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Integration of periodic function and applications on integration formulae of interpolatory type

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Abstract

Some remarks on integration of periodic function are given. A number of inequalities for functions whose $(2r-1)$ derivatives are increasing are proved.

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1 Introduction

In the recent paper [10] the following lemma has been proved:

Lemma 1. *Let* $\varphi(x) \downarrow 0$. *Then*

$$-\int_0^\infty \rho(x)\varphi(x)dx < \frac{1}{8}\varphi(0), \tag{1}$$

where $\rho(x) = x - \lfloor x \rfloor - \frac{1}{2}$.

The aim of this paper is to give a variant of this inequality, which involves generally periodic function ρ, and this will be done in Section 2. In Sections 3,4 and 5 we will use those results to prove a number inequalities for the general Euler two-point formula, Euler-Simpson formula, dual Euler-Simpson formula, Euler Simpson 3/8 formula, Euler Maclaurin formula, Euler-Boole formula and some formulae of Bullen type.

2 Main result

Theorem 1. *Let* $\varphi : I \to \mathbb{R}$, $I \subset \mathbb{R}$, *be a monotonic function, and let* $\rho : \mathbb{R} \to \mathbb{R}$ *be a periodic function with period* P *such that for some* $a \in \mathbb{R}$*and* $n \in \mathbb{N}$ $[a, a+nP] \subset I$. *Suppose that there exists some* $x_0 \in (a, a+P)$ *such that* $\rho(x_0) = 0$, $\rho(x) \geq 0$ *for all* $x \in [a, x_0)$ *and* $\rho(x) \leq 0$ *for all* $x \in (x_0, a+P]$. *Suppose also that* $\int_a^{a+P} \rho(x)\, dx = 0$. *If function* φ *is increasing on* $[a, a+nP]$, *then*

$$-\int_a^{a+nP} \rho(x)\, \varphi(x)\, dx \leq \frac{1}{2n} \left(\varphi(a+nP) - \varphi(a)\right) \int_a^{a+nP} |\rho(x)|\, dx, \qquad (2)$$

and this inequality is sharp. If function φ *is decreasing on* $[a, a+nP]$, *then inequality* (2) *is reversed.*

Proof: First we will consider the case of increasing function φ.

Since function ρ is periodic with period P, from the conditions on ρ we can deduce that for all $k \in \{0, ..., n-1\}$

$$\begin{aligned} \int_{a+kP}^{a+(k+1)P} \rho(x)dx &= 0, \\ \rho(x_k) &= 0, \quad x_k = x_0 + kP \\ \rho(x) &\geq 0, \quad x \in [a+kP, x_k) \\ \rho(x) &\leq 0, \quad x \in (x_k, a+(k+1)P]. \end{aligned}$$

Using these properties, we can easily obtain

$$\begin{aligned} -\int_a^{a+nP} \rho(x)\varphi(x)dx &= -\sum_{k=0}^{n-1} \int_{a+kP}^{a+(k+1)P} \rho(x)\left(\varphi(x) - \varphi(x_k)\right) dx \\ = & \sum_{k=0}^{n-1} \left[\int_{a+kP}^{x_k} \rho(x)\left(\varphi(x_k) - \varphi(a+kP)\right) dx \right. \\ + & \left. \int_{x_k}^{a+(k+1)P} \rho(x)\left(\varphi(x_k) - \varphi(a+(k+1)P)\right) dx + a_k \right] \\ = & \sum_{k=0}^{n-1} \left[\left(\varphi(x_k) - \varphi(a+kP)\right) \int_{a+kP}^{x_k} \rho(x)dx \right. \\ + & \left. \left(\varphi(x_k) - \varphi(a+(k+1)P)\right) \int_{x_k}^{a+(k+1)P} \rho(x)dx + a_k \right] \\ \leq & \left(\varphi(a+nP) - \varphi(a)\right) \frac{1}{2n} \int_a^{a+nP} |\rho(x)|dx + \sum_{k=0}^{n-1} a_k, \end{aligned}$$

where

$$a_k = \int_{a+kP}^{x_k} \rho(x)\left(\varphi(a+kP)-\varphi(x)\right)dx - \int_{x_k}^{a+(k+1)P} \rho(x)\left(\varphi(x)-\varphi(a+(k+1)P)\right)dx.$$

Due to the fact that φ is increasing function on I, we can deduce that for all $k \in \{0,...,n-1\}$ $a_k \le 0$, i.e., $\sum_{k=1}^{n-1} a_k \le 0$. Immediately follows that the inequality (2) is valid.

In order to prove the sharpness we will define function $\varphi : [a, a+nP] \to \mathbb{R}$ with

$$\varphi(x) = \begin{cases} a+kP, & x \in [a+kP, x_k] \\ a+(k+1)P, & x \in (x_k, a+(k+1)P] \end{cases}$$

for all $k \in \{0,...,n-1\}$. It is obvious that function φ is increasing on $[a, a+nP]$, and for any function ρ which fulfils the conditions of this theorem we have:

$$\begin{aligned} &-\int_a^{a+nP} \rho(x)\varphi(x)dx = -\sum_{k=0}^{n-1}\int_{a+kP}^{a+(k+1)P}\rho(x)\varphi(x)dx \\ =\ & -\sum_{k=0}^{n-1}\left[(a+kP)\int_{a+kP}^{x_k}\rho(x)dx + (a+(k+1)P)\int_{x_k}^{a+(k+1)P}\rho(x)dx\right] \\ =\ & -(a+kP)\sum_{k=0}^{n-1}\int_{a+kP}^{a+(k+1)P}\rho(x)dx - P\sum_{k=0}^{n-1}\int_{x_k}^{a+(k+1)P}\rho(x)dx \\ =\ & 0+\frac{P}{2}\sum_{k=0}^{n-1}\int_{a+kP}^{a+(k+1)P}|\rho(x)|dx = \frac{1}{2n}\left(\varphi(a+nP)-\varphi(a)\right)\int_a^{a+nP}|\rho(x)|dx, \end{aligned}$$

and this means that inequality (2) is sharp.

If function φ is decreasing on I, the reverse of (2) can be obtained in the similar way. To prove the sharpness, we can simply choose a decreasing function $\varphi : [a, a+nP] \to \mathbb{R}$ defined with

$$\varphi(x) = \begin{cases} a+(k+1)P, & x \in [a+kP, x_k] \\ a+kP, & x \in (x_k, a+(k+1)P] \end{cases},$$

for all $k \in \{0,...,n-1\}$. This completes the proof. □

Remark: If we consider inequality (2) for a periodic function τ with period P such that $\tau(x) \le 0$ on $[a+kP, x_0)$, $\tau(x_0) = 0$ and $\tau(x) \ge 0$ on $(x_0, a+(k+1)P]$, then we can use inequality (2) with function ρ defined as $\rho(x) = -\tau(x)$, for $x \in \mathbb{R}$.

3 Application on the general Euler two-point formula

In this section we shall show how can Theorem 1 be used in order to obtain a number of inequalities for some integration formulae of interpolatory type.

In the recent paper [11] the following identity, named the general Euler two-point formula, has been proved.

Let $f \in C^{2r-1}([a,b],\mathbf{R})$ for some $r \geq 2$, and let $y \in [a,(a+b)/2]$. We have

$$\begin{aligned}\int_a^b f(t)dt &= \frac{b-a}{2}\left[f(y)+f(a+b-y)\right]-T_{r-1}(y)\\ &+ \frac{(b-a)^{2r-1}}{2(2r-1)!}\int_a^b F_{2r-1}^x\left(\frac{z-a}{b-a}\right)f^{(2r-1)}(z)dz, \end{aligned} \tag{3}$$

where $x=\frac{y-a}{b-a}$. Here we define $T_0(y)=T_1(y)=0$, and for $k\geq 2$

$$T_k(y)=\sum_{j=2}^{k}\frac{(b-a)^{2j}}{(2j)!}B_{2j}\left(\frac{y-a}{b-a}\right)\left[f^{(2j-1)}(b)-f^{(2j-1)}(a)\right],$$

and for $t\in[0,1]$

$$F_k^x(t)=B_k^*(x-t)+B_k^*(1-x-t)-B_k(x)-B_k(1-x).$$

For any $k\geq 0$ $B_k(\cdot)$ denotes the k-th Bernoulli polynomial and $B_k=B_k(0)$ the k-th Bernoulli number. By $B_k^*(\cdot)$ we denote a periodic function with period 1 such that $B_k^*(t)=B_k(t)$, for $0\leq t<1$.

It has been proved in [11] that for any $t\in[0,1/2]$

$$F_k^x(1-t)=(-1)^kF_k^x(t),\quad k\geq 2,$$

and that for $r\geq 2$, $t\in[0,1]$

$$\begin{aligned}(-1)^{r-1}F_{2r-1}^x(t) &\geq 0,\quad x\in\left[0,\frac{1}{2}-\frac{1}{2\sqrt{3}}\right]\\ (-1)^{r}F_{2r-1}^x(t) &\geq 0,\quad x\in\left[\frac{1}{2\sqrt{3}},\frac{1}{2}\right].\end{aligned}$$

Also

$$\int_0^1\left|F_{2r-1}^x(t)\right|dt=\frac{2}{r}\left|B_{2r}\left(\frac{1}{2}-x\right)-B_{2r}(x)\right|.$$

Theorem 2. *Let $f:[a,b]\to\mathbf{R}$ be such that for some $r\geq 2$ derivative $f^{(2r-1)}$ is an increasing function on $[a,b]$. Then for $y\in\left[a,\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}}\right]$ the following inequality holds*

$$\begin{aligned}&(-1)^r\left\{\int_a^b f(t)dt-\frac{b-a}{2}\left[f(y)+f(a+b-y)\right]+T_{r-1}(y)\right\} \\ \leq\ &\frac{(b-a)^{2r}}{(2r)!}\left|B_{2r}\left(\frac{1}{2}-\frac{y-a}{b-a}\right)-B_{2r}\left(\frac{y-a}{b-a}\right)\right|\left[f^{(2r-1)}(b)-f^{(2r-1)}(a)\right].\end{aligned} \tag{4}$$

Also, for $y \in \left[a + \frac{b-a}{2\sqrt{3}}, \frac{a+b}{2}\right]$ *we have*

$$\begin{aligned} & (-1)^{r-1}\left\{\int_a^b f(t)dt - \frac{b-a}{2}\left[f\left(y\right) + f\left(a+b-y\right)\right] + T_{r-1}(y)\right\} \\ \leq\ & \frac{(b-a)^{2r}}{(2r)!}\left|B_{2r}\left(\frac{1}{2} - \frac{y-a}{b-a}\right) - B_{2r}(\frac{y-a}{b-a})\right|\left[f^{(2r-1)}(b) - f^{(2r-1)}(a)\right]. \end{aligned} \tag{5}$$

These two inequalities are sharp.

Proof: We know that function F_{2r-1}^{x} is periodic with period $P = 1$. It can be easily checked that: $F_{2r-1}^{x}(0) = F_{2r-1}^{x}(1/2) = F_{2r-1}^{x}(1) = 0$, $(-1)^{r-1} F_{2r-1}^{x}(t) > 0$ for $t \in (0, 1/2)$, $(-1)^{r-1} F_{2r-1}^{x}(t) < 0$ for $t \in (1/2, 1)$, and also $\int_0^1 F_{2r-1}^{x}(t)\, dt = 0$. This means that if in Theorem 1 we choose $\rho(t) = (-1)^{r-1} F_{2r-1}^{x}(t)$, $\varphi(t) = f^{(2r-1)}(t(b-a)+a)$ and $n = 1$, we obtain

$$\begin{aligned} & (-1)^r \int_a^b F_{2r-1}^{x}\left(\frac{z-a}{b-a}\right) f^{(2r-1)}(z)dz \\ =\ & (-1)^r(b-a)\int_0^1 F_{2r-1}^{x}(t)\, f^{(2r-1)}(t(b-a)+a)dt \\ <\ & \left(f^{(2r-1)}(b) - f^{(2r-1)}(a)\right)\frac{b-a}{2}\int_0^1 \left|F_{2r-1}^{x}(t)\right| dt \\ =\ & \frac{b-a}{r}\left|B_{2r}\left(\frac{1}{2} - x\right) - B_{2r}(x)\right|\left(f^{(2r-1)}(b) - f^{(2r-1)}(a)\right). \end{aligned}$$

and if we combine this with (3), we can easily obtain (5). The proof of the second statement is similar. □

Remark: If in (5) we let $y = a$, we obtain an inequality for trapezoid formula:

$$\begin{aligned} & (-1)^r\left\{\int_a^b f(t)dt - \frac{b-a}{2}\left[f\left(a\right) + f\left(b\right)\right] + T_{r-1}(a)\right\} \\ <\ & \frac{(b-a)^{2r}}{(2r)!}(2 - 2^{1-2r})\left|B_{2r}\right|\left(f^{(2r-1)}(b) - f^{(2r-1)}(a)\right). \end{aligned}$$

If in (6) we let $y = (a+b)/2$, we obtain an inequality for mid-point formula:

$$\begin{aligned} & (-1)^{r-1}\left\{\int_a^b f(t)dt - (b-a)f\left(\frac{a+b}{2}\right) + T_{r-1}\left(\frac{a+b}{2}\right)\right\} \\ <\ & \frac{(b-a)^{2r}}{(2r)!}(2 - 2^{1-2r})\left|B_{2r}\right|\left(f^{(2r-1)}(b) - f^{(2r-1)}(a)\right) \end{aligned}$$

and also for $y = (2a+b)/3$ in inequality (6), we get inequality for two-point Newton-Cotes formula:

$$(-1)^{r-1}\left\{\int_a^b f(t)dt - \frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]+T_{r-1}\left(\frac{2a+b}{3}\right)\right\}$$
$$< \frac{(b-a)^{2r}}{(2r)!}(1-3^{1-2r})(1-2^{-2r})\left|B_{2r}\right|\left(f^{(2r-1)}(b)-f^{(2r-1)}(a)\right),$$

which is an improvement of the Theorem 9 from [7].

4 Application on the some formulae of the higher order

If f is defined on segment $[a,b]$ and has $2r-1$, $r\geq 2$, continuous derivatives there, then the Euler-Simpson formula (see [1]) state that

$$\int_a^b f(t)dt = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$
$$+ \quad T_{r-1}^S(f)+\frac{(b-a)^{2r-1}}{3(2r-1)!}\int_a^b F_{2r-1}^S\left(\frac{z-a}{b-a}\right)f^{(2r-1)}(z)dz,$$

where $T_1^S(f)=T_2^S(f)=0$ and for $k\geq 3$

$$T_k^S(f)=\sum_{j=1}^{k}\frac{(b-a)^{2j}}{3(2j)!}(1-2^{2-2j})B_{2j}\left[f^{(2j-1)}(b)-f^{(2j-1)}(a)\right],$$

and for $k\geq 2$, $t\in[0,1]$

$$F_k^S(t)=B_k^*(1-t)+2B_k^*\left(\frac{1}{2}-t\right)-B_k-2B_k\left(\frac{1}{2}\right).$$

It has been proved in [6] that for any $t\in[0,1/2]$

$$F_k^S(1-t)=(-1)^k F_k^S(t),\ \ k\geq 2,$$
$$(-1)^r F_{2r-1}^S(t)\geq 0,\ r\geq 2.$$

Also (see [1])

$$\int_0^1\left|F_{2r-1}^S(t)\right|dt=\frac{1}{r}\left(2-2^{1-2r}\right)\left|B_{2r}\right|.$$

We can parallel the development of the second section with the following theorem.

Theorem 3. *Let $f:[a,b]\to\mathbf{R}$ be such that for some $r\geq 2$ derivative $f^{(2r-1)}$ is an increasing function on $[a,b]$. Then the following inequality holds*

$$(-1)^{r-1}\left\{\int_a^b f(t)dt-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-T_r^S(f)\right\}$$
$$\leq \frac{(b-a)^{2r}}{3(2r)!}(2-2^{1-2r})|B_{2r}|\left[f^{(2r-1)}(b)-f^{(2r-1)}(a)\right]$$

and this inequality is sharp.

Remark: This theorem is an improvement of the Theorem 5 from [6, p. 225].

If function f is defined on segment $[a,b]$ and has $2r-1$, $r \geq 2$, continuous derivatives there, then the dual Euler-Simpson formula (see [2]) state that

$$\begin{aligned}\int_a^b f(t)dt &= \frac{b-a}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right] - T_{r-1}^D(f) \\ &+ \frac{(b-a)^{2r-1}}{3(2r-1)!}\int_a^b F_{2r-1}^D\left(\frac{z-a}{b-a}\right) f^{(2r-1)}(z)dz,\end{aligned}$$

where $T_0^D(f) = T_1^D(f) = 0$, and for $k \geq 2$

$$T_k^D(f) = \sum_{j=1}^k \frac{(b-a)^{2j}}{3(2j)!}(2^{3-4j} - 3\cdot 2^{1-2j} + 1)B_{2j}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right],$$

and for $t \in [0,1]$

$$\begin{aligned}F_k^D &= 2B_k^*\left(\frac{1}{4}-t\right) - B_k^*\left(\frac{1}{2}-t\right) + 2B_k^*\left(\frac{3}{4}-t\right) \\ &\quad -2B_k\left(\frac{1}{4}\right) + B_k\left(\frac{1}{2}\right) - 2B_k\left(\frac{3}{4}\right).\end{aligned}$$

It has been proved in [2] that for $t \in [0,1/2]$

$$F_k^D(1-t) = (-1)^k F_k^D(t), \quad k \geq 2,$$

$$(-1)^{r-1}F_{2r-1}^D(t) \geq 0, \quad r \geq 2.$$

Also

$$\int_0^1 \left|F_{2r-1}^D(t)\right| dt = \frac{2}{r}(1-2^{-2r})|B_{2r}|.$$

Again, we can parallel the development of the second section with the following theorem.

Theorem 4. *Let $f : [a,b] \to \mathbf{R}$ be such that for some $r \geq 2$ derivative $f^{(2r-1)}$ is an increasing function on $[a,b]$. Then the following inequality holds*

$$\begin{aligned}&(-1)^r\left\{\int_a^b f(t)dt - \frac{b-a}{3}\left[f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + f\left(\frac{a+3b}{4}\right)\right]\right. \\ &+ \left. T_{r-1}^D(f)\right\} \leq \frac{(b-a)^{2r}}{3(2r)!}(2-2^{1-2r})|B_{2r}|\left[f^{(2r-1)}(b) - f^{(2r-1)}(a)\right],\end{aligned}$$

and this inequality is sharp.

Remark: This theorem is an improvement of the Theorem 9 from [2].

If f is defined on segment $[a,b]$ and has $2r-1,\ r\geq 2$, continuous derivatives there, then the Euler-Simpson 3/8 formula (see [4]) state that

$$\begin{aligned}\int_a^b f(t)dt &= \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]+T_{r-1}^{3/8}(f)\\ &+\frac{(b-a)^{2r-1}}{8(2r-1)!}\int_a^b F_{2r-1}^{3/8}\left(\frac{z-a}{b-a}\right)f^{(2r-1)}(z)dz,\end{aligned}$$

where $T_0^{3/8}(f)=T_1^{3/8}(f)=0$ and for $k\geq 2$

$$T_k^{3/8}(f)=\sum_{j=1}^{k}\frac{(b-a)^{2j}}{8(2j)!}(1-3^{2-2j})B_{2j}\left[f^{(2j-1)}(b)-f^{(2j-1)}(a)\right],$$

and for $t\in[0,1]$

$$\begin{aligned}F_k^{3/8}(t) &= 2B_k^*(1-t)+3B_k^*\left(\frac{1}{3}-t\right)\\ &+3B_k^*\left(\frac{2}{3}-t\right)-2B_k-3B_k\left(\frac{1}{3}\right)-3B_k\left(\frac{2}{3}\right).\end{aligned}$$

It has been proved in [4] that for any $t\in[0,1/2]$

$$F_k^{3/8}(1-t)=(-1)^k F_k^{3/8}(t),\ k\geq 2,$$

$$(-1)^r F_{2r-1}^{3/8}(t)\geq 0,\ r\geq 2.$$

Also

$$\int_0^1\left|F_{2r-1}^{3/8}(t)\right|dt=\frac{1}{r}(2-2^{1-2r})(1-3^{2-2r})|B_{2r}|.$$

We can parallel the development of the second section with the following theorem.

Theorem 5. *Let $f:[a,b]\to\mathbf{R}$ be such that for some $r\geq 2$ derivative $f^{(2r-1)}$ is an increasing function on $[a,b]$. Then the following inequality holds*

$$\begin{aligned}&(-1)^{r-1}\left\{\int_a^b f(t)dt-\frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]\right.\\ -&\left.T_{r-1}^{3/8}(f)\right\}\leq\frac{(b-a)^{2r}}{4(2r)!}(1-2^{-2r})(1-3^{2-2r})|B_{2r}|\left[f^{(2r-1)}(b)-f^{(2r-1)}(a)\right],\end{aligned}$$

and this inequality is sharp.

Remark: This theorem is an improvement of the Theorem 11 from [4].

If f is defined on segment $[a,b]$ and has $2r-1,\ r\geq 2$, continuous derivatives there, then the Euler-Maclaurin formula (see [3]) state that

$$\begin{aligned}\int_a^b f(t)dt &= \frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-T_{r-1}^L(f)\\ &+ \frac{(b-a)^{2r}}{8(2r-1)!}\int_a^b F_{2r-1}^L\left(\frac{z-a}{b-a}\right)f^{(2r-1)}(z)dz,\end{aligned}$$

where $T_0^L(f)=T_1^L(f)=T_2^L(f)=0$ and for $k\geq 3$

$$T_k^L(f)=\sum_{j=1}^{k}\frac{(b-a)^{2j}}{8(2j)!}(1-2^{1-2j})(1-3^{2-2j})B_{2j}\left[f^{(2j-1)}(b)-f^{(2j-1)}(a)\right],$$

and for $k\geq 2$, $t\in[0,1]$

$$\begin{aligned}F_k^L &= 3B_k^*\left(\frac{1}{6}-t\right)+2B_k^*\left(\frac{1}{2}-t\right)+3B_k^*\left(\frac{5}{6}-t\right)\\ &\quad -3B_k\left(\frac{1}{6}\right)-2B_k\left(\frac{1}{2}\right)-3B_k\left(\frac{5}{6}\right).\end{aligned}$$

It has been proved in [3] that

$$F_k^L(1-t)=(-1)^kF_k^L(t),\ k\geq 2$$

$$(-1)^{r-1}F_{2r-1}^L(t)\geq 0,\ r\geq 2$$

Also

$$\int_0^1\left|F_{2r-1}^L(t)\right|dt=\frac{1}{r}(2-2^{1-2r})(1-3^{2-2r})|B_{2r}|.$$

We can parallel the development of the second section with the following theorem.

Theorem 6. *Let $f:[a,b]\to\mathbf{R}$ be such that for some $r\geq 2$ derivative $f^{(2r-1)}$ is an increasing function on $[a,b]$. Then the following inequality holds*

$$\begin{aligned}&(-1)^r\left\{\int_a^b f(t)dt-\frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right.\\ &+\left.T_{r-1}^L(f)\right\}\leq\frac{(b-a)^{2r-1}}{4(2r)!}(1-2^{-2r})(1-3^{2-2r})|B_{2r}|\left[f^{(2r-1)}(b)-f^{(2r-1)}(a)\right],\end{aligned}$$

and this inequality is sharp.

Remark: This theorem is an improvement of the Theorem 11 from [3].

If f is defined on segment $[a,b]$ and has $2r-1$, $r \geq 3$, continuous derivatives there, then the Euler-Boole formula (see [12]) state that

$$\begin{aligned}\int_a^b f(t)dt &= \frac{b-a}{90}\left[7f(a)+32f\left(\frac{3a+b}{4}\right)+12f\left(\frac{a+b}{2}\right)+32f\left(\frac{a+3b}{4}\right)\right.\\ &\left.+7f(b)\right]-T_{r-1}^B(f)+\frac{(b-a)^{2r-1}}{90(2r-1)!}\int_a^b F_{2r-1}^B\left(\frac{z-a}{b-a}\right)f^{(2r-1)}(z)dz,\end{aligned}$$

where $T_0^B(f)=T_1^B(f)=0$ and for $k \geq 2$

$$T_k^B(f)=\sum_{j=1}^{k}\frac{(b-a)^{2k}}{90(2k)!}(2-5\cdot 2^{3-2k}+2^{7-4k})B_{2k}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right],$$

and for $t \in [0,1]$

$$\begin{aligned}F_k^B(t) &= 14B_k^*(1-t)+32B_k^*\left(\frac{1}{4}-t\right)+12B_k^*\left(\frac{1}{2}-t\right)\\ &+32B_k^*\left(\frac{3}{4}-t\right)-14B_k-32B_k\left(\frac{1}{4}\right)-12B_k\left(\frac{1}{2}\right)-32B_k^*\left(\frac{3}{4}\right).\end{aligned}$$

It has been proved in [12] that for $t \in [0,1/2]$

$$F_k^B(1-t)=(-1)^k F_k^B(t),\ k \geq 5,$$
$$(-1)^{r-1}F_{2r-1}^B(t) \geq 0,\ r \geq 3.$$

Also

$$\int_0^1 \left|F_{2r-1}^B(t)\right| dt=\frac{2}{r}(2-2^{1-2r})|B_{2r}|.$$

We can parallel the development of the second section with the following theorem.

Theorem 7. *Let $f:[a,b]\to \mathbf{R}$ be such that for some $r \geq 3$ derivative $f^{(2r-1)}$ is an increasing function on $[a,b]$. Then the following inequality holds*

$$\begin{aligned}&(-1)^r\left\{\int_a^b f(t)dt-\frac{b-a}{90}\left[7f(a)+32f\left(\frac{3a+b}{4}\right)+12f\left(\frac{a+b}{2}\right)\right.\right.\\ &+ \left.\left.32f\left(\frac{a+3b}{4}\right)+7f(b)\right]+T_{r-1}^B(f)\right\}\\ &\leq \frac{(b-a)^{2r}}{45(2r)!}(2-2^{1-2r})|B_{2r}|\left[f^{(2r-1)}(b)-f^{(2r-1)}(a)\right],\end{aligned}$$

and this inequality is sharp.

Remark: This theorem is an improvement of the Theorem 7 from [12].

5 Application on the some formulae of Bullen type

If f is defined on segment $[a,b]$ and has $2r-1,\ r\geq 2$, continuous derivatives there, then the Euler bitrapezoid formula (see [5]) state that

$$\begin{aligned}\int_a^b f(t)dt &= \frac{b-a}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]\\ &- T_{r-1}^H(f)+\frac{(b-a)^{2r-1}}{2(2r-1)!}\int_a^b F_{2r-1}^H\left(\frac{z-a}{b-a}\right)f^{(2r-1)}(z)dz,\end{aligned}$$

where $T_0^H(f)=T_1^H(f)=0$ and for $k\geq 2$

$$T_k^H(f)=\sum_{j=1}^{k}\frac{(b-a)^{2j}}{(2j)!}2^{1-2j}B_{2j}\left[f^{(2j-1)}(b)-f^{(2j-1)}(a)\right],$$

and for $t\in[0,1]$

$$F_k^H=B_k^*\left(\frac{1}{2}-t\right)+B_k^*(1-t)-B_k\left(\frac{1}{2}\right)-B_k.$$

It has been proved in [5] that for $t\in[0,1/2]$

$$F_k^H\left(\frac{1}{2}-t\right)=(-1)^kF_k^H(t),\ k\geq 2$$

and for $t\in[0,1/4]$

$$(-1)^{r-1}F_{2r-1}^H(t)\geq 0,\ r\geq 2.$$

Also

$$\int_0^1\left|F_{2r-1}^H(t)\right|dt=\frac{2^{3-2r}}{r}(1-2^{-2r})|B_{2r}|.$$

We can parallel the development of the second section with the following theorem.

Theorem 8. *Let $f:[a,b]\to\mathbf{R}$ be such that for some $r\geq 2$ derivative $f^{(2r-1)}$ is an increasing function on $[a,b]$. Then the following inequality holds*

$$\begin{aligned}&(-1)^r\left\{\int_a^b f(t)dt-\frac{b-a}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]+T_{r-1}^H(f)\right\}\\ \leq\ &\frac{(b-a)^{2r}}{(2r)!}2^{1-2r}(1-2^{-2r})|B_{2r}|\left[f^{(2r-1)}(b)-f^{(2r-1)}(a)\right],\end{aligned}$$

and this inequality is sharp.

Proof: The proof is similar as in Theorem 2, only in this case we have $n=2$. □

Remark: This theorem is an improvement of the Theorem 11 from [5].

If f is defined on segment $[a,b]$ and has $2r-1,\ r\geq 2$, continuous derivatives there, then the Bullen-Simpson formula of Euler type (see [8]) state that

$$\begin{aligned}\int_a^b f(t)dt &= \frac{b-a}{12}\left[f(a)+4f\left(\frac{3a+b}{4}\right)+2f\left(\frac{a+b}{2}\right)+4f\left(\frac{a+3b}{4}\right)+f(b)\right]\\ &+ T_{r-1}^{BS}(f)+\frac{(b-a)^{2r-1}}{6(2r-1)!}\int_a^b F_{2r-1}^{BS}\left(\frac{z-a}{b-a}\right)f^{(2r-1)}(z)dz,\end{aligned}$$

where $T_0^{BS}(f)=T_1^{BS}(f)=0$ and for $k\geq 2$

$$T_k^{BS}(f)=\sum_{j=1}^{k}\frac{(b-a)^{2j}}{3(2j)!}2^{-2j}(1-2^{2-2j})B_{2j}\left[f^{(2j-1)}(b)-f^{(2j-1)}(a)\right],$$

and for $t\in[0,1]$

$$\begin{aligned}F_k^{BS}(t) &= B_k^*(1-t)+2B_k^*\left(\frac{1}{4}-t\right)+B_k^*\left(\frac{1}{2}-t\right)\\ &+2B_k^*\left(\frac{3}{4}-t\right)-B_k-2B_k\left(\frac{1}{4}\right)-B_k\left(\frac{1}{2}\right)-2B_k\left(\frac{3}{4}\right).\end{aligned}$$

It has been proved in [5] that for $t\in[0,1/2]$

$$F_k^{BS}\left(\frac{1}{2}-t\right)=(-1)^kF_k^{BS}(t),\ k\geq 2$$

and for $t\in[0,1/4]$

$$(-1)^rF_{2r-1}^{BS}(t)\geq 0,\ r\geq 2.$$

Also

$$\int_0^1\left|F_{2r-1}^{BS}(t)\right|dt=\frac{2^{3-2r}}{r}(1-2^{-2r})|B_{2r}|.$$

We can parallel the development of the second section with the following theorem.

Theorem 9. *Let $f:[a,b]\to\mathbf{R}$ be such that for some $r\geq 2$ derivative $f^{(2r-1)}$ is an increasing function on $[a,b]$. Then the following inequality holds*

$$\begin{aligned}&(-1)^{r-1}\left\{\int_a^b f(t)dt-\frac{b-a}{12}\left[f(a)+4f\left(\frac{3a+b}{4}\right)+2f\left(\frac{a+b}{2}\right)\right.\right.\\ &+ \left.\left.4f\left(\frac{a+3b}{4}\right)+f(b)\right]-T_{r-1}^{BS}(f)\right\}\\ &\leq \frac{(b-a)^{2r}}{3(2r)!}2^{1-2r}(1-2^{-2r})|B_{2r}|\left(f^{(2r-1)}(b)-f^{(2r-1)}(a)\right),\end{aligned}$$

and this inequality is sharp.

Remark: This theorem is improvement of the Theorem 2 from [8].

If f is defined on segment $[a,b]$ and has $2r-1,\ r \geq 2$, continuous derivatives there, then the Bullen-Simpson 3/8 formula of Euler type (see [9]) state that

$$\begin{aligned}\int_a^b f(t)dt &= \frac{b-a}{16}\left[f(a) + 3f\left(\frac{5a+b}{6}\right) + 3f\left(\frac{2a+b}{3}\right) + 2f\left(\frac{a+b}{2}\right)\right.\\ &+ \left. 3f\left(\frac{a+2b}{3}\right) + 3f\left(\frac{a+5b}{6}\right) + f(b)\right] + T_{r-1}^{BL}(f)\\ &+ \frac{(b-a)^{2r-1}}{16(2r-1)!}\int_a^b F_{2r-1}^{BL}\left(\frac{z-a}{b-a}\right) f^{(2r-1)}(z)dz,\end{aligned}$$

where $T_0^{BL}(f) = T_1^{BL}(f) = 0$ and for $k \geq 2$

$$T_k^{BL}(f) = \sum_{j=1}^{k} \frac{(b-a)^{2j}}{8(2j)!} 2^{-2j}(1-3^{2-2j})B_{2k}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right],$$

and for $t \in [0,1]$

$$\begin{aligned}F_k^{BL}(t) &= 2B_k^*(1-t) + 3B_k^*\left(\frac{1}{6}-t\right) + 3B_k^*\left(\frac{1}{3}-t\right)\\ &\quad +2B_k^*\left(\frac{1}{2}-t\right) + 3B_k^*\left(\frac{2}{3}-t\right) + 3B_k^*\left(\frac{5}{6}-t\right)\\ &\quad -3B_k\left(\frac{1}{6}\right) - 3B_k\left(\frac{1}{3}\right) - 2B_k\left(\frac{1}{2}\right) - 2B_k\left(\frac{2}{3}\right) - 3B_k\left(\frac{5}{6}\right) - 2B_k\end{aligned}$$

It has been proved in [9] that for $t \in [0,1/2]$

$$F_k^{BL}\left(\frac{1}{2}-t\right) = (-1)^k F_k^{BL}(t),\ \ k \geq 2$$

and for $t \in [0,1/4]$

$$(-1)^r F_{2r-1}^{BL}(t) \geq 0,\ \ r \geq 2.$$

Also

$$\int_0^1 \left|F_{2r-1}^{BL}(t)\right| dt = \frac{2^{3-2r}}{r}\left(1-2^{-2r}\right)\left(1-3^{2-2r}\right)|B_{2r}|.$$

We can parallel the development of the second section with the following theorem.

Theorem 10. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(2r-1)}$ is an increasing function on $[a,b]$, for some $r\geq 2$. Then the following inequality holds*

$$\begin{aligned}&(-1)^{r-1}\left\{\int_a^b f(t)dt-\frac{b-a}{16}\left[f(a)+3f\left(\frac{5a+b}{6}\right)+3f\left(\frac{2a+b}{3}\right)\right.\right.\\+\quad&\left.\left.2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+2b}{3}\right)+3f\left(\frac{a+5b}{6}\right)+f(b)\right]-T_{r-1}^{BL}(f)\right\}\\\leq\quad&\frac{(b-a)^{2r}}{4(2r)!}2^{-2r}(1-2^{-2r})(1-3^{2-2r})|B_{2r}|\left[f^{(2r-1)}(b)-f^{(2r-1)}(a)\right],\end{aligned}$$

and this inequality is sharp.

Remark: This theorem is an improvement of the Theorem 2 from [9].

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