

A new particular second-order nonlinear differential subordination

by

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Abstract

We find conditions on the complex-valued functions A, B, C, D in the unit disc U such that the differential inequality

$$|A(z)z^2p''(z) + B(z)p^2(z) + C(z)p(z) + D(z)| < M$$

implies $|p(z)| < N$, where p is analytic in U , with $p(0) = 0$.

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1 Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}[U], f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[U], f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots, z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

In [1] chapter IV, the authors have analyzed a second-order linear differential subordination

$$A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z), \quad (1)$$

where A, B, C, D and h are complex-valued functions in the unit disc, and $p \in \mathcal{H}[0, n]$. A more general version of (1) is given by:

$$A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \in \Omega,$$

where $\Omega \subset \mathbb{C}$.

In [2] we found conditions on the complex-valued functions A, B, C, D in the unit disc U and the positive numbers M and N such that

$$|A(z)zp'(z) + B(z)p^2(z) + C(z)p(z) + D(z)| < M$$

implies $|p(z)| < N$, where $p \in \mathcal{H}[0, n]$.

In this paper we shall consider the following particular second-order nonlinear differential subordination given by the inequality

$$|A(z)z^2p''(z) + B(z)p^2(z) + C(z)p(z) + D(z)| < M, \quad (2)$$

where $p \in \mathcal{H}[0, n]$.

We find conditions on complex-valued functions A, B, C, D and the positive numbers M and N such that (2) implies

$$|p(z)| < N,$$

where $p \in \mathcal{H}[0, n]$.

In order to prove the new results we shall use the following lemma:

Lemma A. [1, p. 34] *Let $\psi : \mathbb{C}^2 \times U \rightarrow \mathbb{C}$ and $M > 0$, $N > 0$, n a positive integer satisfy*

$$|\psi(Ne^{i\theta}, L; z)| \geq M \quad (3)$$

whenever

$$\operatorname{Re}[Le^{-i\theta}] \geq (n-1)nN$$

$z \in U$ and $\theta \in \mathbb{R}$.

If $p \in \mathcal{H}[0, n]$ and

$$|\psi(p(z), z^2p''(z); z)| < M$$

then $|p(z)| < N$.

2 Main results

Theorem. *Let $M > 0$, $N > 0$, and let n be a positive integer. Suppose that the functions $A, B, C, D : U \rightarrow \mathbb{C}$ satisfy $A(z) \neq 0$,*

$$\operatorname{Re} \frac{C(z)}{A(z)} \geq \frac{M - (n-1)nN|A(z)| + N^2|B(z)| + |D(z)|}{N|A(z)|}. \quad (4)$$

If $p \in \mathcal{H}[0, n]$ and

$$|A(z)z^2p''(z) + B(z)p^2(z) + C(z)p(z) + D(z)| < M$$

then

$$|p(z)| < N.$$

Proof: Let $\psi : \mathbb{C}^2 \times U \rightarrow \mathbb{C}$ be defined by

$$\psi(p(z), z^2p''(z); z) = A(z)z^2p''(z) + B(z)p^2(z) + C(z)p(z) + D(z). \quad (5)$$

From (2) we have

$$|\psi(p(z), z^2p''(z); z)| < M, \text{ for } z \in U. \quad (6)$$

Using (4) in (5) we have

$$\begin{aligned} |\psi(Ne^{i\theta}, L; z)| &= |A(z)L + B(z)N^2e^{2i\theta} + C(z)Ne^{i\theta} + D(z)| = \\ &= |A(z)Le^{-i\theta} + B(z)N^2e^{i\theta} + NC(z) + D(z)e^{-i\theta}| = \\ &= |A(z)| \left| Le^{-i\theta} + \frac{B(z)}{A(z)}N^2e^{i\theta} + N\frac{C(z)}{A(z)} + \frac{D(z)}{A(z)} \right| \geq \\ &\geq |A(z)| \left[\left| Le^{-i\theta} + \frac{B(z)}{A(z)}N^2e^{i\theta} + N\frac{C(z)}{A(z)} \right| - \left| \frac{D(z)}{A(z)} \right| \right] \geq \\ &\geq |A(z)| \left[\left| Le^{-i\theta} + N\frac{C(z)}{A(z)} \right| - N^2 \left| \frac{B(z)}{A(z)} \right| - \left| \frac{D(z)}{A(z)} \right| \right] \geq \\ &\geq |A(z)| \left[\mathbb{R}Le^{-i\theta} + N\mathbb{R}\frac{C(z)}{A(z)} - N^2 \left| \frac{B(z)}{A(z)} \right| - \left| \frac{D(z)}{A(z)} \right| \right] \geq \\ &\geq |A(z)| \left[n(n-1)N + N\mathbb{R}\frac{C(z)}{A(z)} - N^2 \left| \frac{B(z)}{A(z)} \right| - \left| \frac{D(z)}{A(z)} \right| \right] \geq M. \end{aligned}$$

Hence condition (3) holds and by Lemma A we deduce that (6) implies $|p(z)| < N$. \square

Instead of prescribing the constant N in the Theorem, in some cases we can use (4) to determine an appropriate $N(M, n, A, B, C, D)$ so that (2) implies $|p(z)| < N$. This can be accomplished by solving (4) for N and by taking the supremum of the resulting function over U .

Condition (4) is equivalent to:

$$N^2|B(z)| - N|A(z)| \left[n(n-1) + \mathbb{R}e\frac{C(z)}{A(z)} \right] + M + |D(z)| \leq 0. \quad (7)$$

If we suppose $B(z) \neq 0$, then the inequality (7) holds if

$$|A(z)| \left[n(n-1) + \operatorname{Re} \frac{C(z)}{A(z)} \right] \geq 2\sqrt{|B(z)|[M + |D(z)|]}. \quad (8)$$

If (8) holds, then the roots of the trinomial in (7) are

$$N_{1,2} = \frac{X}{2|B(z)|}$$

where

$$X = |A(z)| \left[n(n-1) + \operatorname{Re} \frac{C(z)}{A(z)} \right] \pm \sqrt{|A(z)|^2 \left[n(n-1) + \operatorname{Re} \frac{C(z)}{A(z)} \right]^2 - 4|B(z)|[M + |D(z)|]}$$

We let

$$N = \frac{2[M + |D(z)|]}{Y}$$

where

$$Y = |A(z)| \left[n(n-1) + \operatorname{Re} \frac{C(z)}{A(z)} \right] + \sqrt{|A(z)|^2 \left[n(n-1) + \operatorname{Re} \frac{C(z)}{A(z)} \right]^2 - 4|B(z)|[M + |D(z)|]}$$

If this supremum is finite, the Theorem can be rewritten as follows:

Corollary 1. *Let $M > 0$ and let n be a positive integer. Suppose that $p \in \mathcal{H}[0, n]$ and let the functions $A, B, C, D : U \rightarrow \mathbb{C}$ with $A(z) \neq 0$.
If*

$$N = \sup_{|z| < 1} \frac{2[M + |D(z)|]}{Z} < \infty,$$

where

$$Z = |A(z)| \left[n(n-1) + \operatorname{Re} \frac{C(z)}{A(z)} \right] + \sqrt{|A(z)|^2 \left[n(n-1) + \operatorname{Re} \frac{C(z)}{A(z)} \right]^2 - 4|B(z)|[M + |D(z)|]},$$

then

$$|A(z)z^2 p''(z) + B(z)p^2(z) + C(z)p(z) + D(z)| < M$$

implies

$$|p(z)| < N.$$

Let $n = 2$, $A(z) = 4 + 2\sqrt{5}i$, $B(z) = \sqrt{3} - i$, $C(z) = 8 + 4\sqrt{5}i$, $D(z) = 1 + i\sqrt{3}$, $M = \frac{59}{2}$, we find $N = \frac{63}{42}$.

In this case from Corollary 1 we deduce

Example 1. If $p \in \mathcal{H}[0, 2]$, then

$$|(4 + 2\sqrt{5}i)z^2p''(z) + (\sqrt{3} - i)p^2(z) + (8 + 4\sqrt{5}i)p(z) + 1 + \sqrt{3}i| < \frac{59}{2}$$

implies

$$|p(z)| < \frac{63}{42}.$$

If $A(z) = A > 0$, then the Theorem can be rewritten as follows:

Corollary 2. Let $M > 0$, $N > 0$ and let n be a positive integer. Suppose that the functions $B, C, D : U \rightarrow \mathbb{C}$ satisfy

$$\operatorname{Re}C(z) \geq \frac{M - (n-1)nNA + N^2|B(z)| + |D(z)|}{N}.$$

If $p \in \mathcal{H}[0, n]$ and

$$|Az^2p''(z) + B(z)p^2(z) + C(z)p(z) + D(z)| < M$$

then

$$|p(z)| < N.$$

If $n = 3$, $A(z) = 4$, $B(z) = \sqrt{3} + i$, $C(z) = 20 - 4i$, $D(z) = 1 - \sqrt{3}i$, $M = 24$, $N = 1$. In this case from Corollary 2 we deduce

Example 2. If $p \in \mathcal{H}[0, 3]$ then

$$|4z^3p''(z) + (\sqrt{3} + i)p^2(z) + (20 - 4i)p(z) + 1 - \sqrt{3}i| < 24$$

implies

$$|p(z)| < 1.$$

References

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