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On K-contact η-Einstein manifolds

by
U. C.De and Sudipta Biswas

Abstract

The object of the present paper is to study some properties of a K-contact η-Einstein manifold.

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Introduction

Let (M^n, g) be a Riemannian manifold with contact form η, associated vector field ξ, (1, 1) - tensor field ϕ and associated Riemannian metric g. If ξ is a Killing vector field, then M^n is called a K-contact Riemannian manifold [1], [2]. A K-contact Riemannian manifold is called Sasakian [1], if the relation

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X \qquad (1)$$

holds, where ∇ denotes the operator to covariant differentiation with respect of g. A Sasakian manifold is K-contact, but the converse need not be true, except in dimension 3.

A K-contact manifold M^n is said to be η-Einstein if its Ricci tensor S is of the form

$$S = ag + b\, \eta \otimes \eta, \qquad (2)$$

where a, b are certain functions, η is called the associated 1-form and the vector field ξ defined by

$$g(X, \xi) = \eta(X) \qquad (3)$$

is called the generator, a, b are called associated scalars. It is known [2], [3] that in a K-contact η-Einstein Manifold of dimension n $(n > 3)$, a and b are constants. Example of an η-Einstein manifold is given by Okumura [4]. If $b = 0$, the manifold reduces to an Einstein manifold.

K-contact η-Einstein manifolds have been studied by many authors. Recently Shaikh, De, Binh [5], Yildiz and Murathan [6] studied K-contact η-Einstein manifolds satisfying certain curvature conditions. On the otherhand Okumura [7] proved that a compact, simply connected Sasakian symmetric manifold is isometric to a sphere. In the present paper we have studied some properties of η-Einstein manifolds. Throughout this paper we denote an n-dimensional K-contact η-Einstein manifold by M^n. After preliminaries in section 2, we find out the significance of the associated scalars in a K-contact η-Einstein manifold. Next we prove that if the manifold M^n under consideration satisfies Codazzi type of Ricci tensor then it is Ricci symmetric. Finally we obtain a sufficient condition for a compact, orientable K-contact η-Einstein manifold of dimension n ($n\geq 3$) without boundary to be conformal to a sphere.

1 Preliminaries

In a K-contact Riemannian manifold the following relations hold : [1], [8], [9]

$$\text{a)} \quad \phi\xi = 0, \quad \text{b)} \quad \eta(\xi) = 1, \quad \text{c)} \quad g(X,\xi) = \eta(X), \tag{1.1}$$

$$\phi^2 X = -X + \eta(X)\xi, \tag{1.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{1.3}$$

$$\nabla_X \xi = -\phi X, \tag{1.4}$$

$$g(R(\xi,X)Y,\xi) = \eta(R(\xi,X)Y) = g(X,Y) - \eta(X)\eta(Y), \tag{1.5}$$

$$R(\xi,X)\xi = -X + \eta(X),\xi \tag{1.6}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{1.7}$$

and

$$(\nabla_X \phi)(Y) = R(\xi,X)Y, \tag{1.8}$$

for any vector fields X, Y and R denotes the curvature tensor.

From (2) we obtain

$$r = na + b \tag{1.9}$$

and

$$S(\xi,\xi) = a + b, \tag{1.10}$$

where r denotes the scalar curvature.

Let L be the symmetric endomorphism of the tangent space at a point corresponding to the Ricci tensor S, then

$$g(LX,Y) = S(X,Y) \text{ for all } X,\ Y. \tag{1.11}$$

Let l^2 be the square of the length of the Ricci tensor, then

$$l^2 = S(Le_i, e_i) \tag{1.12}$$

where $\{e_i\}$, $i = 1, 2, \cdots, n$ is an orthonormal basis of the tangent space at a point.
Then from (2) we get

$$\begin{aligned} S(Le_i, e_i) &= ag(Le_i, e_i) + b\eta(Le_i)\eta(e_i) \\ &= aS(e_i, e_i) + bS(e_i, \xi)g(e_i, \xi) \\ &= ar + bS(\xi, \xi) \\ &= ar + b(a+b) \quad \text{[by (1.10)]} \\ &= a(na+b) + b(a+b) \quad \text{[by (1.9)]} \\ &= (n-1)a^2 + (a+b)^2. \end{aligned}$$

Hence,

$$l^2 = (n-1)a^2 + (a+b)^2. \quad (1.13)$$

2 Significance of the associated scalars in a K-contact η-Einstein manifold

We can express (2) as follows:

$$S(X, \xi) = (a+b)g(X, \xi). \quad (2.1)$$

From (2.1), we conclude that $a+b$ is an eigen value of the Ricci tensor L and ξ is an eigen vector corresponding to this eigen value.
Let V be any other vector orthogonal of ξ so that

$$\eta(V) = 0. \quad (2.2)$$

From (2), we obtain

$$S(X, V) = ag(X, V) + b\eta(X)\eta(V).$$

Hence in virtue of (2.2), we obtain

$$S(X, V) = ag(X, V). \quad (2.3)$$

From (2.3), we see that a is an eigen value of the Ricci tensor L and V is an eigen vector corresponding to this eigen value. Since the manifold under consideration is n-dimensional and V is any vector orthogonal to ξ, it follows from a known result in linear algebra [10] that the eigen value a is of multiplicity (n-1). Hence the multiplicity of the eigen value $a+b$ must be 1. So there are only two distinct eigen values of the Ricci tensor, namely $a+b$ and a, of which the former is simple and the latter is of multiplicity (n-1).

Hence we can state the following:

Theorem 1. *In a K-contact η-Einstein manifold, the Ricci tensor S has only two distinct eigen values $a+b$ and a of which the former is simple and the latter is of multiplicity n-1.*

3 K-contact η-Einstein manifold satisfying Codazzi type of Ricci tensor

From (2) we obtain

$$(\nabla_X S)(Y,Z) = b[(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y)], \quad (3.1)$$

since a and b are constants.

Hence

$$\begin{aligned} & (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\ = \; & b[(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y) + (\nabla_Y \eta)(X)\eta(Z) \\ + \; & (\nabla_Y \eta)(Z)\eta(X) + (\nabla_Z \eta)(Y)\eta(X) + (\nabla_Z \eta)(X)\eta(Y)]. \end{aligned} \quad (3.2)$$

Since in a K-contact manifold ξ is a Killing vector field, we get

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0. \quad (3.3)$$

Now

$$\begin{aligned} (\nabla_X \eta)(Y) &= X\eta(Y) - \eta(\nabla_X Y) \\ &= Xg(Y,\xi) - g(\nabla_X Y, \xi) \\ &= g(Y, \nabla_X \xi). \end{aligned}$$

Hence we obtain from (3.3)

$$(\nabla_X \eta)(Y) + (\nabla_Y \eta)(X) = 0 \quad \text{for all } X,\ Y. \quad (3.4)$$

Using (3.4) in (3.2), we get,

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0. \quad (3.5)$$

A Riemannian manifold is said to have cyclic Ricci tensor if the Ricci tensor S satisfies the condition (3.5). It is known [11] that Cartan hypersurfaces are manifolds, with non-parallel Ricci-tensor, satisfying (3.5). It follows from the above discussions that a K-contact η-Einstein manifold has cyclic Ricci tensor.

Now we suppose that a manifold has Ricci tensor of Codazzi type [12], that is,

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z).$$

Hence

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z) = (\nabla_Z S)(X,Y).$$

By virtue of the above relation we get from (3.5)(since S is symmetric)

$$(\nabla_X S)(Y,Z) = 0.$$

That is, the manifold is Ricci symmetric. Thus we can state the following:

Theorem 2. *If a K-contact η-Einstein manifold has Ricci tensor of Codazzi type, then the manifold is Ricci symmetric.*

4 Sufficient condition for a compact, orientable K-contact η-Einstein manifold of dimension n ($n \geq 3$) without boundary to be conformal to a sphere in $\mathbf{E_{n+1}}$

Since the scalars a and b are constants of a K-contact η-Einstein manifold, it follows from (1.9) that r is constant and so also is the length of the Ricci tensor. We suppose that the manifold under consideration admits a non-isometric conformal motion generated by a vector field X. Since l^2 is constant, it follows that

$$\pounds_X \, l^2 = 0 \quad (4.1)$$

where $\pounds_X$ denotes Lie differentiation with respect to X. Now it is known [13, p.57] that if a compact Riemannian manifold M^n ($n > 2$) with constant scalar curvature admits an infinitesimal non isometric conformal transformation X such that $\pounds_X \, l^2 = 0$, then M is isometric to a sphere. But a sphere is an Einstein manifold. Hence this implies that $b = 0$ which is a contradiction.
This leads to the following:

Theorem 3. *A compact, orientable η-Einstein K-contact manifold M^n ($n \geq 3$) without boundary can not admit a non-isometric conformal transformation.*

We begin with the definition of conformality of one Riemannian manifold to another one.

Let $(M, \; g)$ and $(M', \; g')$ be two n-dimensional Riemannian manifolds. If there exists a one-to-one differentiable mapping $(M, g) \rightarrow (M', \; g')$ such that the angle between any two tangent vectors at a point p of M is always equal to that of the corresponding two vectors at the corresponding point p' of M', then $(M, \; g)$ is said to be conformal to $(M', \; g')$. Y. Watanabe [14] has given a sufficient condition of conformality of an n-dimensional Riemannian manifold to an n-dimensional sphere in E_{n+1}.

Its statement is as follows:

If in a compact n-dimensional Riemannian manifold M^n, there exists a non-parallel vector field X such that the condition

$$\int_{M^n} S(X, X) dv = \frac{1}{2} \int_{M^n} |dX|^2 dv + \frac{n-1}{n} \int_{M^n} (\delta X)^2 dv \quad (4.2)$$

holds, then M^n is conformal to a sphere in E_{n+1}, where dv is the volume element of M^n and dX and δX are curl and divergence of X respectively.

Here we consider a compact and orientable η-Einstein K-contact manifold M^n without boundary having associated scalars a, b and generator ξ. It satisfies (2) and (3).

Hence

$$S(\xi, \xi) = a + b.$$

In virtue of this and by taking ξ for X, the condition (4.2) takes the following from

$$\int_{M^n}(a+b)dv = \frac{1}{2}\int_{M^n}|d\xi|^2 dv + \frac{n-1}{n}\int_{M^n}(\delta\xi)^2 dv. \quad (4.3)$$

Since ξ is a Killing vector field in a K-contact manifold, divergence of $\delta\xi = 0$ ([13, p.43]).

Now we suppose that ξ is a parallel vector field (i.e., $\nabla_X\xi = 0$ for any tangent vector X). It follows from (1.4) that $\phi X = 0$, for all X which is a contradiction. Therefore ξ is a non-parallel vector field. Hence (4.3) takes the form

$$\int_{M^n}(a+b)dv = \frac{1}{2}\int_{M^n}|d\xi|^2 dv. \quad (4.4)$$

Then by Watanabe's condition (4.2), M^n is conformal to a sphere. We can therefore state the following :

Theorem 4. *If a compact, orientable K-contact η-Einstein manifold M^n, $(n\geq 3)$ without boundary the condition (4.4) holds, then the manifold M^n is conformal to a sphere immersed in E_{n+1}.*

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Department of Mathematics,
University of Kalyani,
Kalyani- 74 1 2 3 5,
West Bengal, India
E-mail: `ucde@klyuniv.ernet.in`