h + g_{l\bar{h}} g_{k\bar{j}}\right) + K_2(z,\eta)\left(C_{k\bar{h}l}\bar{\eta}_j + g_{l\bar{j}} g_{k\bar{h}}\right) \\ & + K_1(z,\eta)|_l g_{k\bar{j}} \bar{\eta}_h + K_2(z,\eta)|_l g_{k\bar{h}} \bar{\eta}_j \\ & + K_1(z,\eta)|_{\bar{h}} \left(L(z,\eta) C_{k\bar{j}l} + g_{k\bar{j}} \eta_l\right) + K_2(z,\eta)|_{\bar{h}} \left(C_{kl}\bar{\eta}_j + g_{l\bar{j}} \eta_k\right) \\ & + K_1(z,\eta)|_{\bar{h}}|_l L g_{k\bar{j}} + K_2(z,\eta)|_{\bar{h}}|_l \eta_k \bar{\eta}_j \\ & + C_{\bar{j}\bar{h}|r|\bar{m}} C^r_{kl} \bar{\eta}^m + C_{\bar{j}\bar{h}|k|\bar{m}}|_l \bar{\eta}^m - C_{\bar{j}\bar{r}|k} C^{\bar{r}}_{l|\bar{h}}. \end{aligned} \tag{3.4}$$

Given any of these equivalent conditions, we have

$$(K_1 - K_2)\, L(z,\eta) h_{k\bar{j}} - L(z,\eta)\,(K_1 + K_2)\,|_k \bar{\eta}_j + C_{\bar{j}\bar{r}|l} C^{\bar{r}}_{k|\bar{h}} \eta^l \bar{\eta}^h + \dot{T}_{\bar{j}k} = 0, \tag{3.5}$$

where

$$\begin{aligned} \eta_i \quad &= \quad g_{i\bar{j}}\bar{\eta}^j \ ; \ \ C_{\bar{i}\bar{j}} := C_{h\bar{i}\bar{j}}\eta^h \ ; \ \ C_{h\bar{i}\bar{j}} := \dot{\partial}_{\bar{j}} g_{h\bar{i}} \ ; \\ C^{\bar{r}}_l \quad &: \quad = g^{\bar{r}j} C_{jl} \ ; \ \ T_{\bar{j}k} := g_{i\bar{j}} T^i_{lk}\eta^l \ ; \ \ \dot{T}_{\bar{j}k} := T_{\bar{j}k|\bar{m}}\bar{\eta}^m. \end{aligned} \tag{3.6}$$

Proof: If (M,F) is $(\eta - E)$, by a direct computation, we obtain:

$$R_{\bar{j}k}|_{\bar{h}} = K_1(z,\eta) g_{k\bar{j}}\bar{\eta}_h + K_2(z,\eta) g_{k\bar{h}}\bar{\eta}_j + K_1(z,\eta)|_{\bar{h}} L g_{k\bar{j}} + K_2(z,\eta)|_{\bar{h}} \eta_k \bar{\eta}_j; \tag{3.7}$$

and

$$\begin{aligned} R_{\bar{j}k}|_{\bar{h}}|_l \quad = \quad & K_1(z,\eta) g_{l\bar{h}} g_{k\bar{j}} + K_2(z,\eta) g_{l\bar{j}} g_{k\bar{h}} \\ & + K_1(z,\eta)|_l g_{k\bar{j}}\bar{\eta}_h + K_2(z,\eta)|_l g_{k\bar{h}}\bar{\eta}_j + K_1(z,\eta)|_{\bar{h}} g_{k\bar{j}} \eta_l \\ & + K_2(z,\eta)|_{\bar{h}} g_{l\bar{j}} \eta_k + K_1(z,\eta)|_{\bar{h}}|_l L g_{k\bar{j}} + K_2(z,\eta)|_{\bar{h}}|_l \eta_k \bar{\eta}_j. \end{aligned} \tag{3.8}$$

If the functions $K_i(z,\eta)$, $i=1,2$, are $0-$ homogeneous with respect to η, we get:

$$K_i(z,\eta)|_l \eta^l = K_i(z,\eta)|_{\bar{h}} \bar{\eta}^h = K_i(z,\eta)|_{\bar{h}}|_l \eta^l = 0. \tag{3.9}$$

Now let us prove that **i**) $\Longleftrightarrow$ **ii**).

Given $R_{\bar{j}k}$ as in (3.1), we can reconstruct $R^i_{\bar{j}k}$. For this, contracting the Bianchi identity, (see [5]), $R^i_{j\bar{h}k}|_{\bar{l}} - P^i_{j\bar{l}k|\bar{h}} - \Xi^i_{j\bar{h}l}P^r_{\bar{l}k} + S^i_{j\bar{l}r}R^r_{\bar{h}k} + R^i_{j\bar{r}k}C^{\bar{r}}_{\bar{h}\bar{l}} = 0$ with $\eta^j\overline{\eta}^h$, we obtain $R^i_{\bar{h}k}|_{\bar{l}}\overline{\eta}^h = -C^i_{\bar{l}|k|\bar{h}}\overline{\eta}^h$. On the other hand, $R^i_{\bar{h}k}|_{\bar{l}}\overline{\eta}^h = R^i_k|_{\bar{l}} - R^i_{\bar{l}k}$, where $R^i_k := R^i_{\bar{h}k}\overline{\eta}^h$. So, $R^i_{\bar{l}k} = C^i_{\bar{l}|k|\bar{h}}\overline{\eta}^h + R^i_k|_{\bar{l}}$. Indeed, $R_{\bar{\imath}\bar{l}k} = C_{\bar{\imath}\bar{l}|k|\bar{h}}\overline{\eta}^h + R_{\bar{\imath}k}|_{\bar{l}}$ which, together with (3.7) implies (3.3). Now, by Proposition 3.1 v), the functions $K_i(z,\eta)$, $i = 1,2$, are $0-$ homogeneous with respect to η.

Conversely, contracting (3.3) by $\overline{\eta}^h$, we have

$R_{\overline{j}\overline{h}k}\overline{\eta}^h = R_{\bar{j}k} = K_1(z,\eta)Lg_{k\bar{j}} + K_2(z,\eta)\eta_k\bar{\eta}_j + K_1(z,\eta)|_{\bar{h}}\overline{\eta}^hLg_{k\bar{j}}$

$+K_2(z,\eta)|_{\bar{h}}\overline{\eta}^h\eta_k\bar{\eta}_j + C_{\overline{j}\overline{h}|k|\bar{m}}\overline{\eta}^m\overline{\eta}^h$. Because $K_i(z,\eta)|_{\bar{h}}\overline{\eta}^h = 0$, $i = 1,2$, and $C_{\overline{j}\overline{h}|k|\bar{m}}\overline{\eta}^m\overline{\eta}^h = 0$, the last relation gives i).

i) $\Longleftrightarrow$ **iii**). Given $R_{\bar{j}k}$ as in (3.1), we use the following Bianchi identity $R^i_{j\bar{h}k}|_l - \Xi^i_{j\bar{h}l|k} - P^i_{j\bar{r}k}P^{\bar{r}}_{l\bar{h}} + S^i_{j\bar{r}l}R^{\bar{r}}_{k\bar{h}} + R^i_{j\bar{h}r}C^r_{kl} = 0$ to reconstruct $R_{\bar{j}l\bar{h}k}$.

If we contract this with η^j, we obtain $R^i_{j\bar{h}k}|_l\eta^j = -C^i_{\bar{r}|k}C^{\bar{r}}_{l|\bar{h}} - R^i_{\bar{h}r}C^r_{kl}$. But, $R^i_{j\bar{h}k}|_l\eta^j = R^i_{\bar{h}k}|_l - R^i_{l\bar{h}k}$, so that $R^i_{l\bar{h}k} = R^i_{\bar{h}k}|_l - C^i_{\bar{r}|k}C^{\bar{r}}_{l|\bar{h}} + R^i_{\bar{h}r}C^r_{kl}$. It results that $R_{\bar{j}l\bar{h}k} = R_{\bar{j}r}|_{\bar{h}}C^r_{kl} + R_{\bar{j}k}|_{\bar{h}}|_l + C_{\overline{j}\overline{h}|r|\overline{m}}C^r_{kl}\overline{\eta}^m - C_{\overline{j}\overline{r}|k}C^{\bar{r}}_{l|\bar{h}} + C_{\overline{j}\overline{h}|k|\overline{m}}|_l\overline{\eta}^m$.

Plugging (3.6) and (3.8) into the last relation, we obtain (3.4). Moreover, taking into account Proposition 3.1 v), the functions $K_i(z,\eta)$, $i = 1,2$, are $0-$ homogeneous with respect to η.

The converse follows from (3.4) by contraction with $\overline{\eta}^h\eta^l$ and using (3.9) and $C^r_{kl}\eta^l = C_{\overline{j}\overline{h}|k|\bar{m}}|_l\eta^l = C^{\bar{r}}_{l|\bar{h}}\eta^l = 0$.

To prove (3.5) we compute $(R_{\bar{j}l\bar{h}k} - R_{\bar{j}k\bar{h}l})\eta^l\overline{\eta}^h$ in two ways. By iii),

$$(R_{\bar{j}l\bar{h}k} - R_{\bar{j}k\bar{h}l})\eta^l\overline{\eta}^h = (K_1 - K_2)\,Lh_{k\bar{j}} - L\,(K_1 + K_2)\,|_k\bar{\eta}_j + C_{\overline{j}\overline{r}|l}C^{\bar{r}}_{k|\bar{h}}\eta^l\bar{\eta}^h,$$

and by Bianchi identity $T^i_{jk|\bar{h}} + \mathcal{A}_{jk}\left\{R^i_{j\bar{h}k} - C^i_{jl}R^l_{\bar{h}k}\right\} = 0$, we obtain $(R_{\bar{j}l\bar{h}k} - R_{\bar{j}k\bar{h}l})\eta^l\overline{\eta}^h = -\dot{T}_{\bar{j}k}$. So, we have (3.5). ∎

Proposition 3.2. *Let* (M,F) *be a* $(\eta - E)$ *complex Finsler space, of complex dimension* ≥ 2. *Then*

i) $\mathcal{K}_F$ *depends on* z *alone, i.e.* $K_1 + K_2 := K(z)$*;*

ii) $C_{\overline{j}\overline{h}|k|\bar{m}}\eta^k\overline{\eta}^m - (K_1 + K_2)\,L(z,\eta)C_{\overline{j}\overline{h}} = 0$;

iii) $(K_1 - K_2)\,L(z,\eta)h_{k\bar{j}} + \dot{T}_{\bar{j}k} = 0$,

iv) $C_{\overline{j}\overline{r}|l}C^{\bar{r}}_{k|\bar{h}}\eta^l\overline{\eta}^h = 0$.

Proof: Since $\overline{R_{j\bar{l}h\bar{k}}} = R_{\bar{j}l\bar{h}k}$, then $\overline{R_{j\bar{l}h\bar{k}}\overline{\eta}^l\overline{\eta}^k} = R_{\bar{j}l\bar{h}k}\eta^l\eta^k$. If we contract (3.4) by $\eta^l\eta^k$, taking into account Theorem 3.1, ii) we deduce

$$R_{\bar{j}l\bar{h}k}\eta^l\eta^k = C_{\overline{j}\overline{h}|k|\bar{m}}\eta^k\overline{\eta}^m + L(z,\eta)\,(K_1 + K_2)\,|_{\bar{h}}\bar{\eta}_j + (K_1 + K_2)\,\bar{\eta}_j\bar{\eta}_h. \quad (3.10)$$

On the other hand, $R_{j\bar{l}h\bar{k}}\bar{\eta}^l\bar{\eta}^k = R_{\bar{l}j\bar{k}h}\bar{\eta}^l\bar{\eta}^k$ and by (3.4), we have
$R_{\bar{l}j\bar{k}h}\bar{\eta}^l\bar{\eta}^k = (K_1 + K_2)(L(z,\eta)C_{jh} + \eta_j\eta_k) + 2L(z,\eta)(K_1 + K_2)|_j\eta_h$.
By conjugation,

$$\overline{R_{\bar{l}j\bar{k}h}\bar{\eta}^l\bar{\eta}^k} = (K_1 + K_2)\left(L(z,\eta)C_{\overline{jh}} + \overline{\eta}_j\overline{\eta}_k\right) + 2L(z,\eta)(K_1 + K_2)|_{\bar{j}}\overline{\eta}_h. \quad (3.11)$$

So, (3.10) and (3.11) lead to

$$C_{\overline{jh}|k|\bar{m}}\eta^k\bar{\eta}^m -$$

$$L(z,\eta)\left((K_1 + K_2)C_{\overline{jh}} - (K_1 + K_2)|_{\overline{h}}\overline{\eta}_j + (K_1 + K_2)|_{\bar{j}}\overline{\eta}_h\right) = 0. \quad (3.12)$$

To prove $i)$ we contract (3.12) with $\bar{\eta}^j$ and we have $L^2(z,\eta)(K_1 + K_2)|_{\overline{h}} = 0$. Hence $(K_1 + K_2)|_{\overline{h}} = 0$, i.e. $\frac{\partial(K_1+K_2)}{\partial\bar{\eta}^h} = 0$ By conjugation, $\frac{\partial(K_1+K_2)}{\partial\eta^h} = 0$, and so $K_1 + K_2$ does not depends on η. As a consequence of $i)$, the relation (3.12) brings to $ii)$.

$iii)$ By Jacobi identity $\left[\dot{\partial}_i, [\delta_j, \delta_{\bar{k}}]\right] + \left[\delta_j, [\delta_{\bar{k}}, \dot{\partial}_i]\right] + \left[\delta_{\bar{k}}, [\dot{\partial}_i, \delta_j]\right] = 0$, we have $-\dot{\partial}_i(R^l_{\bar{k}j}\bar{\eta}^k) + \dot{\partial}_j(R^l_{\bar{k}i}\bar{\eta}^k) - T^l_{ij|\bar{k}}\bar{\eta}^k = 0$. Taking into account (3.1), we obtain

$$(K_1 - K_2)(\delta^l_i\eta_j - \delta^l_j\eta_i) + K_{1|i}(\delta^l_j L - \eta_j\eta^l) + K_{1|j}(\delta^l_i L - \eta_i\eta^l) - T^l_{ij|\bar{k}}\bar{\eta}^k = 0. \quad (3.13)$$

Contracting above relation by $g_{l\bar{r}}\eta^j$, it became $(K_1 - K_2)L(z,\eta)h_{i\bar{r}} + \dot{T}_{i\bar{r}} = 0$, *i.e.* $iii)$. From (3.5), $ii)$ and $iii)$ we obtain $iv)$. □

Corollary 3.1. *Let (M,F) be a $(\eta - E)$ complex Finsler space, of complex dimension ≥ 2. Then*

$i)$ $Ric = (nK_1 + K_2)L(z,\eta)$ is real valued;

$ii)$ $Ric_{i\bar{j}} = [(n-1)K_1 + K(z)]g_{i\bar{j}} + (n-1)\left(K_1|_{\bar{j}}\eta_i + K_1|_i\bar{\eta}_j + LK_1|_i|_{\bar{j}}\right)$.

Proof: $Ric := g^{\bar{j}k}R_{\bar{j}k} = g^{\bar{j}k}\left(K_1 L g_{k\bar{j}} + K_2\eta_k\bar{\eta}_j\right) = K_1 L\delta^k_k + K_2 L$
$= (nK_1 + K_2)L$.
We compute:

$$\tfrac{\partial Ric}{\partial\eta^i} = (nK_1 + K_2)\eta_i + (nK_1|_i + K_2|_i)L;$$

$$Ric_{i\bar{j}} := \tfrac{\partial^2 Ric}{\partial\eta^i\partial\bar{\eta}^j} = (nK_1 + K_2)g_{i\bar{j}} + (nK_1|_{\bar{j}} + K_2|_{\bar{j}})\eta_i$$

$$+(nK_1|_i + K_2|_i)\bar{\eta}_j + (nK_1|_i|_{\bar{j}} + K_2|_i|_{\bar{j}})L.$$

But, by Proposition 3.2 $i)$ we have $K_2 = K(z) - K_1$. It results $K_2|_{\bar{j}} = -K_1|_{\bar{j}}$, $K_2|_i = -K_1|_i$ and $K_2|_i|_{\bar{j}} = -K_1|_i|_{\bar{j}}$. All these relations lead to $ii)$. □

We emphasize that in any $(\eta - E)$ complex Finsler space the holomorphic curvature depends on z only, $\mathcal{K}_F(z) := \mathcal{K}_F(z,\eta) = 2K(z)$ and a $(\eta - E)$ complex Finsler space is $(g.E.)$ if $K_1 = K_2$. It is natural for us to inquire when $K_1 = K_2$? The answer came below.

Corollary 3.2. *If* (M, F) *is a Kähler* $(\eta - E)$ *complex Finsler space, of complex dimension* ≥ 2*, then it is* $(g.E.)$.

Proof: It follows immediately from Proposition 3.2 *iii*). Indeed, because (M, F) is Kähler, we have $T^i_{jk}\eta^j = 0$ and so, $\dot{T}_{\bar{j}k} = 0$. We obtain
$(K_1 - K_2)\, L(z,\eta) h_{k\bar{j}} = 0$, and from here $(n-1)(K_1 - K_2)\, L(z,\eta) = 0$. It results $K_1 = K_2$. □

From this and Theorem 2.1. (*iv*) it follows immediately the following

Corollary 3.3. *If* (M, F) *is a Kähler* $(\eta - E)$ *complex Finsler space, of complex dimension* ≥ 2, *with* $K(z) \neq 0$, *then* F *is purely Hermitian.*

4 $\eta-$Einstein spaces with constant holomorphic curvature

In the sequel, our goal is to determine conditions under which a $(\eta - E)$ complex Finsler space has constant holomorphic curvature, i.e. when $K(z) := K_1(z,\eta) + K_2(z,\eta)$ is constant. At first we prove a Schur type theorem for $(\eta - E)$ complex Finsler space, namely:

Theorem 4.1. *Let* (M, F) *be a* $(\eta - E)$ *connected complex Finsler space, weakly Kähler, of complex dimension* ≥ 2. *Then it is a space with constant holomorphic curvature.*

Proof: By a direct computation, we obtain

$$\begin{aligned}
R_{\bar{j}k|l} &= Lh_{k\bar{j}}K_1(z,\eta)_{|l} + K(z)_{|l}\eta_k\bar{\eta}_j; \qquad (4.1)\\
R^{\bar{s}}_{k\bar{h}} &= C^{\bar{s}}_{k|\bar{h}|m}\eta^m + R^{\bar{s}}_{\bar{h}}|_k; \quad \text{where}\\
R^{\bar{s}}_{\bar{h}} :=& \; R_{\bar{h}k}g^{\bar{s}k} = \left(L(z,\eta)\delta^{\bar{s}}_{\bar{h}} - \bar{\eta}_h\bar{\eta}^s\right)K_1(z,\eta) + K(z)\bar{\eta}_h\bar{\eta}^s;\\
R^{\bar{s}}_{\bar{h}}|_k &= K_1(z,\eta)\left(\eta_k\delta^{\bar{s}}_{\bar{h}} - g_{k\bar{h}}\bar{\eta}^s\right) + K_1(z,\eta)|_k\left(L\delta^{\bar{s}}_{\bar{h}} - \eta_{\bar{h}}\bar{\eta}^s\right) + K(z)g_{k\bar{h}}\bar{\eta}^s.
\end{aligned}$$

The contraction of the Bianchi identity $\mathcal{A}_{kl}\left\{R^i_{j\bar{h}k|l} - P^i_{j\bar{r}k}R^{\bar{r}}_{l\bar{h}}\right\} + R^i_{j\bar{h}m}T^m_{kl} = 0$ with $g_{i\bar{r}}\eta^j\eta^l\bar{\eta}^h$, leads to
$R_{\bar{r}k|l}\eta^l - R_{\bar{r}l|k}\eta^l + C_{\bar{r}\bar{s}|k}R^{\bar{s}}_{\bar{h}}\bar{\eta}^h - C_{\bar{r}\bar{s}|l}R^{\bar{s}}_{k\bar{h}}\eta^l\bar{\eta}^h + R_{\bar{r}m}T^m_{kl}\eta^l = 0.$
The last result and (4.1) give

$$K(z)g_{m\bar{r}}T^m_{kl}\eta^l\bar{\eta}^r + \eta_k K(z)_{|l}\eta^l - L(z,\eta)K(z)_{|k} = 0. \qquad (4.2)$$

Since F is weakly Kähler, then from (4.2) we get $\eta_k K(z)_{|l}\eta^l - LK(z)_{|k} = 0$. So, by conjugation we have

$$K(z)_{|\bar{h}} = \frac{1}{L(z,\eta)}\bar{\eta}_h K(z)_{|\bar{l}}\bar{\eta}^l. \tag{4.3}$$

Because of $K(z)_{|\bar{h}}|_j = K(z)|_{j|\bar{h}} = 0$, deriving (4.3) we easily deduce $0 = K(z)_{|\bar{h}}|_j = \frac{K(z)_{|\bar{l}}\bar{\eta}^l}{L(z,\eta)}h_{j\bar{h}}$, which multiplied by $g^{\bar{h}j}$, we obtain $K(z)_{|\bar{l}}\bar{\eta}^l = 0$. Plugging it into (4.3), it follows that $K(z)_{|\bar{h}} = 0$, i.e. $\frac{\partial K(z)}{\partial \bar{z}^h} = 0$. By conjugation, $\frac{\partial K(z)}{\partial z^h} = 0$ and so, $K(z)$ is a constant on M. □

By (4.2), we deduce the following

Proposition 4.1. *If (M,F) is a $(\eta - E)$ complex Finsler space, of complex dimension ≥ 2, with $K(z)$ a nonzero constant, then F is weakly Kähler.*

Proof: Since F is $(\eta - E)$, with $K(z)$ a nonzero constant, then $K(z)_{|l} = 0$ and (4.2) becomes $g_{m\bar{r}}T^m_{kl}\eta^l\bar{\eta}^r = 0$, i.e. F is weakly Kähler. □

Particularly, if (M,F) is a $(\eta - E)$ complex Finsler space, with $K(z) = 0$, then it is a flat complex Finsler space, i.e. $\mathcal{K}_F = 0$, and $C_{\overline{jh}|k|\bar{m}}\eta^k\bar{\eta}^m = 0$. Moreover, using Theorem 3.1 we can prove

Theorem 4.2. *Let (M,F) be a complex Finsler space, of complex dimension ≥ 2. The following statements are equivalent:*

i) (M,F) is $(\eta - E)$ with constant curvature $\mathcal{K}_F = 2(K_1 + K_2) = 2c$, $c \in \mathbf{R}$;

ii) There exists two smooth functions $K_i(z,\eta) : T'M \to \mathbf{R}$, $i = 1,2$, such that $K_1(z,\eta)$ is $0-$ homogeneous with respect to η, $K_1(z,\eta) + K_2(z,\eta) = c$ and

$$\begin{aligned} R_{\bar{j}\bar{h}k} &:= R^l_{\bar{h}k}g_{l\bar{j}} = K_1(g_{k\bar{j}}\bar{\eta}_h - g_{k\bar{h}}\bar{\eta}_j) \\ &\quad + cg_{k\bar{h}}\bar{\eta}_j + K_1|_{\bar{h}}Lh_{k\bar{j}} + C_{\overline{jh}|k|\bar{m}}\bar{\eta}^m. \end{aligned} \tag{4.4}$$

iii) There exists two smooth functions $K_i(z,\eta) : T'M \to \mathbf{R}$, $i = 1,2$, such that $K_1(z,\eta)$ is $0-$ homogeneous with respect to η, $K_1(z,\eta) + K_2(z,\eta) = c$ and

$$\begin{aligned} R_{\bar{j}l\bar{h}k} = {} & K_1\left(C_{k\bar{j}l}\bar{\eta}_h - C_{k\bar{h}l}\bar{\eta}_j + g_{l\bar{h}}g_{k\bar{j}} - g_{l\bar{j}}g_{k\bar{h}}\right) \\ & + c\left(C_{k\bar{h}l}\bar{\eta}_j + g_{l\bar{j}}g_{k\bar{h}}\right) + K_1|_l\left(g_{k\bar{j}}\bar{\eta}_h - g_{k\bar{h}}\bar{\eta}_j\right) \\ & + K_1|_{\bar{h}}\left(L(z,\eta)C_{k\bar{j}l} - C_{kl}\bar{\eta}_j + g_{k\bar{j}}\eta_l - g_{l\bar{j}}\eta_k\right) \\ & + K_1|_{\bar{h}}|_l Lh_{k\bar{j}} + C_{\overline{jh}|r|\bar{m}}C^r_{kl}\bar{\eta}^m + C_{\overline{jh}|k|\bar{m}}|_l\bar{\eta}^m - C_{\overline{jr}|k}C^{\bar{r}}_{l|\bar{h}}. \end{aligned} \tag{4.5}$$

Proof: By Theorem 3.1, if (M, F) is $(\eta - E)$ then there exists the smoothly functions $K_i(z, \eta)$, $i = 1, 2$, which are $0-$ homogeneous with respect to η and satisfy (3.3) and (3.4). Moreover, $K(z) = K_1(z, \eta) + K_2(z, \eta) = c$ and plugging it into (3.3) and (3.4) we obtain (4.4) and (4.5). So, the requirements $i) \Rightarrow ii)$ and $i) \Rightarrow iii)$ are true.

Conversely, contracting (4.4) by $\overline{\eta}^h$ and (4.5) by $\overline{\eta}^h \eta^l$ and taking into account $K_1(z, \eta) + K_2(z, \eta) = c$ and $K_1(z, \eta)$ is $0-$ homogeneous with respect to η, we obtain $i)$. So we have proved $ii) \Rightarrow i)$ and $iii) \Rightarrow i)$. ∎

Proposition 4.2. *Let (M, F) be a $(\eta - E)$ complex Finsler space, of complex dimension ≥ 2, of constant holomorphic curvature $2c$. Then,*

i) $R^l_{\bar{j}k} \overline{\eta}^j \eta^k = cL(z, \eta)\eta^l$; $R_{\bar{j}l\bar{h}k} \overline{\eta}^j \eta^l \overline{\eta}^h = cL(z, \eta)\eta_k$;

ii) $(R_{\bar{j}l\bar{h}k} - R_{\bar{j}k\bar{h}l})\overline{\eta}^j \eta^l \overline{\eta}^h = 0$;

iii) $C_{\overline{jh}|k|\bar{m}} \eta^k \overline{\eta}^m - cLC_{\overline{jh}} = 0$.

Proof: It follows from Theorem 4.2. ∎

We note that the above conditions $i)$ and $ii)$, with $c = -2$, are equivalent to the conditions of Theorem 3.1.15, from [1], p. 146. Therefore, the following Proposition gives a particular form of that Theorem.

Proposition 4.3. *Let (M, F) be a complex Finsler space, of complex dimension ≥ 2. If one of equivalent conditions from Theorem 4.2 holds for $c = -2$, then F is the Kobayashi metric on M.*

An example. We give an example which illustrate our theory. Let

$$L := \frac{|\eta|^2 + \varepsilon(|z|^2|\eta|^2 - < z, \eta > \overline{< z, \eta >})}{(1 + \varepsilon|z|^2)^2}, \tag{4.6}$$

be a complex Finsler metric, where $|z|^2 := \sum_{k=1}^n z^k \overline{z}^k$, $< z, \eta > := \sum_{k=1}^n z^k \overline{\eta}^k$, defined on the disk $\Delta^n_r = \left\{ z \in \mathbf{C}^n,\ |z| < r,\ \ r := \sqrt{\frac{1}{|\varepsilon|}} \right\}$ if $\varepsilon < 0$, on $\mathbf{C}^n$ if $\varepsilon = 0$ and on the complex projective space $P^n(\mathbf{C})$ if $\varepsilon > 0$. In particular, for $\varepsilon = -1$ we obtain the *Bergman metric* on the unit disk $\Delta^n := \Delta^n_1$; for $\varepsilon = 0$ the *Euclidean metric* on $\mathbf{C}^n$, and for $\varepsilon = 1$ the *Fubini-Study metric* on $P^n(\mathbf{C})$. They are purely Hermitian. Indeed, they are the well known metrics of the simply connected homogeneous Kähler manifolds of constant holomorphic sectional curvature $\mathcal{K}_F = 4\varepsilon$.

Now, let us consider a Finsler metric which is conformal to (4.6), i.e. $g'_{i\bar{j}} = e^{\rho(z)} g_{i\bar{j}} = \frac{e^{\rho(z)}}{1+\varepsilon|z|^2} \left(\delta_{i\bar{j}} - \varepsilon \frac{\overline{z}^i z^j}{1+\varepsilon|z|^2} \right)$. Clearly, $g'_{i\bar{j}}$ is purely Hermitian and an immediate computation shows that $R'_{\bar{j}k} = e^{\rho(z)} \left(\varepsilon t_{k\bar{j}} - \frac{\partial^2 \rho}{\partial z^k \partial \overline{z}^h} \overline{\eta}_j \overline{\eta}^h \right)$.

We suppose that $\rho(z) = \alpha \log(1 + \varepsilon|z|^2)$, $\varepsilon, \alpha \in \mathbf{R}^*$. Therefore,

$g'_{i\bar{j}} = (1+\varepsilon|z|^2)^{\alpha-1}\left(\delta_{i\bar{j}} - \varepsilon\frac{\bar{z}^i z^j}{1+\varepsilon|z|^2}\right)$, and it is not Kähler. Furthermore, we have $R'_{\bar{j}k} = \frac{\varepsilon}{(1+\varepsilon|z|^2)^\alpha}\left(Lg'_{k\bar{j}} - (1-\alpha)\eta'_k\bar{\eta}'_j\right)$. This last relation shows that $R'_{\bar{j}k} = K_1 Lg'_{k\bar{j}} + K_2\eta'_k\bar{\eta}'_j$, where $K_1 = \frac{\varepsilon}{(1+\varepsilon|z|^2)^\alpha}$ and $K_2 = \frac{(1-\alpha)\varepsilon}{(1+\varepsilon|z|^2)^\alpha}$. So the metric $g'_{i\bar{j}}$ is $(\eta - E)$ with holomorphic curvature $\mathcal{K}'_{F'} = 2(K_1+K_2) = \frac{2\varepsilon(2-\alpha)}{(1+\varepsilon|z|^2)^\alpha}$. Moreover, if $\varepsilon < 0$ and $\alpha < 2$, or $\varepsilon > 0$ and $\alpha > 2$, then $\mathcal{K}'_{F'} < 0$. If $\varepsilon < 0$ and $\alpha \geq 2$, or $\varepsilon > 0$ and $\alpha \leq 2$, then $\mathcal{K}'_{F'} \geq 0$.

These are examples of $(\eta - E)$ purely Hermitian complex Finsler spaces that are not Kähler nor $(g.E.)$.

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