On the Ramsey numbers for paths and generalized Jahangir graphs $J_{s,m}$

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Abstract

For given graphs $G$ and $H$, the Ramsey number $R(G, H)$ is the least natural number $n$ such that for every graph $F$ of order $n$ the following condition holds: either $F$ contains $G$ or the complement of $F$ contains $H$. In this paper, we determine the Ramsey number of paths versus generalized Jahangir graphs. We also derive the Ramsey number $R(tP_n, H)$, where $H$ is a generalized Jahangir graph $J_{s,m}$ where $s \geq 2$ is even, $m \geq 3$ and $t \geq 1$ is any integer.

Key Words: Ramsey number, path, generalized Jahangir graph.

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1 Introduction

The study of Ramsey numbers for (general) graphs have received tremendous efforts in the last two decades, see few related papers [1]-[4], [6, 8] and a nice survey paper [7].

Let $G(V, E)$ be a graph with vertex-set $V(G)$ and edge-set $E(G)$. If $xy \in E(G)$ then $x$ is called adjacent to $y$, and $y$ is a neighbor of $x$ and vice versa. For any $A \subseteq V(G)$, we use $N_A(x)$ to denote the set of all neighbors of $x$ in $A$, namely $N_A(x) = \{y \in A | xy \in E(G)\}$. Let $P_n$ be a path with $n$ vertices, $C_n$ be a cycle with $n$ vertices, $W_k$ be a wheel of $k + 1$ vertices, i.e., a graph consisting of a cycle $C_k$ with one additional vertex adjacent to all vertices of $C_k$. For $s, m \geq 2$, the generalized Jahangir graph $J_{s,m}$ is a graph on $sm + 1$ vertices i.e., a graph consisting of a cycle $C_{sm}$ with one additional vertex which is adjacent to $m$ vertices of $C_{sm}$ at distance $s$ to each other on $C_{sm}$.
Recently, Surahmat and Tomescu [9] studied the Ramsey number of a combination of paths $P_n$ versus $J_{2,m}$, and obtained the following result.

**Theorem A.** [9].

$$R(P_n, J_{2,m}) = \begin{cases} 
6 & \text{if } (n, m) = (4, 2), \\
n + 1 & \text{if } m = 2 \text{ and } n \geq 5, \\
n + m - 1 & \text{if } m \geq 3 \text{ and } n \geq (4m - 1)(m - 1) + 1.
\end{cases}$$

For the Ramsey number of $P_n$ with respect to wheel $W_m$, Surahmat and Baskoro [1] showed the following result.

**Theorem B.** [1].

$$R(P_n, W_m) = \begin{cases} 
2n - 1 & \text{if } m \geq 4 \text{ is even and } n \geq \frac{m}{2}(m - 2), \\
3n - 2 & \text{if } m \geq 5 \text{ is odd and } n \geq \frac{m - 1}{2}(m - 3).
\end{cases}$$

In this paper, we determine the Ramsey numbers involving paths $P_n$ and generalized Jahangir graphs $J_{s,m}$. We also find the Ramsey number $R(tP_n, H)$, where $H$ is a generalized Jahangir graph $J_{s,m}$ where $s \geq 2$ is even, $m \geq 3$. In the following section we prove our main results.

### 2 Main Results

**Theorem 1.** For even $s \geq 2$ and $m \geq 3$, $R(P_n, J_{s,m}) = n + \frac{sm}{2} - 1$, where $n \geq (2sm - 1)(\frac{sm}{2} - 1) + 1$.

**Proof:** Let $G = K_{n-1} \cup K_{\frac{sm}{2}-1}$. We have $R(P_n, J_{s,m}) \geq n + \frac{sm}{2} - 1$ since $P_n \not\subseteq G$ and $J_{s,m} \not\subseteq G$. It remains to prove that $R(P_n, J_{s,m}) \leq n + \frac{sm}{2} - 1$. Let $F$ be a graph of order $n + \frac{sm}{2} - 1$ and containing no path $P_n$, we will show that $F \supseteq J_{s,m}$.

Let $L_1 = l_{1,1}, l_{1,2}, \ldots, l_{1,k}$ be the longest path in $F$ and so $k \leq n - 1$. If $k = 1$ we have $F \cong K_n + \frac{sm}{2} - 1$, which contains $J_{s,m}$. Suppose that $k \geq 2$ and $J_{s,m} \not\subseteq F$. We have $z_l \not\in E(F)$ for each $z \in V_1 = V(F) \setminus V(L_1)$. We distinguish two cases:

**Case 1.** $k \leq 2sm - 1$. Let $L_2 = l_{2,1}, l_{2,2}, \ldots, l_{2,t}$ be a longest path in $F[V_1]$. It is clear that $1 \leq t \leq k$. If $t = 1$ then the vertices in $V_1$ induce a subgraph having only isolated vertices. In this case we shall add an edge $uv$ to $F$, where $u, v \in V_1$ and denote $L_2 = u, v$. In this way we can define inductively the system of paths $L_1, L_2, \ldots, L_{\frac{sm}{2}-1}$ such that $L_i$ is a longest path in $F[V_{i-1}]$, where $V_{i-1} = V(F) \setminus \bigcup_{j=1}^{i-1} V(L_j)$ or an edge added to $F$ as above. By denoting the set of remaining vertices by $B$, we have $|B| \geq n + \frac{sm}{2} - 1 - \frac{(sm)}{2}(-1)(2sm - 1) \geq \frac{sm}{2} \geq 3$ since $s \geq 2$ and $m \geq 3$. Let $x, y, z \in B$ be three distinct vertices which are not in any $L_j$ for $j = 1, 2, \ldots, \frac{sm}{2}-1$. Clearly, $x, y, z$ are not adjacent to all endpoints of these $L_j$. If $F_1$ denotes the graph $F$ or the graph $F$ plus some edges added
in the process of defining the system of paths, it follows that the endpoints of these \( L_j \) induce in \( F_1 \) a complete graph \( K_{sm-2} \) minus a matching having at most \( \frac{sm}{2} - 1 \) edges if some of the endpoints of same \( L_j \) are adjacent in \( F_1 \). Since \( x, y, z \) are not adjacent to all endpoints of these \( L_j \) it is easy to see that vertices \( x, y, z \) and endpoints of the paths \( L_j \) form a \( J_{s,m} \subseteq F_1 \subseteq F \).

**Case 2.** \( k > 2sm - 1 \). In this case we define \( \frac{sm}{2} - 1 \) quadruple of consecutive vertices of \( L_1 \) as follows:

\[
C_1 = \{l_{1,2}, l_{1,3}, l_{1,4}, l_{1,5}\},
\]

\[
C_2 = \{l_{1,6}, l_{1,7}, l_{1,8}, l_{1,9}\},
\]

\[
\vdots
\]

\[
C_{\frac{sm}{2} - 1} = \{l_{1,2sm-6}; l_{1,2sm-5}, l_{1,2sm-4}, l_{1,2sm-3}\}.
\]

Let \( Y = V(F) \setminus V(L_1) \). We have \(|Y| = n + \frac{sm}{2} - 1 - k \geq \frac{sm}{2} \) since \( k \leq n - 1 \). Hence we can consider \( \frac{sm}{2} \) distinct elements in \( Y \): \( y_1, y_2, \ldots, y_{sm} \) and \( \frac{sm}{2} - 1 \) pairs of elements \( Y_i = \{y_i, y_{i+1}\} \) for \( i = 1, \ldots, \frac{sm}{2} - 1 \). By the maximality of \( L_1 \) it follows that for each \( i = 1, \ldots, \frac{sm}{2} - 1 \) at least one vertex in \( C_i \) is not adjacent to any vertex in \( Y_i \). Denote by \( c_i \) the vertex in \( C_i \) which is not adjacent to any vertex in \( Y_i \) for \( i = 1, \ldots, \frac{sm}{2} - 1 \). We have \( F \supseteq J_{s,m} \), where \( J_{s,m} \) consists of the cycle \( C_{sm} \) having \( V(C_{sm}) = \{y_1, c_1, y_2, c_2, \ldots, y_{\frac{sm}{2} - 1}, c_{\frac{sm}{2} - 1}, y_{\frac{sm}{2}}, l_{1,1}\} \) and the hub \( l_{1,1} \).

**Theorem 2.** *For odd \( s \geq 3 \),

\[
R(P_n, J_{s,m}) = \begin{cases} 
2n - 1 & \text{if } n \geq \frac{sm}{2}(sm - 2), \text{ and } m \geq 2 \text{ is even,} \\
2n & \text{if } n \geq \frac{sm-1}{2}(sm - 1), \text{ and } m \geq 3 \text{ is odd.}
\end{cases}
\]

**Proof:** To show the lower bound, consider graphs \( 2K_{n-1} \) and \( K_1 \cup 2K_{n-1} \) for the first and second cases of Theorem respectively.

For the reverse inequality, firstly we will prove the result for the first case of Theorem. Let \( F \) be a graph of order \( 2n - 1 \) containing no path \( P_n \) where \( n \geq \frac{sm}{2}(sm - 2) \). We will show that \( F \supseteq J_{s,m} \). Since \( F \) does not contain \( P_n \), by Theorem B, \( F \) will contain a wheel \( W_{sm} \), and so clearly \( F \supseteq J_{s,m} \).

For the second case, to prove \( R(P_n, J_{s,m}) \leq 2n \) let \( F \) be a graph on \( 2n \) vertices containing no \( P_n \). Let \( L_1 = (l_{11}, l_{12}, \ldots, l_{1k-1}, l_{1k}) \) be a longest path in \( F \) and so \( k \leq n - 1 \). If \( k = 1 \) we have \( F \supseteq K_{2n} \), which contains \( J_{s,m} \). Suppose that \( k \geq 2 \) and \( F \) does not contain \( J_{s,m} \). Obviously, \( zl_{11}, zl_{1k} \) are not in \( E(F) \) for each \( z \in V_1 \), where \( V_1 = V(F) \setminus V(L_1) \). Let \( L_2 = (l_{21}, l_{22}, \ldots, l_{2t-1}, l_{2t}) \) be a longest path in \( F[V_1] \). It is clear that \( 1 \leq t \leq k \). Let \( V_2 = V(F) \setminus (V(L_1) \cup V(L_2)) \). We distinguish three cases.
**Case 1:** $k < sm - 1$. If $t = 1$ then the vertices in $V_1$ induce a subgraph having only isolated vertices. In this case we shall add an edge $uv$ to $F$, where $u, v \in V_1$ and denote $L_2 = u, v$. In this way we can define inductively the system of paths $L_1, L_2, \ldots, L_{sm-1}$ such that $L_i$ is a longest path in $F[V_{i-1}]$, where $V_{i-1} = V(F) \setminus \bigcup_{j=1}^{i-1} V(L_j)$ or an edge added to $F$ as above. If $F_1$ denotes the graph $F$ or the graph $F$ plus some edges added in the process of defining the system of paths, it follows that endpoints of these $L_j$, where $j = 1, 2, \ldots, sm - 1$ induce in $F_1$ a complete graph $K_{sm-1}$ minus a matching having at most $\frac{sm-1}{2}$ edges if some of the endpoints of same $L_j$ are adjacent in $F_1$. Since $s, m \geq 3$ there exist at least two vertices $x, y$ which are not adjacent to all endpoints of these $L_j$. Thus, it is easy to see that vertices $x, y$ together with all endpoints of paths $L_j$ form a $J_{s, m} \subseteq \overline{F_1} \subseteq \overline{F}$.

**Case 2:** $k \geq sm - 1$ and $t \geq sm - 1$. For $i = 1, 2, \ldots, \frac{sm-3}{2}$ define the couples $A_i$ in path $L_1$ as follows:

$$A_i = \begin{cases} 
{l_{1i+1}, l_{1i+2}} & \text{for } i \text{ odd}, \\
{l_{1k-i}, l_{1k-i+1}} & \text{for } i \text{ even}.
\end{cases}$$

Similarly, define couples $B_i$ in path $L_2$ as follows:

$$B_i = \begin{cases} 
{l_{2i+1}, l_{2i+2}} & \text{for } i \text{ odd}, \\
{l_{2t-i}, l_{2t-i+1}} & \text{for } i \text{ even}.
\end{cases}$$

Since $t \leq k \leq n - 1$ and $|F| = 2n$, there exist at least two vertices $x, y$ which are not in $L_1 \cup L_2$. Since $L_1$ is a longest path in $F$, there exists one vertex of $A_i$ for each $i$, say $a_i$ which is not adjacent with $x$. Similarly, since $L_2$ is a longest path in $V(F) \setminus V(L_1)$ there must be one vertex, say $b_i$, in couple $B_i$ which is not adjacent to $x$ for each $i$. By maximality of path $L_1$, $b_i a_i$ and $a_i b_{i+1}$ are not in $E(F)$ for each $i$. Thus $\{l_1, b_1, a_1, b_2, a_2, \ldots, b_{sm-3}, a_{sm-3}, l_{2i}, y\}$ will form a cycle $C_{sm}$ in $\overline{F}$ and since $x$ is adjacent with at least $sm - 1$ vertices of cycle $C_{sm}$ in $\overline{F}$, we have a subgraph in $\overline{F}$ which contain $J_{s, m}$, so $J_{s, m} \subseteq \overline{F}$.

**Case 3:** $k \geq sm - 1$ and $t < sm - 1$. Since $k \leq n - 1$ ($F$ has no $P_n$), $V_1$ will have at least $n + 1$ vertices. Then, we can define the same process as in Case 1, since $n + 1 - (sm - 2) \frac{sm-1}{2} \geq \frac{sm+1}{2} \geq 5$. 

In the following theorem we derive Ramsey number $R(tP_n, J_{s, m})$ for any integer $t \geq 1$, even $s$ and $m \geq 3$, where $n$ is large enough with respect to $s$ and $m$ as follows.

**Theorem 3.** $R(tP_n, J_{s, m}) = tn + \frac{sm}{2} - 1$ if $n \geq (\frac{sm}{2} - 1)(2sm - 1) + 1$, $s \geq 2$ is even, $m \geq 3$ and $t$ is any positive integer.

**Proof:** Since graph $G = K_{sm-1} \cup K_{m-1}$ contains no $tP_n$ and $\overline{G}$ contains no $J_{s, m}$, then $R(tP_n, J_{s, m}) \geq tn + \frac{sm}{2} - 1$. For proving the upper bound, let $F$ be
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a graph of order \( tn + \frac{nm}{2} - 1 \) such that \( F \) contains no \( J_{s,m} \). We will show that \( F \) contains \( tP_n \). We use induction on \( t \). For \( t = 1 \) this is true from Theorem 1. Now, let assume that the theorem is true for all \( t' \leq t - 1 \). Take any graph \( F \) of \( tn + \frac{nm}{2} - 1 \) vertices such that its complement contains no \( J_{s,m} \). By the induction hypothesis, \( F \) must contain \( t - 1 \) disjoint copies of \( P_n \). Remove these copies from \( F \), then by Theorem 1 the subgraph \( F[H] \) on remaining vertices will induce another \( P_n \) in \( F \) since \( F \not\supseteq J_{s,m} \), so \( F[H] \not\supseteq J_{s,m} \). Therefore \( F \supseteq tP_n \). The proof is complete. \( \square \)

References


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