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Starlike curves with regular refraction property

by
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Abstract

We obtain the regular refraction interval for a given starlike curve Γ, with respect to a point $p \notin \Gamma$, denoted by $RRI[\Gamma, p]$ and we give several examples. In particular, we deduce that if Γ is a circle and the point p lies inside of Γ, then $RRI[\Gamma, p] = [0, r/d]$, where r is the radius of Γ and d is the distance from p to the center of Γ.

Key Words: Starlike curve, refraction property.
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1 Preliminaries

Let Γ be a smooth curve with parametrization $z = z(t)$, $t \in [\alpha, \beta]$ and suppose that Γ is a directed arc, the direction being that determined as t increases. The arc Γ is said to be starlike, with respect to a point $p \notin \Gamma$, if $\arg[z(t) - p]$ is an increasing function of t on $[\alpha, \beta]$, i.e. if

$$\frac{\mathrm{d}}{\mathrm{d}\,t} \arg[z(t) - p] > 0,\ t \in [\alpha, \beta].$$

For simplicity we will suppose that $p = 0$. Let Γ be starlike and let R be the radius vector from the origin to the point $z(t) \in \Gamma$. Let N be the outer normal to Γ at the point $z(t)$ and let denote by ω the angle between R and N. Let γ be a positive number and let consider the vector V starting from $z(t)$ such that

$$\sin \psi = \gamma \sin \omega \tag{1}$$

where ψ is the angle between V and N. We remark that if Γ separates two media with different indices of refraction and R and V are the trajectories of the light in these two media, then (1) is the well-known refraction law.

Definition 1. *We say that the curve Γ has the regular refraction property of index γ, with respect to the point $p=0$, written $\Gamma \in RP(\gamma,0)$, if the argument of the vector V is a nondecreasing function of t on $[\alpha,\beta]$, i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\arg V(t) \geq 0,\ t\in[\alpha,\beta]. \tag{2}$$

The above definition was given in [1] and [2] in the case $\gamma\in[0,1]$.

For a given starlike curve, with respect to the origin, we will find the largest interval $[\gamma_0,\gamma_1]$, $\gamma_0<1<\gamma_1$, such that $\Gamma\in RP(\gamma,0)$, for all $\gamma\in[\gamma_0,\gamma_1]$.

2 Main results

If we let $\chi=\arg V(t)$ and $\varphi=\arg z(t)$, then

$$\chi=\varphi+\omega-\psi. \tag{3}$$

We have

$$\varphi'=\operatorname{Im}\frac{z'}{z},\ \omega'=\operatorname{Im}\left[\frac{z''}{z'}-\frac{z'}{z}\right],\ \psi'=\frac{\gamma\cos\omega}{(1-\gamma^2+\gamma^2\cos^2\omega)^{1/2}}\omega',$$

where

$$\cos\omega=\frac{\operatorname{Im}\frac{z'}{z}}{\left|\frac{z'}{z}\right|}.$$

By using (3), the inequality (2), i.e. $\chi'\geq 0$, is equivalent to

$$\gamma\left(\operatorname{Im}\frac{z'}{z}\right)^2+\left[\sqrt{\Delta}-\gamma\operatorname{Im}\frac{z'}{z}\right]\operatorname{Im}\frac{z''}{z'}\geq 0,\ t\in[\alpha,\beta], \tag{4}$$

where

$$\Delta=(1-\gamma^2)\left|\frac{z'}{z}\right|^2+\gamma^2\left(\operatorname{Im}\frac{z'}{z}\right)^2\geq 0. \tag{5}$$

The first condition that we will impose is $\Delta\geq 0$, which is equivalent to $\gamma\mid\sin\omega\mid\leq 1$. This means that to have regular refraction property of Γ, we first put the condition (5) in order to have refraction. Otherwise at the points $z(t)$, where $\Delta<0$ we have total reflection.

Since Γ is starlike we have $\varphi'=\operatorname{Im}\dfrac{z'}{z}>0$.

If we let

$$F=F(\gamma,t)=1-\gamma\frac{\operatorname{Im}\frac{z'}{z}}{\sqrt{\Delta}}, \tag{6}$$

where $\operatorname{Im}\frac{z'}{z}>0$ and $\Delta>0$, then condition (4) can be rewritten as

$$(1-F)\operatorname{Im}\frac{z'}{z}+F\operatorname{Im}\frac{z''}{z'}\geq 0. \tag{7}$$

We remark that $0 \leq F \leq 1$, for $\gamma \in [0,1]$. If Γ is convex, then $\operatorname{Im} \dfrac{z''}{z'} \geq 0$, and we deduce that (7) holds for all $\gamma \in [0,1]$. Hence we deduce the following interesting property (see [1] and [2]):

Theorem 1. *Any convex curve has the regular refraction property, with respect to the origin, for all* $\gamma \in [0,1]$.

Since $F(1,t) = 0$ and F is decreasing with respect to γ, we deduce that if (7) holds for certain γ, then it also holds for all γ' between γ and 1. Hence, for a given starlike curve Γ, a natural problem is to find the largest interval $I = [\gamma_0, \gamma_1]$, with $1 \in I$, such that Γ has the regular refraction property for all $\gamma \in I$.

Definition 2. *We call regular refraction interval of the starlike curve* Γ*, with respect to the point* $p = 0$*, written* $RRI[\Gamma, 0]$*, the largest interval* $[\gamma_0, \gamma_1]$*,* $\gamma_0 \leq 1 \leq \gamma_1$*, such that* $\Gamma \in RP(\gamma, 0)$*, for all* $\gamma \in [\gamma_0, \gamma_1]$*.*

If Γ is convex then $\gamma_0 = 0$ and in this case we only have to find the maximum value of $\gamma_1 \geq 1$ such that $\Gamma \in RP(\gamma, 0)$ for all $\gamma \in [0, \gamma_1]$. We solved this problem when Γ is an ellipse [3] and a parabolic lens [4].

The following result will allow us to obtain the interval $RRI[\Gamma, p]$ for any starlike curve Γ (with respect to the point $p \notin \Gamma$).

Theorem 2. *Let* Γ *be a starlike curve,with respect to the point* $p = 0 \notin \Gamma$*, defined by the equation* $z = z(t)$*,* $t \in [\alpha, \beta]$*. Let* $RRI[\gamma, 0] = [\gamma_0, \gamma_1]$ *and let denote*

$$A = A(t) = \operatorname{Re} \frac{z'(t)}{z(t)},\ B = B(t) = \operatorname{Im} \frac{z'(t)}{z(t)},\ C = C(t) = \operatorname{Im} \frac{z''(t)}{z'(t)}, \tag{8}$$

with $B(t) > 0$*, for* $t \in [\alpha, \beta]$*. Let*

$$\delta = \delta(t) = \sqrt{1 + \frac{B^2}{A^2}} \tag{9}$$

and

$$\sigma = \sigma(t) = \sqrt{\frac{(A^2 + B^2)C^2}{B^2(B - C)^2 + A^2C^2}}. \tag{10}$$

I) If $C(t) \geq 0$ *and* $B(t) - C(t) \geq 0$*, for all* $t \in [\alpha, \beta]$*, then* $\gamma_0 = 0$ *and* $\gamma_1 = \delta_1 = \min\{\delta(t), t \in [\alpha, \beta]\}$*.*

II) If $C(t) \geq 0$*, for all* $t \in [\alpha, \beta]$ *and* $T_1 = \{t \in [\alpha, \beta], B(t) - C(t) < 0\} \neq \emptyset$*, then* $\gamma_0 = 0$ *and* $\gamma_1 = \sigma_1 = \min\{\sigma(t), t \in T_1\}$*.*

III) If $T_0 = \{t \in [\alpha, \beta], C(t) < 0\} \neq \emptyset$*, then* $\gamma_0 = \sigma_0 = \max\{\sigma(t), t \in T_0\}$ *and* $\gamma_1 = \min\{\delta_1, \sigma_1\}$*.*

Proof: From (5) we deduce $\Delta = A^2 + B^2 - \gamma^2 A^2$ and the condition (5) holds if and only if $\gamma \leq \delta_1$. With the notations (8) condition (7) becomes

$$\gamma B(B - C) + C\sqrt{\Delta} \geq 0. \tag{11}$$

In the case I), the curve Γ is convex and so $\gamma_0 = 0$. Since $C \geq 0$ and $B - C \geq 0$, the inequality (11) holds and we deduce that $\gamma_1 = \delta_1$.

In the case II) we also have $\gamma_0 = 0$ and by using (11) we successively have, for $t \in T_1$,

$$C\sqrt{\Delta} \geq \gamma B(C - B)$$

$$C^2(A^2 + B^2 - \gamma^2 A^2) \geq \gamma^2 B^2 (B - C)^2$$

and

$$\gamma^2 \leq \sigma^2(t),$$

where σ is given by (10). Hence $\gamma_1 = \sigma_1$.

In the case III), by using (11) and $C < 0$, $B - C > 0$, we successively have, for $t \in T_0$,

$$-C\sqrt{\Delta} \leq \gamma B(B - C)$$

$$C^2(A^2 + B^2 - \gamma^2 A^2) \geq \gamma^2 B^2 (B - C)^2$$

and

$$\gamma^2 \geq \sigma^2(t).$$

Hence $\gamma_0 = \sigma_0$ and $\gamma_1 = \min\{\delta_1, \sigma_1]$. □

In the next sections we will give some applications of Theorem 2.

3 The regular refraction interval of a circle

Theorem 3. *The regular refraction interval of a circle* Γ*, with respect to a point* p *inside* Γ*, is given by* $[0, r/d]$*, where* r *is the radius of* Γ *and* d *is the distance from* p *to the center of* Γ.

Proof: Without loss of generality we can suppose that the equation of the circle Γ is given by

$$z = \cos t - \lambda + i \sin t, \ t \in [-\pi, \pi], \tag{12}$$

where $\lambda \in (0, 1)$.

We remark that Γ is a circle with center at $-\lambda$ and radius 1. The origin lies inside Γ and $r/d = 1/\lambda$. We shall show that

$$RRI[\Gamma, 0] = \frac{1}{\lambda}.$$

By using (8), from (12) we deduce

$$A = \frac{\lambda \sin t}{1+\lambda^2 - 2\lambda \cos t}, \quad B = \frac{1-\lambda \cos t}{1+\lambda^2 - 2\lambda \cos t}, \quad C = 1.$$

We also have

$$B - C = \lambda \frac{\cos t - \lambda}{1+\lambda^2 - 2\lambda \cos t}$$

and from (9) we obtain

$$\delta(t) = \sqrt{\frac{1+\lambda^2 - 2\lambda \cos t}{\lambda^2 \sin^2 t}}$$

and

$$\delta_1 = \min\{\delta(t), t \in [-\pi, \pi]\} = \frac{1}{\lambda}.$$

If $\cos t \geq \lambda$, then $B - C \geq 0$ and by Theorem 2 we deduce that $\gamma_1 = \delta_1 = 1/\lambda$ (for the arc of Γ corresponding to $B - C \geq 0$).

If $\cos t < \lambda$, then $B - C < 0$ and by Theorem 2 we deduce that $\gamma_1 = \sigma_1 = \min\{\sigma(t), t \in T_1\}$ (for the arc of Γ corresponding to $B - C < 0$), where $\sigma(t)$ is given by (10).

We shall show that $\sigma_1 = \delta_1 = 1/\lambda$, i.e.

$$\sigma^2(t) \geq \frac{1}{\lambda^2}, \text{ for } \cos t < \lambda. \tag{13}$$

By using (10) the inequality (13) becomes

$$\sigma^2(t) = \frac{1}{\lambda^2} \frac{(1+\lambda^2 - 2\lambda \cos t)^2}{(1-\lambda \cos t)^2 (\cos t - \lambda)^2 + \sin^2 t (1+\lambda^2 - 2\lambda \cos t)^2} \geq \frac{1}{\lambda^2},$$

which is equivalent to

$$(1+\lambda^2 - 2\lambda \cos t)^3 \geq (1-\lambda \cos t)^2 (\cos t - \lambda)^2 + (1 - \cos^2 t)(1+\lambda^2 - 2\lambda \cos t)^2$$

or

$$(1+\lambda^2 - 2\lambda \cos t)^2 (\cos t - \lambda)^2 \geq (1-\lambda \cos t)^2 (\cos t - \lambda)^2,$$

which becomes $1+\lambda^2 - 2\lambda \cos t \geq 1 - \lambda \cos t$, i.e. $\lambda - \cos t \geq 0$. Hence $\gamma_1 = 1/\lambda$. Since Γ is convex, we have $\gamma_0 = 0$.

Therefore, by Definition 2, we obtain $RRI[\Gamma, 0] = [0, 1/\lambda] = [0, r/d]$. □

For $\gamma_1 = r/d = 2$, the regular refraction property of the circle Γ is illustrated in Figure 1.

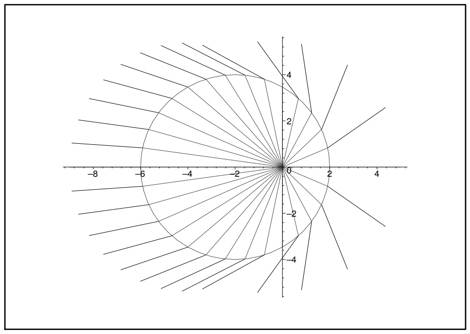

Figure 1.

4 Other examples

1) Let Γ be the arc of hyperbola given by the equation

$$z = -t + i\sqrt{1+t^2},\ t \in [-1,1].$$

In this case we have

$$A = \frac{2t}{1+2t^2},\ B = \frac{1}{(1+2t^2)\sqrt{1+t^2}},\ C = -\frac{1}{(1+2t^2)\sqrt{1+t^2}},$$

$$\delta(t) = \frac{1+2t^2}{2|t|\sqrt{1+t^2}}$$

and

$$\sigma(t) = \frac{1}{2}\frac{1+2t^2}{\sqrt{1+t^2+t^4}}.$$

Since

$$\delta_1 = \min\{\delta(t), t \in [-1,1]\} = \delta(1) = \frac{3\sqrt{2}}{4} = 1.06\ldots$$

and

$$\sigma_0 = \max\{\sigma(t), t \in [-1,1]\} = \sigma(1) = \frac{\sqrt{3}}{2} = 0.86\ldots,$$

from Theorem 2 we deduce that $RRI[\Gamma, 0] = \left[\frac{\sqrt{3}}{2}, \frac{3\sqrt{2}}{4}\right]$. For $\gamma_0 = \frac{\sqrt{3}}{2}$ and $\gamma_1 = \frac{3\sqrt{2}}{4}$ the regular refraction property of Γ is illustrated in Figure 2 and Figure 3, respectively.

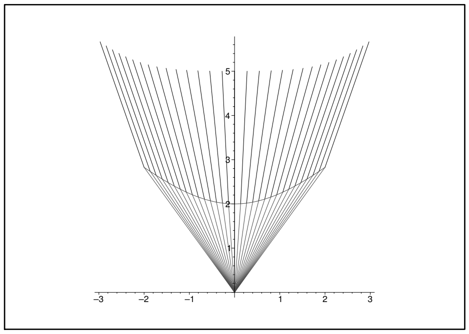

Figure 2.

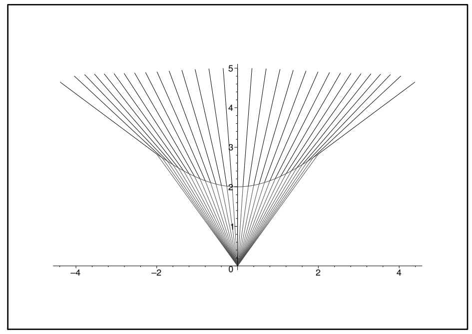

Figure 3.

2) Let Γ be the arc of spiral defined by

$$z = t(\cos t + i \sin t),\ t \in [2\pi, 3\pi].$$

In this case we have

$$A = \frac{1}{t},\ B = 1,\ C = \frac{2+t^2}{1+t^2} > 0,\ B - C = -\frac{1}{1+t^2} < 0$$

$$\delta(t) = \sqrt{1+t^2}$$

and

$$\sigma(t) = (2+t^2)\sqrt{\frac{1+t^2}{t^2+(2+t^2)^2}}.$$

Since $C > 0$ and $B - C < 0$, by Theorem 2 we deduce $\gamma_0 = 0$ and $\gamma_1 = \sigma_1 = \min\{\sigma(t), t \in [2\pi, 3\pi]\} = \sigma(2\pi) = 6.29\ldots$. Hence $RRI[\Gamma, 0] = [0, 6.29\ldots]$ (See Figure 4 for $\gamma = 6.29$).

We remark that $\delta(2\pi) = 6.36\ldots > \gamma_1$.

3) Let Γ be given by the equation

$$z = (2\pi^2 - t^2)(\cos t + i \sin t),\ t \in [0, \pi].$$

In this case we have

$$A = -\frac{2t}{2\pi^2 - t^2},\ B = 1,\ C = \frac{6t^2 + (2\pi^2 - t^2)^2}{4t^2 + (2\pi^2 - t^2)^2} > 0,\ B - C = -2\frac{2\pi^2 + t^2}{4t^2 + (2\pi^2 - t^2)^2}.$$

and we deduce $\gamma_0 = 0$ and $\gamma_1 = \sigma_1 = \sigma(\pi) = 1.682\ldots$. Hence $RRI[\Gamma, 0] = [0, 1.682\ldots]$ (See Figure 5, for $\gamma = 1.68$).

We remark that $\delta_1 = \delta(\pi) = 1.862\ldots > \gamma_1$.

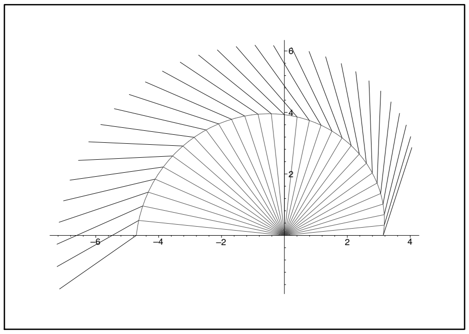

Figure 4.

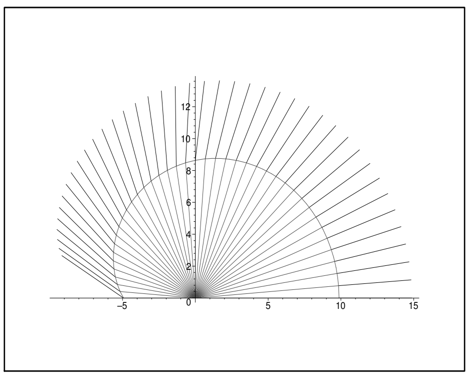

Figure 5.

4) Let Γ be given by the equation

$$z = -t + i(1 + t^2 - t^4),\ t \in [-1, 1].$$

In this case we have

$$A = \frac{t[1 + 2(1 - 2t^2)(1 + t^2 - t^4)]}{t^2 + (1 + t^2 - t^4)^2},\ B = \frac{1 - t^2 + 3t^4}{t^2 + (1 + t^2 - t^4)^2},\ C = \frac{2(6t^2 - 1)}{1 + 4t^2(1 - 2t^2)^2}.$$

By using Theorem 2, a straightforward calculation shows that $\gamma_0 = \sigma(0) = \dfrac{2}{3}$ and $\gamma_1 = 1.122\ldots$. (See Figure 6 and Figure 7, for $\gamma = \dfrac{2}{3}$ and $\gamma = 1.12$ respectively)

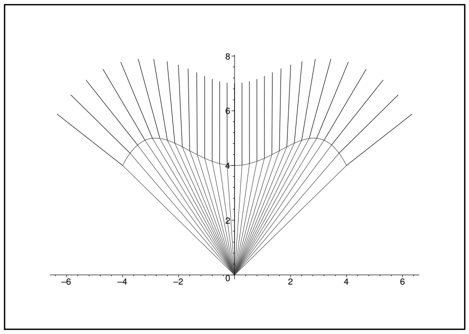

Figure 6.

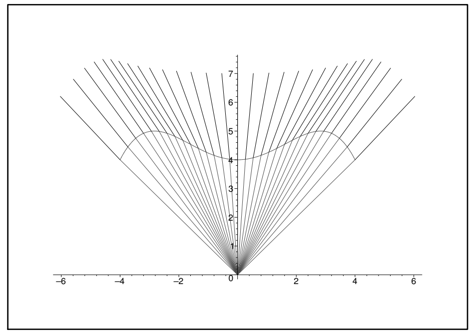

Figure 7.

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